

# Analysis of Typical Mistakes

## Evaluation of minimal (maximal) value

One of the most popular class of Tournament of the Towns problems is the one where it is required to find a minimal (maximal) value.

There are two kinds of mistakes:

- Either a (correct) example is given but there is no proof that this value is a minimal (maximal) one.
- Or there is an estimate for some value; however, there is no example that the minimal (maximal) value can be reached indeed.

A correct approach includes both steps:

- Obtaining an estimate
- Producing an example

## Problems

1. Five 15 liter vessels contain 1, 2, 3, 4, 5 liter of liquid correspondingly. It is allowed to tripple amount of liquid in any vessel by pouring liquid from another vessel. Find the maximal amount of liquid that can be collected in one of the vessels.
2. Each square of a  $4 \times 4$  board is occupied by a couple of cockroaches. At some moment each cockroach moves to an adjacent (by side) square so that all the couples are separated. What is the maximal number of cockroach-free squares?
3. What is the minimal number of paper triangles needed to cover a cube (without overlapping)?
4. (TT Spring Round 2006 Juniors and Seniors). Peter has  $n^3$  white  $1 \times 1 \times 1$ -cubes. He wants to make a  $n \times n \times n$ -cube using them, and he wants to make this cube totally white from the outside. What is the minimum number of sides of the cubes Basil has to paint in black to prevent Peter from doing this?
  - (a)  $n = 2$ ;
  - (b)  $n = 3$ .

ANSWERS (a) 2 sides (b) 12 sides

5. (TT SR 2003 Juniors) Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence? ANSWER 5
6. One should plant four pear trees and several apple trees in such a way that for any apple tree there are exactly two pear trees at the distance 10 meters. Find the maximal possible number of apple trees one can plant. ANSWER 12

7. Every inhabitant of the Wonderland is either Truth Teller (who always tells the truth) or Liar (who always lies). The new Governor came for a short visit with purpose to evaluate the number of Liars. Each of aboriginals aware about everyone who is who, but the Governor only knows that not all of them are Liars. Each day he may choose a group of aboriginals and ask everyone in this group about the number of Liars among them. Find the minimum number of days of the Governor's stay in Wonderland. ANSWER 2 days
8. Council of Wizards is tested in the following way: the King lines up all the wizards in a line and places either white or black hat on each of them. Every wizard can see the hats of all the wizards ahead of him, but not on himself and on those who is behind. According to testing each wizard in backwards order must call the color of his hat. When the last wizard called, the King executes all wizards who failed to name the color of the hats correctly. Prior to the testing, wizards worked out a system, which allows to minimize the number of executions. Find the maximal number of wizards that stay alive (guaranteed).
9. (TT Fall Round 2006 A-Level Juniors) A Magician has a deck of 52 cards. Spectators want to know the order of cards in the deck (without specifying face-up or face-down). They are allowed to ask questions : "How many cards are there between such-and-such card and such-and-such card"? One of the spectators knows the card order. Find the minimal number of questions he needs to ask to be sure that the other spectators can learn the card order. ANSWER 34.

## More problems to practice

1. Joe keeps all his money in Loony Bank. On his account he has \$500. The Bank allows two kinds of operations: a withdraw of \$300 or deposit of \$198. What is the maximal amount Joe can withdraw (in a few steps)? ANSWER: 498.
2. A family of four: a man, his wife, a boy, and an old man need to cross a river at night. Dilapidated bridge can be safe for only two people at the time. The family has one flashlight with them. It takes one minute, two minutes, 5 minutes and 10 minutes for the man, the boy, the woman, and the old man respectively to cross the bridge. Going in pairs, they go with the minimal speed of two. Find the minimal time it takes for the family to cross the river.  
ANSWER:17 min.
3. (TT Fall Round 2006 Juniors) A square is dissected into  $n$  congruent non-convex polygons whose sides are parallel to the sides of the square, and no two of these polygons are parallel translates of each other. What is the maximum value of  $n$ ? ANSWER: 8
4. (TT Fall Round 2003 Juniors) 25 checkers are placed on 25 leftmost squares of  $1 \times N$  board. Checker can either move to the empty adjacent square to its right or jump over adjacent right checker to the next square if it is empty. Moves to the left are not allowed. Find minimal  $N$  such that all the checkers could be placed in the row of 25 successive squares but in the reverse order. ANSWER: 50.
5. (TT Spring Round 2005 Juniors) There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens

are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens? ANSWER 23.

6. (TT Fall Round 2005 Juniors) A chess piece moves as follows: it can jump 8 or 9 squares either vertically or horizontally. It is not allowed to visit the same square twice. At most, how many squares can this piece visit on a  $15 \times 15$  board (it can start from any square)? ANSWER: 144.

## Solutions

1. SOLUTION. Since both 300 and 198 are multiples of 6, then Joe can withdraw only a multiple of 6. The largest multiple of 6 not exceeding 500 is 498.

*Example* (sequence of steps to withdraw \$498):

$$500 - 300 = 200$$

$$200 + 198 = 398$$

$$398 - 300 = 98$$

$$98 + 198 = 296$$

$$296 + 198 = 494.$$

Note, that initial amount of \$500 decreased by \$6. Therefore, if Joe repeats the above procedure 15 more times, his account would be 404 (without being overdraft). Finally,

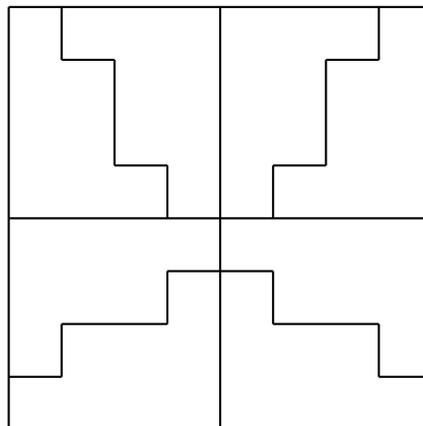
$$404 - 300 = 104$$

$$104 + 198 = 302$$

$$302 - 300 = 2.$$

2. SOLUTION. It is easy to check by observation that the best scenario takes 17 minutes. First step: the father and the boy, father is coming back. Second step: the old man and the mother, the boy is coming back. Last step: the father and the boy.
3. (TT Fall Round 2006 Juniors)

SOLUTION. The maximum value of  $n$  is at most 8 because such a polygon can only have 8 possible orientations. We may use each of them once as otherwise we would have two copies which are parallel translates of each other. One can construct an example demonstrating that 8 works. The maximum value is in fact 8 as it is attained by the polygon in the diagram below.



4. (TT Fall Round 2003 Juniors) SOLUTION. Let us prove that the arrangement in question is impossible, if  $N < 50$ . Case  $N = 49$  means that checker “25” stays on its initial place, so the only possible move is that checker “24” jumps through checker “25” on the 26-th place (its final position). After that any movement to the right is impossible.

Example, that  $N = 50$  works. (a) Checker “25” moves on the 26-th place (its final position).

(b) Checker “23” jumps through “24” and “25”, then moves on the 28-th place (its final position).

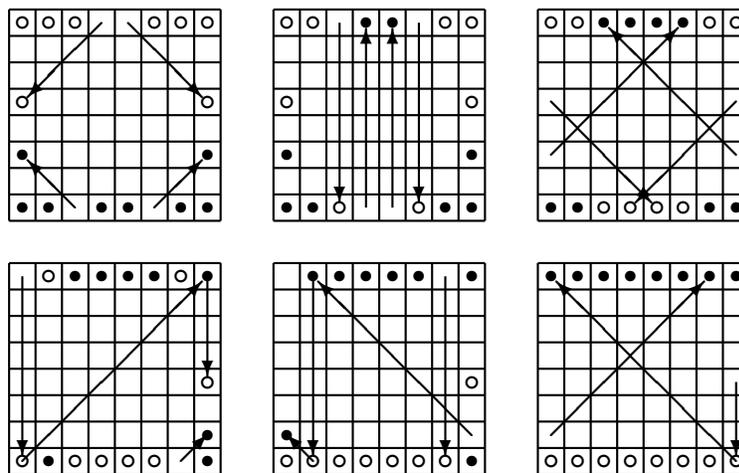
(c) Checker “21” jumps through “22”, “24”, “25” and “23”, then moves on the 30-th place (its final position) and so on.

Finally, checker “1” jumps through “2”, “4” ,... “22”, “24”, “24”, “25”, “213” ,... , “3” and then moves on the 50-th place (its final position). Now,

(a) Checker “2” moves one space to the right and then jumps through “4” ,... , “22,” “24” , “25” , “23” ,... , “3” occupying the 49-th place (its final position).

(b) Checker “4” moves one space to the right and then jumps through “6” ,... , “22” , “24” , “25” , “23” ,... , “5” occupying the 47-th place (its final position) and so on.

5. (TT Spring Round 2005 Juniors) SOLUTION. We first show that the task can be accomplished in 23 moves.



We now prove that we need at least 23 moves. Each the 16 Queens must move at least once. Of the two Queens on each inside column, at most one can move only once. This means at least 6 extra moves. Of the four Queens at the corners, at most three can move only once. This means at least 1 extra move. Hence the minimum is 23 moves.

6. (TT Fall Round 2005 Juniors) SOLUTION.

The diagram below shows the  $15 \times 15$  chessboard divided into a central cross of width 3 and four quadrants each a  $6 \times 6$  square. The numbering shows that all 144 squares in the four quadrants may be visited. If more squares may be visited, then the chess piece must visit one of the squares of the central cross. However, from any such square, the piece can never get to any of the 144 squares in the four quadrants. Even if it can visit all squares in the central cross, the total of 81 is well short of 144.

14	16	18	20	22	24				13	15	17	19	21	23
38	40	42	44	46	48				37	39	41	43	45	47
62	64	66	68	70	72				61	63	65	67	69	71
86	88	90	92	94	96				85	87	89	91	93	95
110	112	114	116	118	120				109	111	113	115	117	119
134	136	138	140	142	144				133	135	137	139	141	143
11	9	7	5	3	1				12	10	8	6	4	2
35	33	31	29	27	25				36	34	32	30	28	26
59	57	55	53	51	49				60	58	56	54	52	50
83	81	79	77	75	73				84	82	80	78	76	74
107	105	103	101	99	97				108	106	104	102	100	98
131	129	127	125	123	121				132	130	128	126	124	122

7.

SOLUTION. In his first question, spectator (S) calls the top and bottom cards. The answer “50” reveals two outmost cards. Let us number either of them by 1 and the other by 52, defining the order of the deck. Then, S calls the pair (1,3). The answer “1” reveals the card numbered 3 (S intentionally skips card 2, which we refer to as the “space”). He continues to ask questions in pairs. In odd questions he calls two farthest unmentioned cards (one of them is “space”, we refer to the other as the “antispaces”), reassigns unmentioned card adjacent to “antispaces” as new “space” and in his next (even) question he calls two cards adjacent to “space”.

Thus, in the second pair of the questions, S calls the pairs (2, 51) ( two farthest unmentioned cards) and (51, 49) ( two cards adjacent to “space”.) Next pair of questions is (50, 4), (4, 6), then (5, 48), (48, 46) and so on. After the first pair of questions, the audience is aware that the situation is the following (revealed cards are boxed):

$\boxed{1}$  2  $\boxed{3}$  4 5 6 ... 48 49 50 51  $\boxed{52}$

Notice that while the first pair of questions reveals all three of mentioned cards, the second pair of questions would leave two possible arrangements. We refer to the actual arrangement of cards as the “main case” and to the other possible arrangement of cards as the “auxiliary case”.

So, after the second pair of questions, the situation is one of the following ( cards that have been mentioned but not yet revealed are underlined):

$\boxed{1}$   $\underline{2}$   $\boxed{3}$  4 5 6 ... 48  $\underline{49}$  50  $\underline{51}$   $\boxed{52}$  (main case)

$\boxed{1}$   $\underline{2}$   $\boxed{3}$   $\underline{4}$  5 6 ... 48 49 50  $\underline{51}$   $\boxed{52}$  (auxiliary case)

To distinguish the main and auxiliary cases, we observe that the distance between the two farthest unmentioned cards is different (it is lower in the auxiliary case). The answer on next question would eliminate the auxiliary case.

Really, the fifth question names cards (50, 4). The answer “46’ leaves the main case; otherwise, the answer would be “45”.

Thus, after 5 questions (in total) we come to the following situation:

$\boxed{1}$   $\boxed{2}$   $\boxed{3}$   $\underline{4}$  5 6 ... 48  $\boxed{49}$   $\underline{50}$   $\boxed{51}$   $\boxed{52}$

One may check that after 33 questions we get:

$\boxed{1}$  ...  $\boxed{24}$   $\underline{25}$  26 27  $\boxed{28}$   $\underline{29}$   $\boxed{30}$  ...  $\boxed{52}$

In his last question S calls (25, 26). The answer 0 eliminates an auxiliary case, and (27) card is revealed as the last card left.

Now let us show that 33 (or less) questions are not enough. Let us assume that originally all the cards are split into 52 groups; one card in each group. In case of a question when named cards belong to different groups, we combine these groups into one. So, each question decreases the number of groups maximum by one. Therefore, after 33 questions the number of groups left is no less than  $52 - 33 = 19$ . Among them the number of groups consisting of at least 3 cards is no more than 17. Thus, there are either two groups consisting of one card or there is a group consisting of exactly 2 cards. In either case if these two cards trade places, while the rest of the cards remain untouched, then the answers will be the same. This means that the order of cards can not be restored uniquely.