

ALMOST SURE LOCAL WELL-POSEDNESS OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION BELOW L^2

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ABSTRACT. We consider the local well-posedness of the periodic cubic nonlinear Schrödinger equation with initial data below L^2 . In particular, we exhibit nonlinear smoothing when the initial data are randomized. Then, we prove that it is locally well-posed almost surely for the initial data in the support of the canonical Gaussian measures on $H^s(\mathbb{T})$ for each $s > -\frac{1}{3}$.

1. INTRODUCTION

Consider the initial value problem for the one dimensional periodic cubic nonlinear Schrödinger equation (NLS):

$$(1.1) \quad \begin{cases} iu_t - u_{xx} \pm u|u|^2 = 0 \\ u|_{t=0} = u_0, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}. \end{cases}$$

In this paper, we establish almost sure local well-posedness for¹ (1.1) with respect to the canonical Gaussian measure supported on $H^s(\mathbb{T})$ in the range $-\frac{1}{3} < s < 0$. This result is motivated by (a) the well-posedness theory of nonlinear dispersive equations with low regularity initial conditions and (b) construction of measures on phase spaces which are invariant under the (1.1) evolution.

1.1. Low Regularity Well-Posedness Theory. The well-posedness theory for the Cauchy problem (1.1) for rough data has been the subject of recent studies. In particular, detailed studies of (1.1) have revealed diverse phenomena of the associated data-to-solution map leading to ramified notions of ill-posedness and well-posedness. It is known that:

- The data-to-solution map $H^s \ni u_0 \mapsto u(t) \in H^s$ (for some $t \neq 0$) is well-defined and analytic provided $s \geq 0$ [26, 2].
- Uniform continuity of the data-to-solution map from H^s to H^s fails for $s < 0$ [14, 5, 8]. Moreover, when $s < 0$, the data-to-solution map is discontinuous from $H^s(\mathbb{T})$ even to the space of distributions $(C^\infty(\mathbb{T}))^*$ [9, 19].
- The data-to-solution map is unbounded from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ provided $s < -\frac{1}{2}$. For example, the norm inflation phenomena identified in [8] shows there exist initial data arbitrarily small in $H^s(\mathbb{R})$ which evolve into solutions which are arbitrarily large in $H^s(\mathbb{R})$ in an arbitrarily short time.

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¹We actually consider the Wick ordered version (1.10) instead of (1.1) below.

- The data-to-solution map is bounded from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})^2$ provided $-\frac{1}{6} \leq s < 0$ [16]. Moreover, there exist weak solutions associated to every $u_0 \in H^s(\mathbb{R})$ in this range. These weak solutions are not known to be unique.

It is unknown whether well-posedness with merely continuous dependence upon the initial data for (1.1) holds true in H^s for $s \geq -\frac{1}{2}$. In contrast to these negative results, this paper establishes positive results on subsets of $\dot{H}^s(\mathbb{T})$ for certain $s < 0$ which are full with respect to natural Gaussian measures.

1.2. Invariant Gibbs Measures. Inspired by [18] and following an approach from [29], Bourgain constructed [3] the Gibbs measure for³ (1.1) and established its invariance under the (1.1) flow. Sufficiently regular solutions of (1.1) satisfy mass conservation

$$(1.2) \quad \|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})},$$

and Hamiltonian conservation

$$(1.3) \quad H[u(t)] = \int_{\mathbb{T}} \frac{1}{2} |u_x(t)|^2 \pm \frac{1}{4} |u(t)|^4 dx = H[u_0].$$

By the Hamiltonian structure of the equation, the Gibbs measure

$$(1.4) \quad d\mu = e^{-H[u]} \prod_{x \in \mathbb{T}} du(x)$$

is formally invariant. The Gibbs measure is rewritten as a weighted Wiener measure

$$(1.5) \quad d\mu = Z^{-1} e^{\mp \frac{1}{4} \int |u|^4 dx} d\rho$$

where

$$(1.6) \quad d\rho = Z_0^{-1} e^{-\frac{1}{2} \int |u_x|^2 dx} \prod_{x \in \mathbb{T}} du(x)$$

is the Wiener measure.

The construction of the Gibbs measure proceeds by showing that the density $e^{\mp \frac{1}{4} \int |u|^4 dx}$ is in $L^1(d\mu)$. Expressed in terms of Fourier coefficients, the Wiener measure describes a Gaussian distribution for each $|n| \hat{u}(n)$. Thus, a typical element in the support of the Wiener measure may be represented⁴

$$(1.7) \quad u = u^\omega = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|n|} e^{inx}$$

where the $\{g_n\}_{n \in \mathbb{Z}}$ are independent standard complex valued Gaussian random variables. Almost surely in ω , the series (1.7) defines a function $u^\omega \in H^{\frac{1}{2}-}(\mathbb{T})$. Thus, $\int |u|^4 dx$ is well-defined and the density $e^{\mp \frac{1}{4} \int |u|^4 dx}$ may be shown⁵ to be in $L^1(\omega)$.

The invariance of the Gibbs measure is established by studying a sequence of finite dimensional approximations obtained by Dirichlet-projecting the dynamics of (1.1) onto

²M. Christ (with J. Holmer and D. Tataru) announced similar results [10] on \mathbb{T} in April 2009 at IHP in Paris.

³In fact, the construction and invariance of the Gibbs measure is proved for a family of (sub-)quintic NLS equations containing (1.1) in [3].

⁴There is an issue regarding the zero Fourier mode which the reader is invited to ignore. The Wiener measure will soon be adjusted using the conserved L^2 norm into another formally invariant Gaussian measure which avoids the $n = 0$ issue.

⁵In the defocusing case, this step is clear. The focusing case requires a more delicate analysis exploiting an (invariant) $L^2(\mathbb{T})$ size cutoff (See [18] and [3]).

finitely many modes using the fact that the (1.1) evolution is well-defined on the support of the Wiener measure. Recall that the evolution for (1.1) is well-defined for all $u_0 \in L^2(\mathbb{T})$ so it is certainly well-defined on the support of the Gibbs measure living in $H^{\frac{1}{2}^-}(\mathbb{T})$.

The questions of existence and invariance of the Gibbs measure associated to (1.1) (in fact, associated to the Wick ordered version (1.10)) posed on the two dimensional torus \mathbb{T}^2 were investigated in [4]. In the two dimensional case, the representation (1.7) almost surely in ω defines a distribution in $H^{0^-}(\mathbb{T}^2)$ but not in $L^2(\mathbb{T}^2)$. More precisely, u defined in (1.7) is almost surely in $B_{2,\infty}^0(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$. Since the data-to-solution map is not well-defined on even $L^2(\mathbb{T}^2)$, the issue of well-defined dynamics on the support of the Gibbs measure is not at all obvious. Bourgain established [4] a well-defined local-in-time dynamics on the support of the Wiener measure. In the defocusing case, he proved global well-posedness almost surely on the support, exploiting the invariance of the (finite dimensional) Gibbs measure.

1.3. Almost Sure Local Well-Posedness. Consider the canonical Gaussian measure on $H^\alpha(\mathbb{T})$:

$$d\tilde{\rho}_\alpha = \tilde{Z}_\alpha^{-1} e^{-\frac{1}{2} \int |D^\alpha u|^2 dx} \prod_{x \in \mathbb{T}} du(x)$$

where $D = \sqrt{-\partial_x^2}$. The Gaussian measure $d\rho_\alpha$ corresponds to a collection of Gaussian distributions of $\{|n|^\alpha \hat{u}(n)\}_{n \in \mathbb{Z}}$, so a typical element in the support may be represented⁶ as a random Fourier series

$$(1.8) \quad u = u^\omega = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|n|^\alpha} e^{inx}.$$

This series almost surely in ω defines a function in $H^{\alpha-\frac{1}{2}^-}(\mathbb{T})$ but not in $H^{\alpha-\frac{1}{2}}(\mathbb{T})$. Note that u_0^ω in (1.8) can also be expressed as $u_0^\omega = \sum \tilde{g}_n e_n$ where e_n is another orthonormal basis in $H^\alpha(\mathbb{T})$ and $\{\tilde{g}_n\}$ is another family of independent standard complex-valued Gaussian random variables. In this respect, the Gaussian measure $\tilde{\rho}_\alpha$ is canonical. See [17] for discussions on the Gaussian measures on Banach spaces. Also, see [30].

Since $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ under the flow of (1.1), we formally expect the Gaussian measure on $L^2(\mathbb{T})$

$$(1.9) \quad d\rho_0 = Z_0^{-1} e^{-\frac{1}{2} \int |u|^2 dx} \prod_{x \in \mathbb{T}} du(x)$$

to be invariant in view of the Hamiltonian structure of (1.1). This measure ρ_0 is the white noise on the distributions on \mathbb{T} and is supported on $H^{-\frac{1}{2}^-}(\mathbb{T}) \setminus H^{-\frac{1}{2}}(\mathbb{T})$, i.e. in the scaling supercritical regime for (1.1). Nonetheless, it was shown in [23] that the white noise ρ_0 is a weak limit of the invariant measures under the flow of (1.1). However, this result does not establish the invariance of the white noise ρ_0 since the flow is not well-defined on its support. See Remark 1.3. Invariance of white noise has recently been established for the KdV equation on \mathbb{T} [24, 21, 23]. See [22] for a summary of these results.

If we define $v(t) = e^{i\gamma t} u(t)$, with $\gamma \in \mathbb{R}$, where u solves (1.1), then v satisfies $i\partial_t v - v_{xx} \pm |v|^2 v + \gamma v = 0$. Recall that $\int |u|^2 dx := \frac{1}{2\pi} \int |u|^2 dx$ is conserved under the flow of (1.1) for

⁶The issue with the zero mode should be ignored; see (1.12) below.

$u_0 \in L^2(\mathbb{T})$. Hence, by letting $\gamma = \mp 2 \int |u|^2 dx$, (1.1) is equivalent to

$$(1.10) \quad \begin{cases} iu_t - u_{xx} \pm (|u|^2 - 2u \int |u|^2 dx) = 0 \\ u|_{t=0} = u_0, \end{cases}$$

at least for $u_0 \in L^2(\mathbb{T})$. However, for $u_0 \notin L^2(\mathbb{T})$, we can't freely convert solutions of (1.10) into solutions of (1.1). Bourgain [4] refers to (1.10) as the Wick ordered cubic NLS since it may also be obtained from the Wick ordered Hamiltonian.

In the following, we choose to study (1.10) *instead* of (1.1) for $u_0 \notin L^2(\mathbb{T})$. See Remark 1.6. In particular, we consider u_0 of the form (slightly adjusted compared with (1.8))

$$(1.11) \quad u_0 = u_0^\omega = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1 + |n|^{2\alpha}}} e^{inx}$$

which can be regarded as a typical element in the support of the Gaussian measure

$$(1.12) \quad d\rho_\alpha = Z_\alpha^{-1} \exp\left(-\frac{1}{2} \int |u|^2 dx - \frac{1}{2} \int |D^\alpha u|^2 dx\right) \prod_{x \in \mathbb{T}} du(x).$$

By shifting the Laplacian as in [3, 4], i.e. replacing $-u_{xx}$ by $-u_{xx} + u$ in (1.1) or (1.10), we can also regard u_0 of the form (1.11) as the functions in the support of the Gaussian measure $\tilde{\rho}_\alpha$. (Strictly speaking, one needs to replace the denominator in (1.11) by $(1 + |n|^2)^{\frac{\alpha}{2}}$.)

Note that u_0^ω in (1.11) is not in $L^2(\mathbb{T})$ almost surely in ω for $\alpha \leq \frac{1}{2}$. The following theorem states the almost sure local well-posedness for each $\alpha \in (\frac{1}{6}, \frac{1}{2}]$.

Theorem 1. *Let $\alpha \in (\max(\frac{s}{3} + \frac{1}{6}, s), \frac{1}{2}]$ with $s \in [0, \frac{1}{2}]$. Then, the periodic (Wick ordered) cubic NLS (1.10) is locally well-posed almost surely in $H^{\alpha - \frac{1}{2}^-}(\mathbb{T})$. More precisely, there exist $c, \delta > 0$ such that for each $T \ll 1$, there exists a set $\Omega_T \in \mathcal{F}$ with the following properties:*

- (i) $\mathbb{P}(\Omega_T^c) = \rho_\alpha \circ u_0(\Omega_T^c) < e^{-\frac{c}{T^\delta}}$, where $u_0 : \Omega \rightarrow H^{\alpha - \frac{1}{2}^-}(\mathbb{T})$.
- (ii) For each $\omega \in \Omega_T$ there exists a unique solution u of (1.10) in

$$e^{-i\partial_x^2 t} u_0 + C([-T, T]; H^s(\mathbb{T})) \subset C([-T, T]; H^{\alpha - \frac{1}{2}^-}(\mathbb{T}))$$

with the initial condition u_0^ω given by (1.11).

In particular, we have a.s. local well-posedness with respect to the Gaussian measure (1.12) supported in $H^\sigma(\mathbb{T})$ for each $\sigma > -\frac{1}{3}$.

We conclude the introduction by stating several important remarks.

Remark 1.1. A linear part of a solution constructed in Theorem 1 indeed lies in $C([-T, T]; B(\mathbb{T}))$ for any Banach space $B(\mathbb{T}) \supset H^\alpha(\mathbb{T})$ such that $(H^\alpha, B, \rho_\alpha)$ is an abstract Wiener space (roughly speaking, any Banach space B containing H^α where the Gaussian measure ρ_α makes sense as a countable additive probability measure.) In this case, a solution u to (1.10) lies in

$$u = e^{-i\partial_x^2 t} u_0 + (-i\partial_t + \partial_x^2)^{-1} u \in C([-T, T]; B(\mathbb{T})) + C([-T, T]; H^s(\mathbb{T})).$$

As examples of B , we can take the Sobolev spaces $W^{\sigma, p}$ with $\sigma < \alpha - \frac{1}{2}$, and the Fourier-Lebesgue spaces $\mathcal{FL}^{\sigma, p}$ with $\sigma < \alpha - \frac{1}{p}$, where $\mathcal{FL}^{\sigma, p}$ is defined via the norm $\|f\|_{\mathcal{FL}^{\sigma, p}} = \|\langle n \rangle^\sigma \hat{f}(n)\|_{L_n^p}$. See Bényi-Oh [1] for regularity of ρ_α (and u_0 in (1.11)) in different function spaces. We can also take the Besov spaces $B_{p, \infty}^{\alpha - \frac{1}{2}}$ with $p < \infty$. In [1], we study the regularity of ρ_α for $\alpha = 1$ but it can be easily adjusted for any α .

Remark 1.2. Uniqueness holds only in the ball centered at $S(t)u_0^\omega$ of radius 1 in $Z^{s, \frac{1}{2}}$ defined in (2.1) below. Continuous dependence on the initial data holds in $H^s(\mathbb{T})$ for some $s \geq 0$. See Section 3. Also, Theorem 1 can not be applied to (1.1), since u_0^ω is almost surely not in $L^2(\mathbb{T})$.

Remark 1.3. One should regard Theorem 1 as a local-well posedness result for a typical element in $H^{\alpha - \frac{1}{2}^-}(\mathbb{T})$ (independent of choice of a basis of $H^\alpha(T)$ as mentioned before.) Theorem 1 may also be viewed as progress towards showing local well-posedness of (1.10) on the support of the white noise ρ_0 , corresponding to $\alpha = 0$.

Remark 1.4. Note that $u_{N,a}(x, t) = ae^{i(Nx + N^2t \mp |a|^2t)}$ solves the Wick ordered cubic NLS (1.10) for $a \in \mathbb{C}$ and $N \in \mathbb{N}$. Hence, by following the argument by Burq-Gérard-Tzvetkov [5], we can show failure of uniform continuity of the solution map of (1.10) below $L^2(\mathbb{T})$. Thus, it is nontrivial to construct solutions of (1.10) in the negative Sobolev spaces.

As mentioned earlier, Molinet [19] showed that (1.1) is not well-posedness below $L^2(\mathbb{T})$ by proving the weak discontinuity of the flow map in $L^2(\mathbb{T})$. We point out that his argument does not apply to (1.10).

Remark 1.5. On the one hand, it is known that u_0^ω of the form (1.11) is in $\mathcal{FL}^{s,p}$ almost surely for $s < \alpha - \frac{1}{p}$ and not in the smoother spaces. See [21, 1]. On the other hand, Christ [7] constructed local-in-time solutions in $\mathcal{FL}^{0,p}$ for $2 < p < \infty$ by his power series method. Also see Grünrock-Herr [13] for the same result via the fixed point argument. Hence, it follows from their result that (1.10) with u_0^ω in (1.11) is almost surely locally well-posed for $\alpha > 0$, but the solution u lies in $C([-T, T]; \mathcal{FL}^{0, \frac{1}{\alpha}+}(\mathbb{T}))$.

In the following, we construct local-in-time solutions in $C([-T, T]; H^{\alpha - \frac{1}{2}^-}(\mathbb{T}))$ by exhibiting nonlinear smoothing under randomization. See Section 3. In the forthcoming paper [11], we extend the local solutions to the global ones (in the absence of invariant measures) by exploiting such nonlinear smoothing.

Remark 1.6. In [4], the two dimensional Wick ordered (defocusing) cubic NLS appeared as an equivalent formulation of (the limit of the finite dimensional) Hamiltonian equation, arising from the Wick ordered Hamiltonian. Such renormalization on the nonlinearity was a natural consequence of the Euclidean φ_2^4 quantum field theory. In our case, by taking the initial data u_0^ω to be of the form (1.11) with $\alpha \leq \frac{1}{2}$, (1.10) also arises as an equivalent formulation of (the limit of the finite dimensional) Hamiltonian equation from the Wick ordered Hamiltonian, (at least for $\alpha > \frac{1}{4}$.) Moreover, such renormalization is needed to obtain the continuous dependence on the initial data [7, 13].

Remark 1.7. In [4], the local solutions were constructed via the fixed point argument around the linear solution $z_1(t) := S(t)u_0$ with probabilistic arguments. Also see Burq-Tzvetkov [6] and Thomann [25] for related arguments. While the basic probabilistic argument (e.g. Lemma 4.4) is similar, the argument in [6, 25] further exploits the properties of the eigenfunctions, and the argument in [4] and this paper exploits more properties of the product of Gaussians via the hypercontractivity of the Ornstein-Uhlenbeck semigroup.

This paper is organized as follows. In Section 2, we introduce the basic function spaces and notations. In Section 3, we set up the basic strategy for proving Theorem 1. In Section 4, we list some deterministic and probabilistic lemmata. Then, we prove the estimates of the nonlinear term in Sections 5–7.

2. NOTATION

First, recall the Bourgain space $X^{s,b}(\mathbb{T} \times \mathbb{R})$, c.f. [2], whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^2 \rangle^b \widehat{u}(n, \tau)\|_{l_n^2 L_\tau^2}.$$

Since the $X^{s,\frac{1}{2}}$ norm fails to control $L_t^\infty H_x^s$ norm, we use a smaller space $Z^{s,b}(\mathbb{T} \times \mathbb{R})$ whose norm is given by

$$(2.1) \quad \|u\|_{Z^{s,b}(\mathbb{T} \times \mathbb{R})} = \|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} + \|u\|_{Y^{s,b-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})}$$

where $\langle \cdot \rangle = 1 + |\cdot|$ and $\|u\|_{Y^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^2 \rangle^b \widehat{u}(n, \tau)\|_{l_n^2 L_\tau^1}$. We also define the local-in-time version $Z^{s,b,T}$ on $\mathbb{T} \times [-T, T]$, by

$$\|u\|_{Z^{s,b,T}} = \inf \{ \|\tilde{u}\|_{Z^{s,b}(\mathbb{T} \times \mathbb{R})} : \tilde{u}|_{[-T,T]} = u \}.$$

The local-in-time versions of other function spaces are defined analogously.

If a function depends on both x and t , we use $\widehat{\cdot}^x$ (and $\widehat{\cdot}^t$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $\widehat{\cdot}$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context. For simplicity, we often drop 2π in dealing with the Fourier transforms. If a function f is random, we may use the superscript f^ω to show the dependence on ω .

Lastly, let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function supported on $[-2, 2]$ with $\eta \equiv 1$ on $[-1, 1]$ and let $\eta_T(t) = \eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on α and s . If a constant depends on other quantities, we will make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. We also use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

3. BASIC SETUP

In this paper, we follow the argument by Bourgain [4]. First, write (1.10) as an integral equation:

$$(3.1) \quad u(t) = \Gamma u(t) := S(t)u_0 \pm i \int_0^t S(t-t')\mathcal{N}(u)(t')dt'$$

where $S(t) = e^{-i\partial_x^2 t}$, u_0 is as in (1.11), and $\mathcal{N}(u) = u|u|^2 - 2u f |u|^2$. Note that $S(t)u_0$ has the same regularity as u_0 for each fixed $t \in \mathbb{R}$. i.e. $S(t)u_0^\omega \in H^{\alpha-\frac{1}{2}-}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$ a.s. Hence, $S(t)u_0$ is strictly in the negative Sobolev space for $\alpha \leq \frac{1}{2}$ a.s.

It turns out that $\int_0^t S(t-t')\mathcal{N}(u)(t')dt'$ lies in a smoother space $H^s(\mathbb{T})$ for some $s \geq 0$ even for $\alpha \leq \frac{1}{2}$. (c.f. [4], [6].) We indeed show that for each $T > 0$ there exists Ω_T with complementary measure $< e^{-\frac{c}{T^\delta}}$ such that Γ defined in (3.1) is a contraction on $S(t)u_0^\omega + B$ for $\omega \in \Omega_T$, where B denotes the ball of radius 1 in $Z^{s,\frac{1}{2},T}$ for some $s \geq 0$.

Recall the following linear estimate [2], [12]:

$$(3.2) \quad \left\| \eta_T(t) \int_0^t S(t-t')\mathcal{N}(u)(t')dt' \right\|_{Z^{s,\frac{1}{2},T}} \lesssim T^{0-} \|\mathcal{N}(u)\|_{Z^{s,-\frac{1}{2},T}},$$

where η_T is a smooth cutoff on $[-2T, 2T]$. Then, it suffices to prove

$$(3.3) \quad \|\mathcal{N}(u)\|_{Z^{s,-\frac{1}{2},T}} \lesssim T^\theta, \quad \theta > 0$$

for $\omega \in \Omega_T$ with $\mathbb{P}(\Omega_T^c) < e^{-\frac{c}{T^\delta}}$. This also shows that the difference of the solution u^ω to (3.1) and the linear solution $S(t)u_0^\omega$ is in $C([-T, T] : H^s)$.

Now, write $\mathcal{N}(u)$ as follows:

$$(3.4) \quad \begin{aligned} \mathcal{N}(u) &= u|u|^2 - 2u \int |u|^2 \\ &= \sum_{n_2 \neq n_1, n_3} \widehat{u}(n_1) \overline{\widehat{u}(n_2)} \widehat{u}(n_3) e^{i(n_1 - n_2 + n_3)x} - \sum_n \widehat{u}(n) |\widehat{u}(n)|^2 e^{inx} =: \mathcal{N}_1(u) - \mathcal{N}_2(u). \end{aligned}$$

In the following sections, we will prove (3.3) by separately estimating the contributions from $\mathcal{N}_1(u)$ and $\mathcal{N}_2(u)$ with $u \in S(t)u_0^\omega + B$. i.e. $u = S(t)u_0^\omega + v$ for some v with $\|v\|_{Z^{s, \frac{1}{2}, T}} \leq 1$.

Note that (3.2) and (3.3) imply only the boundedness of the map Γ from $S(t)u_0^\omega + B$ into itself (for $T > 0$ small). In establishing the contraction property, one needs to consider $\Gamma u_1 - \Gamma u_2$ for $u_1, u_2 \in S(t)u_0^\omega + B$. We omit the details since the argument is standard. Lastly, suppose that $u_0 = u_0^\omega$ is a good initial condition such that Γ is a contraction on $S(t)u_0 + B$. Let \tilde{u}_0 be a function on \mathbb{T} such that $\|u_0 - \tilde{u}_0\|_{H^s} < \frac{1}{10}$. Denote by $\tilde{\Gamma}$ the solution map corresponding to the initial condition \tilde{u}_0 . Then, one can show that $\tilde{\Gamma}$ is also a contraction on $S(t)u_0 + B$ for T sufficiently small. Moreover, we have

$$\|u(t) - \tilde{u}(t)\|_{H^s} \leq C \|u_0 - \tilde{u}_0\|_{H^s}$$

for $|t| \leq T$, where \tilde{u} is the solution with the initial condition \tilde{u}_0 . For details, see [2], [4].

4. DETERMINISTIC AND PROBABILISTIC LEMMATA

First, recall the following algebraic identity related to the cubic NLS:

$$(4.1) \quad n^2 - (n_1^2 - n_2^2 + n_3^2) = 2(n_2 - n_1)(n_2 - n_3)$$

for $n = n_1 - n_2 + n_3$. Let N^1, N^2, N^3 be the decreasing ordering of N_1, N_2, N_3 , where $|n_j| \sim N_j$, and let n^j denote the corresponding frequency.

Lemma 4.1. *Let*

$$S_\mu = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \sim N_j, n_2 \neq n_1, n_3, \text{ and } 2(n_2 - n_1)(n_2 - n_3) = \mu\}.$$

Then, we have

$$(4.2) \quad \#S_\mu \lesssim (N^1)^{0+} N^3.$$

Proof. By assumption, we have $|\mu| \lesssim (N^1)^2$. Hence, the number of the divisors of μ is $o((N^1)^\varepsilon)$ for any $\varepsilon > 0$. Without loss of generality, assume $N^3 \sim \min(|n_2|, |n_3|)$.

First, suppose $|n_2| \sim N^3$. For fixed n_2 , there are at most $o((N^1)^{0+})$ many choices for $d := n_2 - n_1$. Then, there are at most $o((N^1)^{0+})$ many choices for n_1 and n_3 since $n_1 = n_2 - d$ and $n_3 = n_2 - \frac{\mu}{d}$.

Next, suppose $|n_3| \sim N^3$. For fixed n_3 , there are at most $o((N^1)^{0+})$ many choices for $d := n_2 - n_3$. Then, there are at most $o((N^1)^{0+})$ many choices for n_1 and n_2 since $n_2 = d + n_3$ and $n_1 = d + n_3 - \frac{\mu}{d}$. Hence, (4.2) holds in both cases. \square

Next, recall that by restricting the Bourgain spaces onto a small time interval $[-T, T]$, we can gain a small power of T (at a slight loss of regularity on $\langle \tau - n^2 \rangle$). See [2].

Lemma 4.2. *For $b < \frac{1}{2}$, we have*

$$(4.3) \quad \|u\|_{X^{s, b, T}} = \|\eta_T u\|_{X^{s, b, T}} \lesssim T^{\frac{1}{2} - b -} \|u\|_{X^{s, \frac{1}{2}, T}}.$$

Proof. By interpolation, we have

$$(4.4) \quad \|u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,0}}^\alpha \|u\|_{X^{s,\frac{1}{2}}}^{1-\alpha},$$

where $\alpha = 1 - 2b \in (0, 1)$. Recall $\widehat{\eta}_T(\tau) = T\widehat{\eta}(T\tau)$. Hence, we have $\|\widehat{\eta}_T\|_{L_T^q} \sim T^{\frac{q-1}{q}} \|\widehat{\eta}\|_{L_T^q} \sim T^{\frac{q-1}{q}}$. i.e. we can gain a positive power of T as long as $q > 1$. For fixed n , by Young and Hölder inequalities, we have

$$\|\widehat{\eta}_T * \widehat{u}(n, \cdot)\|_{L_T^{2'}} \leq \|\widehat{\eta}_T\|_{L_T^{2-}} \|\widehat{u}(n, \cdot)\|_{L_T^{1+}} \lesssim T^{\frac{1}{2}-} \|\langle \tau - n^2 \rangle^{-\frac{1}{2}}\|_{L_T^{2+}} \|\langle \tau - n^2 \rangle^{\frac{1}{2}} \widehat{u}(n, \cdot)\|_{L_T^2}$$

for some $\theta' > 0$. Hence, for $p > 2$, we have

$$(4.5) \quad \|u\|_{X^{s,0}} \lesssim T^{\frac{1}{2}-} \|u\|_{X^{s,\frac{1}{2}}}.$$

Then, (4.3) follows from (4.4) and (4.5). \square

Lastly, we present several probabilistic lemmata related to the Gaussians.

Lemma 4.3. *Let $\varepsilon, \delta > 0$. Then, we have*

$$(4.6) \quad |g_n(\omega)| \leq CT^{-\frac{\delta}{2}} \langle n \rangle^\varepsilon$$

for all $n \in \mathbb{Z}$ for ω outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$.

Proof. Recall from [20] we have $\mathbb{P}(\sup_n \langle n \rangle^{-\varepsilon} |g_n(\omega)| > K) \leq e^{-cK^2}$. Now, choose $K \sim T^{-\frac{\delta}{2}}$. \square

Lemma 4.4. *Let $f^\omega(x, t) = \sum c_n g_n(\omega) e^{i(nx+n^2t)}$, where $\{g_n\}$ is a family of complex valued standard i.i.d. Gaussian random variables. Then, for $p \geq 2$, there exists $\delta, T_0 > 0$ such that*

$$\mathbb{P}(\|f^\omega\|_{L^p(\mathbb{T} \times [-T, T])} > C \|c_n\|_{l_n^2}) < e^{-\frac{c}{T^\delta}}$$

for $T \leq T_0$.

Proof. By separating the real and imaginary parts, assume that g_n is real-valued without loss of generality. From the general Gaussian bound (c.f. Burq-Tzvetkov [6]), there exists $C > 0$ such that

$$\left\| \sum_n c_n g_n(\omega) \right\|_{L^r(\Omega)} \leq C \sqrt{r} \|c_n\|_{l_n^2}$$

for every $r \geq 2$ and every $\{c_n\}_{n \in \mathbb{Z}} \in l_n^2$. (This is also immediate from the hypercontractivity property as well. See [27].) By Minkowski integral inequality, we have

$$\begin{aligned} \mathbb{E}(\|f^\omega\|_{L_{x,t}^p(\mathbb{T} \times [-T, T])}^r)^{\frac{1}{r}} &\leq \left\| \|f^\omega\|_{L^r(\Omega)} \right\|_{L_{x,t}^p} \lesssim \sqrt{r} \left\| \|c_n\|_{l_n^2} \right\|_{L_{x,t}^p(\mathbb{T} \times [-T, T])} \\ &\lesssim \sqrt{r} T^{\frac{1}{p}} \|c_n\|_{l_n^2} \end{aligned}$$

for $r \geq p$. Then, by Chebyshev inequality, we have

$$\mathbb{P}(\|f^\omega\|_{L^p(\mathbb{T} \times [-T, T])} > \lambda) \leq C^r \lambda^{-r} r^{\frac{r}{2}} T^{\frac{r}{p}} \|c_n\|_{l_n^2}^r.$$

Let $\lambda = CrT^{\frac{1}{p}} \|c_n\|_{l_n^2}$ and $r = T^{-\delta}$ with $\delta = \frac{1}{p}$. Then, we have

$$\mathbb{P}(\|f^\omega\|_{L^p(\mathbb{T} \times [-T, T])} > \lambda) \leq e^{-r \ln \sqrt{r}} \leq e^{-\frac{c}{T^\delta}},$$

for T sufficiently small such that $r \geq p$. \square

In Section 7, we also use the hypercontractivity properties related to the Gaussians. See Sections 3 and 4 in Tzvetkov [27].

5. ESTIMATE ON $\mathcal{N}_2(u)$

In this section, we prove the easier part of the estimate (3.3):

$$(5.1) \quad \|\mathcal{N}_2(u_1, u_2, u_3)\|_{Z^{s, -\frac{1}{2}, T}} \lesssim T^\theta$$

for some $\theta > 0$, where

$$\mathcal{N}_2(u_1, u_2, u_3) = \sum_n \widehat{u}_1(n, t) \overline{\widehat{u}_2(n, t)} \widehat{u}_3(n, t) e^{inx}$$

with $u_j \in S(t)u_0^\omega + B$ outside an exceptional set of measure $e^{-\frac{c}{T^\delta}}$. By Hölder inequality, we have

$$(5.2) \quad \text{LHS of (5.1)} \lesssim \left\| \frac{\langle n \rangle^s}{\langle \tau - n^2 \rangle^{\frac{1}{2}-}} \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{u}_1(n, \tau_1) \overline{\widehat{u}_2(n, \tau_2)} \widehat{u}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_n^2, \tau}.$$

Then, it suffices to show (5.2) $\lesssim T^\theta$ assuming u_j is either of the form

$$(5.3) \quad \text{(I)} : u_j(x, t) = \sum_n \frac{g_n(\omega)}{1 + |n|^\alpha} e^{i(nx + n^2 t)},$$

$$\text{or (II)} : u_j = v_j \text{ with } \|v_j\|_{Z^{s, \frac{1}{2}, T}} \leq 1.$$

We will carry out a case-by-case analysis pivoting on these forms.

We may insert the smooth cutoff function η_T supported on $[-2T, 2T]$ if necessary.

- **Case (a):** u_j of type (II), $j = 1, \dots, 3$.

By Hölder inequality with p large ($\frac{1}{2} = \frac{1}{2+} + \frac{1}{p}$), we have

$$(5.2) \lesssim \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_\tau^{2+}} \left\| \langle n \rangle^s \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{u}_1(n, \tau_1) \overline{\widehat{u}_2(n, \tau_2)} \widehat{u}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_n^2 L_\tau^p}.$$

By Young and Hölder inequalities,

$$\leq \|\langle n \rangle^s \prod_{j=1}^3 \|\widehat{v}_j(n, \tau)\|_{L_\tau^{\frac{3}{2}-}}\|_{L_n^2} \lesssim \|\langle n \rangle^s \prod_{j=1}^3 \|\langle \tau - n^2 \rangle^{\frac{1}{6}+} \widehat{v}_j(n, \tau)\|_{L_\tau^2}\|_{L_n^2}.$$

By Lemma 4.2, Hölder and $L_n^2 \subset L_n^6$, we have for $s \geq 0$

$$\lesssim T^{1-} \prod_{j=1}^3 \|\langle n \rangle^{\frac{s}{3}} \langle \tau - n^2 \rangle^{\frac{1}{2}} \widehat{v}_j(n, \tau)\|_{L_n^6 L_\tau^2} \leq T^{1-} \prod_{j=1}^3 \|\widehat{v}_j\|_{Z^{\frac{s}{3}, \frac{1}{2}, T}} \leq T^{1-}.$$

- **Case (b):** u_j of type (I), $j = 1, \dots, 3$.

By Lemma 4.3, we have $|g_n(\omega)| \leq CT^{-\frac{\delta}{2}} \langle n \rangle^\varepsilon$ for ω outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. Then, by Hölder inequality with p large ($\frac{1}{2} = \frac{1}{2+} + \frac{1}{p}$) and Young's inequality,

$$(5.2) \lesssim \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_\tau^{2+}} \times \left\| \langle n \rangle^{s-3\alpha} |g_n|^3 \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\eta}(\tau_1 - n^2) \overline{\widehat{\eta}(\tau_2 - n^2)} \widehat{\eta}(\tau_3 - n^2) d\tau_1 d\tau_2 \right\|_{L_n^2 L_\tau^p} \\ \lesssim T^1 \|\langle n \rangle^{s-3\alpha} |g_n(\omega)|^3\|_{L_n^2} \lesssim T^{1-\frac{3}{2}\delta} \|\langle n \rangle^{s-3\alpha+3\varepsilon}\|_{L_n^2} \lesssim T^{1-}$$

as long as $2s - 6\alpha + 6\varepsilon < -1$ or $\alpha > \frac{1}{3}s + \frac{1}{6}$.

- **Case (c):** Exactly two u_j 's of type (I). Say $u_1(\text{I})$, $u_2(\text{I})$, and $u_3(\text{II})$.

By Hölder inequality with p large, Young's inequality, and Lemmata 4.2 and 4.3, we have

$$\begin{aligned}
(5.2) &\lesssim \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_\tau^{2+}} \\
&\quad \times \left\| \langle n \rangle^{s-2\alpha} |g_n|^2 \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\eta}(\tau_1 - n^2) \overline{\widehat{\eta}(\tau_2 - n^2)} \widehat{v}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_n^2 L_\tau^p} \\
&\lesssim T^{\frac{1}{2}-} \left(\sup_n \langle n \rangle^{-2\alpha} |g_n|^2 \right) \|\langle n \rangle^s \widehat{v}_3(n, \tau)\|_{L_{n,\tau}^2} \lesssim T^{1-\delta} \|v_3\|_{Z^{s, \frac{1}{2}, T}} \leq T^{1-}
\end{aligned}$$

for ω outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$ as long as $\alpha > 0$.

• **Case (d):** Exactly one u_j of type (I). Say $u_1(\text{I})$, $u_2(\text{II})$, and $u_3(\text{II})$.

By Hölder with p large, Young's inequality, and Lemma 4.3, we have

$$\begin{aligned}
(5.2) &\lesssim \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_\tau^{2+}} \\
&\quad \times \left\| \langle n \rangle^{s-\alpha} |g_n| \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\eta}(\tau_1 - n^2) \overline{\widehat{v}_2(n, \tau_2)} \widehat{v}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_n^2 L_\tau^p} \\
&\lesssim T^{\frac{1}{2}-} \left(\sup_n \langle n \rangle^{-s-\alpha} |g_n| \right) \left\| \prod_{j=2}^3 \|\langle n \rangle^s \widehat{v}_j(n, \tau)\|_{L_\tau^{\frac{4}{3}}} \right\|_{L_n^2}
\end{aligned}$$

By Hölder inequality in n ($\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$) and in τ ($\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$) followed by Lemma 4.2,

$$\begin{aligned}
&\lesssim T^{\frac{1}{2}-\delta-} \prod_{j=2}^3 \|\langle n \rangle^s \langle \tau - n^2 \rangle^{\frac{1}{4}+} \widehat{v}_j(n, \tau)\|_{L_n^4 L_\tau^2} \\
&\lesssim T^{1-\frac{\delta}{2}-} \prod_{j=2}^3 \|\langle n \rangle^s \langle \tau - n^2 \rangle^{\frac{1}{2}} \widehat{v}_j(n, \tau)\|_{L_n^4 L_\tau^2} \lesssim T^{1-}
\end{aligned}$$

for ω outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$ as long as $\alpha > -s$.

6. ESTIMATE ON $\mathcal{N}_1(u)$: BASIC SETUP

In the next two sections, we prove the main part of the estimate (3.3):

$$(6.1) \quad \|\mathcal{N}_1(u_1, u_2, u_3)\|_{Z^{s, -\frac{1}{2}, T}} \lesssim T^\theta$$

for some $\theta > 0$, where

$$\mathcal{N}_1(u_1, u_2, u_3) = \sum_{n_2 \neq n_1, n_3} \widehat{u}_1(n_1, t) \overline{\widehat{u}_2(n_2, t)} \widehat{u}_3(n_3, t) e^{i(n_1 - n_2 + n_3)x}$$

with $u_j \in S(t)u_0^\omega + B$ outside an exceptional set of measure $e^{-\frac{c}{T^\delta}}$. In the following, we assume that u_j is of type (I) or (II) as in (5.3). Note that we may simply write \mathcal{N}_1 for $\mathcal{N}_1(u_1, u_2, u_3)$.

As in [4], let N^1, N^2, N^3 be the decreasing ordering of N_1, N_2, N_3 and u^1, u^2, u^3 be the corresponding u_j -factors. Also, let $\sigma^1, \sigma^2, \sigma^3$ denote the corresponding $\sigma_j := \langle \tau_j - n_j^2 \rangle$. In the following, we use superscripts to imply that the functions (or variables) are arranged in the decreasing order of the spatial frequencies N_1, N_2, N_3 .

In the rest of this section, we consider basic cases. Recall the periodic L^4 Strichartz estimate from [2]:

$$(6.2) \quad \|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{3}{8}}}.$$

Interpolating this with $\|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}}$, we have

$$(6.3) \quad \|u\|_{L^{3+}_{x,t}} \lesssim \|u\|_{X^{0,\frac{1}{4}+}}, \text{ and } \|u\|_{L^{2+}_{x,t}} \lesssim \|u\|_{X^{0,0+}}.$$

By Hölder's inequality, we have $\|\mathcal{N}_1\|_{Z^{s,-\frac{1}{2},T}} \lesssim \|\mathcal{N}_1\|_{X^{s,-\frac{1}{2}+,T}}$ as used above. Then, using duality, we can estimate (6.1) by

$$(6.4) \quad \int (\langle \partial_x \rangle^s u^1) u^2 u^3 \cdot v \, dx dt$$

where $\|v\|_{X^{0,\frac{1}{2}-,T}} \leq 1$ (with the complex conjugate on an appropriate u^j .)

• **Case (A):** u^1 and u^2 are of type (II).

If u^3 is of type (II), then by Hölder inequality, (6.2), and Lemma 4.2, we have

$$(6.4) \leq \|\langle \partial_x \rangle^s u^1\|_{L^4_{x,t}} \|u^2\|_{L^4_{x,t}} \|u^3\|_{L^4_{x,t}} \|v\|_{L^4_{x,t}} \lesssim T^{\frac{1}{2}-} \|u^1\|_{Z^{s,\frac{1}{2},T}} \|u^2\|_{Z^{0,\frac{1}{2},T}} \|u^3\|_{Z^{0,\frac{1}{2},T}} \leq T^{\frac{1}{2}-}$$

as long as $s \geq 0$. If u^3 is of type (I) i.e. $u^3 = S(t)u_0$, then apply dyadic decompositions on N^2 and N^3 . Then, by Hölder inequality with p large, (6.3), and Lemmata 4.4 and 4.2, we have

$$(6.4) \leq \|\langle \partial_x \rangle^s u^1\|_{L^{3+}} \|u^2\|_{L^{3+}} \|u^3\|_{L^p} \|v\|_{L^{3+}} \\ \lesssim (N^3)^{\frac{1}{2}-\alpha+} \|u^1\|_{Z^{s,\frac{1}{4}+,T}} \|u^2\|_{Z^{0,\frac{1}{4}+,T}} \|v\|_{Z^{0,\frac{1}{4}+,T}}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. If $\langle \tau_j - n_j^2 \rangle^{\frac{1}{4}-} \gtrsim (N^3)^{\frac{1}{2}-\alpha+}$ for u_j of type (II), or if $\langle \tau - n^2 \rangle^{\frac{1}{4}-} \gtrsim (N^3)^{\frac{1}{2}-\alpha+}$, then (6.1) follows with $\theta = \frac{1}{2}-$ in view of Lemma 4.2. Hence, in the following, we may assume

$$(6.5) \quad \langle \tau - n^2 \rangle \ll (N^3)^{2-4\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^3)^{2-4\alpha+} \text{ if } u_j \text{ of type (II).}$$

• **Case (B):** u^1 of type (II), and u^2 of type (I).

Dyadically decompose the spatial frequencies for N^2 and N^3 . First, suppose that u^3 is of type (II). By Hölder inequality with p large, (6.3), and Lemma 4.4, we have

$$(6.4) \leq \|\langle \partial_x \rangle^s u^1\|_{L^{3+}} \|u^2\|_{L^p} \|u^3\|_{L^{3+}} \|v\|_{L^{3+}} \\ \lesssim (N^2)^{\frac{1}{2}-\alpha+} \|u^1\|_{X^{s,\frac{1}{4}+,T}} \|u^3\|_{X^{0,\frac{1}{4}+,T}} \|v\|_{X^{0,\frac{1}{4}+,T}}$$

outside an exceptional set of size $< e^{-\frac{c}{T^\delta}}$. If $\langle \tau_j - n_j^2 \rangle^{\frac{1}{4}-} \gtrsim (N^2)^{\frac{1}{2}-\alpha+}$ for u_j of type (II), or if $\langle \tau - n^2 \rangle^{\frac{1}{4}-} \gtrsim (N^2)^{\frac{1}{2}-\alpha+}$, then (6.1) follows with $\theta = \frac{1}{2}-$ in view of Lemma 4.2. Hence, we may assume

$$(6.6) \quad \langle \tau - n^2 \rangle \ll (N^2)^{2-4\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^2)^{2-4\alpha+} \text{ if } u_j \text{ of type (II)}$$

in the following.

Next, suppose that u^3 is of type (I). Again, by Hölder inequality with p large, (6.3), and Lemma 4.4, we have

$$(6.4) \leq \|\langle \partial_x \rangle^s u^1\|_{L^{2+}} \|u^2\|_{L^p} \|u^3\|_{L^p} \|v\|_{L^{2+}} \\ \lesssim (N^2)^{1-2\alpha+} \|u^1\|_{X^{s,0+,T}} \|v\|_{X^{0,0+,T}}.$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. If $(\sigma^1)^{\frac{1}{2}-} \gtrsim (N^2)^{1-2\alpha+}$ or if $\langle \tau - n^2 \rangle^{\frac{1}{2}-} \gtrsim (N^2)^{1-2\alpha+}$, then (6.1) follows with $\theta = \frac{1}{2}-$ in view of Lemma 4.2. Hence, we can assume (6.6) as well.

• **Case (C):** u^1 of type (I), and u^2, u^3 of type (II).

Dyadically decompose all the spatial frequencies. Suppose $\langle \tau - n^2 \rangle \gg \sigma^2, \sigma^3$. By Hölder inequality with p large and Lemmata 4.4 and 4.2, we have

$$(6.4) \leq (N^1)^s \|u^1\|_{L^p} \|u^2\|_{L^4} \|u^3\|_{L^4} \|v\|_{L^{2+}} \lesssim (N^1)^{s+\frac{1}{2}-\alpha+} \|u^2\|_{X^{0, \frac{3}{8}, T}} \|u^3\|_{X^{0, \frac{3}{8}, T}} \|v\|_{X^{0, 0+, T}} \\ \lesssim T^{\frac{1}{4}-} (N^1)^{s+\frac{1}{2}-\alpha+} \|u^2\|_{X^{0, \frac{1}{2}, T}} \|u^3\|_{X^{0, \frac{1}{2}, T}} \|v\|_{X^{0, 0+, T}}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. Hence, (6.1) follows as long as $\langle \tau - n^2 \rangle \gtrsim (N^1)^{2s+1-2\alpha+}$. Similar results hold if $\sigma^2 \gg \sigma^3, \langle \tau - n^2 \rangle$ or $\sigma^3 \gtrsim \sigma^2, \langle \tau - n^2 \rangle$. Hence, we assume

$$(6.7) \quad \langle \tau - n^2 \rangle \ll (N^1)^{2s+1-2\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^1)^{2s+1-2\alpha+} \text{ if } u_j \text{ of type (II).}$$

• **Case (D):** u^1 of type (I), and either $u^2(\text{I}), u^3(\text{II})$ or $u^2(\text{II}), u^3(\text{I})$.

Suppose that u^2 is of type (I) and that u^3 is of type (II). Moreover, suppose $\langle \tau - n^2 \rangle \gg \sigma^3$. By Hölder inequality with p large and Lemmata 4.4 and 4.2, we have

$$(6.4) \leq (N^1)^s \|u^1\|_{L^p} \|u^2\|_{L^p} \|u^3\|_{L^{2+}} \|v\|_{L^2} \lesssim (N^1)^{s+1-2\alpha+} \|u^3\|_{X^{0, 0+, T}} \|v\|_{X^{0, 0, T}} \\ \lesssim T^{\frac{1}{2}-} (N^1)^{s+1-2\alpha+} \|u^3\|_{X^{0, \frac{1}{2}, T}} \|v\|_{X^{0, 0, T}}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. Hence, (6.1) follows as long as $\langle \tau - n^2 \rangle \gtrsim (N^1)^{2s+2-4\alpha+}$. Similar results hold if $\sigma^3 \gtrsim \langle \tau - n^2 \rangle$, (or u^2 is of type (II) and u^3 is of type (I).) Hence, we assume

$$(6.8) \quad \langle \tau - n^2 \rangle \ll (N^1)^{2s+2-4\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^1)^{2s+2-4\alpha+} \text{ if } u_j \text{ of type (II).}$$

Summary: Given a function $v(x, t)$, we can write v as

$$(6.9) \quad v(x, t) = \int \langle \lambda \rangle^{-\frac{1}{2}} \left(\sum_n \langle n \rangle^{2s} \langle \lambda \rangle |\widehat{v}(n, n^2 + \lambda)|^2 \right)^{\frac{1}{2}} \left\{ e^{i\lambda t} \sum_n a_\lambda(n) e^{i(nx + n^2 t)} \right\} d\lambda$$

where $a_\lambda(n) = \frac{\widehat{v}(n, n^2 + \lambda)}{(\sum_m \langle m \rangle^{2s} |\widehat{v}(m, m^2 + \lambda)|^2)^{\frac{1}{2}}}$. Note that $\sum_n \langle n \rangle^{2s} |a_\lambda(n)|^2 = 1$. For $\|v\|_{X^{s, \frac{1}{2}}} \leq 1$, we have

$$(6.10) \quad \int_{|\lambda| < K} \langle \lambda \rangle^{-\frac{1}{2}} \left(\sum_n \langle n \rangle^{2s} \langle \lambda \rangle |\widehat{v}(n, n^2 + \lambda)|^2 \right)^{\frac{1}{2}} d\lambda \lesssim (\log K)^{\frac{1}{2}}$$

by Hölder inequality. See [4, (22) and (23)]. Note that (6.9) is a standard representation for functions in $X^{s, b}$ for $b > \frac{1}{2}$. e.g. Klainerman-Selberg [15]. We have a logarithmic loss in (6.9) since $b = \frac{1}{2}$ in our case.

Now, consider (6.1) on a small time interval $[-T, T]$. First, replace $\widehat{\mathcal{N}}_1$ with $\widehat{\eta}_T * \widehat{\mathcal{N}}_1$. Then, by Hölder and Young's inequalities, we have, for each $n \in \mathbb{Z}$,

$$\|\langle \tau - n^2 \rangle^{-\frac{1}{2}+} \widehat{\eta}_T * \widehat{\mathcal{N}}_1(n, \cdot)\|_{L_T^2} \leq \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_T^{2+}} \|\widehat{\eta}_T\|_{L_T^2} \|\widehat{\mathcal{N}}_1(n, \cdot)\|_{L_T^2} \lesssim T^{\frac{1}{2}-} \|\widehat{\mathcal{N}}_1(n, \cdot)\|_{L_T^2}.$$

Then, letting $*$ = $\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3\}$ and $**$ = $\{(\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 : \tau = \tau_1 - \tau_2 + \tau_3\}$, we have

$$(6.11) \quad \text{LHS of (6.1)} \lesssim T^{\frac{1}{2}-} \|\mathcal{N}_1\|_{X^{s,0,T}} \leq T^{\frac{1}{2}-} \left\| \sum_* \langle n^1 \rangle^s \left\| \int_{**} \prod_{j=1}^3 \widehat{u}_j(n_j, \tau_j) d\tau_1 d\tau_2 \right\|_{L_\tau^2} \right\|_{L_n^2}$$

where $\widehat{u}_j(n_j, \tau_j) = \frac{g_{n_j}(\omega) \delta(\tau_j - n_j^2)}{1 + |n_j|^\alpha}$ or

$$\widehat{u}_j(n_j, \tau_j) = \int_{\{|\lambda_j| < K\}} \langle \lambda_j \rangle^{-\frac{1}{2}} c_j(\lambda_j) a_{\lambda_j}(n_j) \delta(\tau_j - n_j^2 - \lambda_j) d\lambda_j$$

with $\sum_{n_j} \langle n_j \rangle^{2s} |a_{\lambda_j}(n_j)|^2 \leq 1$, where $c_j(\lambda_j) = \left(\sum_{m_j} \langle m_j \rangle^{2s} \langle \lambda_j \rangle |\widehat{u}_j(m_j, n_j^2 + \lambda_j)|^2 \right)^{\frac{1}{2}}$. If any of u_j is of type (II), then, we can pull the integral in the corresponding λ_j outside via Minkowski integral inequality. For example, when u_1 and u_2 are of type (II) and u_3 is of type (I), we have

$$(6.11) \lesssim T^{\frac{1}{2}-} \int \prod_{j=1}^2 \chi_{\{|\lambda_j| < K\}} \langle \lambda_j \rangle^{-\frac{1}{2}} c_j(\lambda_j) \left\| \sum_* \langle n^1 \rangle^s \prod_{k=1}^2 a_{\lambda_k}(n_k) \frac{g_{n_3}(\omega)}{1 + |n_3|^\alpha} \right\|_{L_n^2} d\lambda_1 d\lambda_2 d$$

$$(6.12) \lesssim (\log K)^{\frac{3}{2}} \sup_{\lambda_1, \lambda_2} \left\| \sum_* \langle n^1 \rangle^s \prod_{j=1}^2 a_{\lambda_j}(n_j) \frac{g_{n_3}(\omega)}{1 + |n_3|^\alpha} \right\|_{L_n^2},$$

where the last inequiaty follows from Hölder inequality and (6.10). Strictly speaking, we need to put an absolute value on the summand of \sum_* in (6.12), which may prevent us from exploiting the properties of the Gaussians g_n . However, this can be avoided by integrating only in two of the τ_j variables and keeping L_τ^2 outside \sum_* in (6.11). Then, we can integrate in τ after estimating the sum \sum_* using the properties of the Gaussians as in the next section.

Therefore, we can reduce the estimate into the following two cases (with $\theta = \frac{1}{2}-$):

- u^1 is of type (II):

From (6.5) and (6.6), we can assume that $\sigma_j \ll (N^3)^{2-4\alpha+}$ or $(N^2)^{2-4\alpha+}$ for u_j of type (II). Then, by (6.9) and (6.10), we can bound (6.1) as follows:

$$(6.13) \quad (6.1) \lesssim T^\theta M(N^2, N^3) \left(\sum_n \left| \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3 \\ n^2=n_1^2-n_2^2+n_3^2+\mu}} a_1(n_1) \overline{a_2(n_2)} a_3(n_3) \right|^2 \right)^{\frac{1}{2}},$$

where $\sum_n |a^1(n)|^2 \leq 1$, $a^j(n) = \frac{g_n(\omega)}{1 + |n^j|^\alpha}$ or $\sum_{|n| \sim N^j} |a^j(n)|^2 \leq (N^j)^{-2s}$ for $j = 2, 3$, and

- Case (A): $M(N^2, N^3) = (N^3)^{0+}$ and $|\mu| \ll (N^3)^{2-4\alpha+}$
- Case (B): $M(N^2, N^3) = (N^2)^{0+}$ and $|\mu| \ll (N^2)^{2-4\alpha+}$.

Note that we did not apply dyadic decomposition on N^1 .

- u^1 is of type (I):

From (6.7) and (6.8), we can assume that $\sigma_j \ll (N^1)^{2s+2-4\alpha+}$ for u_j of type (I) since $\alpha \leq \frac{1}{2}$. Then, by (6.9) and (6.10), we can bound (6.1) as follows:

$$(6.14) \quad (6.1) \lesssim T^\theta (N^1)^{s+} \left(\sum_{|n| \lesssim N^1} \left| \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3 \\ n^2=n_1^2-n_2^2+n_3^2+\mu}} a_1(n_1) \overline{a_2(n_2)} a_3(n_3) \right|^2 \right)^{\frac{1}{2}},$$

where $a^1(n) = \frac{g_n(\omega)}{1+|n|^\alpha}$, $a^j(n) = \frac{g_n(\omega)}{1+|n^j|^\alpha}$ or $\sum_{|n| \sim N^j} |a^j(n)|^2 \leq (N^j)^{-2s}$ for $j = 2, 3$, and

$$\begin{aligned} \text{Case (C):} & \quad |\mu| \ll (N^1)^{2s+1-2\alpha+} \\ \text{Case (D):} & \quad |\mu| \ll (N^1)^{2s+2-4\alpha+} \\ \text{All type (I):} & \quad |\mu| \ll (N^1)^2, \end{aligned}$$

Note that all the spatial frequencies are dyadically decomposed.

Suppose $|n_2| > 10(|n_1| + |n_3|)$. Then, $|\mu| \sim |(n_2 - n_1)(n_2 - n_3)| \sim |n_2|^2 \sim (N^1)^2$ by (4.1). If $u^1 = u_2$ is of type (II), we have $|\mu| \ll (N^2)^{2-4\alpha+} \ll (N^1)^2$ as long as $\alpha > 0$. If $u^1 = u_2$ is of type (I), we have $|\mu| \ll (N^1)^{2s+2-4\alpha+} \ll (N^1)^2$ since $\alpha > \frac{s}{2}$. In both cases, we would have a contradiction. Hence, we can assume that $|n_1| \sim N^1$ or $|n_3| \sim N^1$. Moreover, by symmetry between u_1 and u_3 , we assume $|n_1| \sim N^1$ in the following.

Lastly, we list all the different cases following [4]. We consider these cases in details in the next section.

- Case (a): $n_1 = N^1(\text{II}), n_2 = N^2(\text{I}), n_3 = N^3(\text{II})$ or $n_2 = N^3(\text{I}), n_3 = N^2(\text{II})$.
- Case (b): $n_1 = N^1(\text{II}), n_2 = N^3(\text{II}), n_3 = N^2(\text{I})$ or $n_2 = N^2(\text{II}), n_3 = N^3(\text{I})$.
- Case (c): $n_1 = N^1(\text{I}), n_2 = N^2(\text{II}), n_3 = N^3(\text{II})$.
- Case (d): $n_1 = N^1(\text{I}), n_2 = N^3(\text{II}), n_3 = N^2(\text{II})$.
- Case (e): $n_1 = N^1(\text{II}), n_2 = N^2(\text{I}), n_3 = N^3(\text{I})$.
- Case (f): $n_1 = N^1(\text{II}), n_2 = N^3(\text{I}), n_3 = N^2(\text{I})$.
- Case (g): $n_1 = N^1(\text{I}), n_2 = N^2(\text{I}), n_3 = N^3(\text{II})$.
- Case (h): $n_1 = N^1(\text{I}), n_2 = N^3(\text{I}), n_3 = N^2(\text{II})$.
- Case (i): $n_1 = N^1(\text{I}), n_2 = N^2(\text{II}), n_3 = N^3(\text{I})$.
- Case (j): $n_1 = N^1(\text{I}), n_2 = N^3(\text{II}), n_3 = N^2(\text{I})$.
- Case (k): $n_1 = N^1(\text{I}), n_2 = N^2(\text{I}), n_3 = N^3(\text{I})$.
- Case (l): $n_1 = N^1(\text{I}), n_2 = N^3(\text{I}), n_3 = N^2(\text{I})$.

7. ESTIMATE ON $\mathcal{N}_1(u_1, u_2, u_3)$: ALL DIFFERENT CASES

For notational simplicity, we use $|n|^\alpha$ for $1 + |n|^\alpha$. Also, we may drop the complex conjugate on u_2 when it plays no significant role.

Let

$$\begin{aligned} A_n = \{ & (n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3, |n_j| \sim N_j, j = 2, 3, \\ & n_2 \neq n_1, n_3, \text{ and } n^2 = n_1^2 - n_2^2 + n_3^2 + \mu \} \end{aligned}$$

and $B_n = A_n \cap \{|n_1| \sim N_1\}$. Also, recall

$$(7.1) \quad \mu = 2(n_2 - n_1)(n_2 - n_3) = 2(n - n_1)(n - n_3).$$

• **Cases (k), (l)** : u_1, u_2, u_3 of type (I). In this case, we have

$$(7.2) \quad (6.14) \lesssim T^\theta N_1^{s+} \left(\sum_{|n| \lesssim N_1} \left| \sum_{B_n} \frac{g_{n_1}}{|n_1|^\alpha} \frac{\overline{g_{n_2}}}{|n_2|^\alpha} \frac{g_{n_3}}{|n_3|^\alpha} \right|^2 \right)^{\frac{1}{2}}.$$

First, we consider the contribution from $n_1 \neq n_3$. Let

$$F_n(\omega) := \sum_{C_n} \frac{g_{n_1}(\omega)}{|n_1|^\alpha} \frac{\overline{g_{n_2}(\omega)}}{|n_2|^\alpha} \frac{g_{n_3}(\omega)}{|n_3|^\alpha},$$

where $C_n = B_n \cap \{n_1 \neq n_3\}$. Then, by hypercontractivity property related to the product of Gaussians (cf. [27, Propositions 3.1 and 3.3]), we have $\|F_n\|_{L^p(\Omega)} \leq p^{\frac{3}{2}} \|F_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, it follows from Lemma 4.5 in [27] that $\mathbb{P}(|F_n(\omega)| \geq \lambda) \leq \exp(-c\|F_n\|_{L^2(\Omega)}^{\frac{2}{3}} \lambda^{\frac{2}{3}})$.

By choosing $\lambda = T^{-\frac{3}{2}\delta} N_1^{\frac{3}{2}\varepsilon} \|F_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$(7.3) \quad \mathbb{P}(|F_n(\omega)| \geq T^{-\frac{3}{2}\delta} N_1^{\frac{3}{2}\varepsilon} \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{cN_1^\varepsilon}{T^\delta}}.$$

By Lemma 4.1, we have

$$\begin{aligned} \text{RHS of (7.2)} &\lesssim T^{\theta - \frac{3}{2}\delta} N_1^{s + \frac{3}{2}\varepsilon +} \left(\sum_{|n| \lesssim N_1} \sum_{C_n} \frac{1}{|n_1|^{2\alpha} |n_2|^{2\alpha} |n_3|^{2\alpha}} \right)^{\frac{1}{2}} \\ &\lesssim T^{\theta - \frac{3}{2}\delta} N_1^{s - \alpha + \frac{3}{2}\varepsilon +} (N^2)^{-\alpha} (N^3)^{-\alpha + \frac{1}{2}} \\ &\lesssim \begin{cases} T^{\theta - \frac{3}{2}\delta} N_1^{s - 3\alpha + \frac{1}{2} + \frac{3}{2}\varepsilon +} & \text{for } \alpha \leq \frac{1}{4} \\ T^{\theta - \frac{3}{2}\delta} N_1^{s - \alpha + \frac{3}{2}\varepsilon +} & \text{for } \alpha \geq \frac{1}{4} \end{cases} \\ &\leq T^{\theta - \frac{3}{2}\delta} \prod_{j=1}^3 N_j^{0-}, \quad \begin{cases} \text{for } \alpha > \frac{s}{3} + \frac{1}{6} & (\text{with } \alpha \leq \frac{1}{4}) \\ \text{for } \alpha > s & (\text{with } \alpha \geq \frac{1}{4}) \end{cases} \end{aligned}$$

outside an exceptional set of measure

$$< \sum_{|n| \lesssim N_1} e^{-\frac{cN_1^\varepsilon}{T^\delta}} \lesssim N_1 e^{-\frac{cN_1^\varepsilon}{T^\delta}} \leq N_1^{0-} e^{-\frac{c}{T^\delta} N_1^\varepsilon + (1+)\log N_1} < N_1^{0-} e^{-\frac{c'}{T^\delta}}.$$

Note that in this case we need to make sure that the measures of these exceptional sets corresponding to different dyadic blocks are indeed summable and bounded by $e^{-\frac{c}{T^\delta}}$. We may not be explicit about this point in other cases. e.g. Cases (A)–(D) in Section 6. We do not encounter this issue in using Lemma 4.3 since it gives one exceptional set of measure $< e^{-\frac{c}{T^\delta}}$ for all the frequencies.

Now, consider the contribution from $n_1 = n_3$. It follows from (7.1) that there is at most one choice of (n_1, n_2, n_3) for each fixed n . Thus, $\sum_{|n| \lesssim N_1} \left| \sum_{B_n, n_1=n_3} 1 \right|^2 = \sum_{|n| \lesssim N_1} \sum_{B_n, n_1=n_3} 1$. Hence, by Lemmata 4.1 and 4.3, we have

$$\text{RHS of (7.2)} \lesssim T^{\theta - \frac{3}{2}\delta} N_1^{s - 2\alpha + 2\varepsilon +} N_2^{-\alpha + \frac{1}{2} + \varepsilon} \leq T^{\theta - \frac{3}{2}\delta} \prod_{j=1}^3 N_j^{0-}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$ for $\alpha > \frac{s}{3} + \frac{1}{6}$.

• **Case (a)** : (Case (b) can be treated in a similar way by replacing n_2 and n_3 .)

In this case, we have $\mu = 2(n_2 - n_1)(n_2 - n_3) = o((N_2)^{2-4\alpha+})$. This implies that $|n|, |n_1|, |n_3| \lesssim N_2^\beta$ for some $\beta > 0$ since $n_2 \neq n_1, n_3$. Now, let

$$G_n(\omega) := \sum_{A_n} a_1(n_1) \frac{\overline{g_{n_2}(\omega)}}{|n_2|^\alpha} a_3(n_3),$$

Note that the above sum is over a finite set. Letting $A_n = \cup_{|n_2| \sim N_2} A_{n, n_2}$ (i.e. a disjoint union over distinct n_2), write

$$G_n(\omega) = \sum_{n_2} c_{n, n_2} \overline{g_{n_2}}, \quad \text{where } c_{n, n_2} = \sum_{A_{n, n_2}} |n_2|^{-\alpha} a_1(n_1) a_3(n_3).$$

Then, by hypercontractivity of the Gaussians, we have $\|G_n\|_{L^p(\Omega)} \leq p^{\frac{1}{2}} \|G_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, by Lemma 4.5 in [27], we have $\mathbb{P}(|G_n(\omega)| \geq \lambda) \leq \exp(-c\|G_n\|_{L^2(\Omega)}^2 \lambda^2)$. By choosing $\lambda = T^{-\frac{1}{2}\delta} (N^2)^\varepsilon \|G_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$(7.4) \quad \mathbb{P}(|G_n(\omega)| \geq T^{-\frac{1}{2}\delta} (N^2)^\varepsilon \|G_n\|_{L^2(\Omega)}) \leq e^{-\frac{c(N^2)^{2\varepsilon}}{T^\delta}}.$$

Note that $(N^2)^\varepsilon \lesssim N_2^{\beta\varepsilon}$ since $|n_3| \lesssim N_2^\beta$. Hence, we have

$$(6.13) \lesssim T^{\theta - \frac{\delta}{2}} (N_2)^{-\alpha + \beta\varepsilon +} \left(\sum_{n, n_2} \left| \sum_{A_{n, n_2}} |a_1(n_1) a_3(n_3)| \right|^2 \right)^{\frac{1}{2}}$$

by Hölder inequality (note that $\#A_{n, n_2} = O(N_2^{0+})$),

$$(7.5) \lesssim T^{\theta - \frac{\delta}{2}} N_2^{-\alpha + \beta\varepsilon +} \left(\sum_n \sum_{A_n} |a_1(n_1)|^2 |a_3(n_3)|^2 \right)^{\frac{1}{2}} \\ \leq T^{\theta - \frac{\delta}{2}} N_2^{-\alpha + \beta\varepsilon +} N_3^{-s} \leq T^{\theta'} N_2^{0-} N_3^{0-}$$

for $\alpha > 0$ and $s \geq 0$ outside an exceptional set of measure

$$< \sum_{|n| \lesssim N_2^\beta} e^{-\frac{c(N^2)^{2\varepsilon}}{T^\delta}} \lesssim N_2^\beta e^{-\frac{c(N^2)^{2\varepsilon}}{T^\delta}} \leq N_2^{0-} e^{-\frac{c}{T^\delta} N_2^\varepsilon + (\beta+) \log N_2} < N_2^{0-} e^{-\frac{c'}{T^\delta}}.$$

Note that $|n_3| \lesssim N_2^\beta$ is crucial in the last inequality of (7.5) when $s = 0$, $n_2 = N^3$, and $n_3 = N^2$.

In Case (b), we have $\mu = 2(n_2 - n_1)(n_2 - n_3) = o((N_3)^{2-4\alpha+})$, which implies that $|n|, |n_1|, |n_2| \lesssim N_3^\beta$ for some $\beta > 0$ since $n_2 \neq n_1, n_3$. The rest of the argument follows as above by replacing n_2 and n_3 .

• **Case (c)** : (Case (d) can be treated in a similar way by replacing n_2 and n_3 .)

Let $b_2(n_2) = |n_2|^s a_2(n_2)$. Then, we have $\sum_{|n_2| \sim N_2} |b_2(n_2)|^2 \lesssim 1$. By Lemma 4.3 and Hölder inequality on n_3 in the inner sum,

$$(6.14) \lesssim T^{\theta - \frac{\delta}{2}} N_1^{s - \alpha + \varepsilon +} N_2^{-s} N_3^{-s} \left(\sum_{|n| \lesssim N^1} \sum_{B_n} |b_2(n_2)|^2 \right)^{\frac{1}{2}}$$

outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$. For fixed n_2 , it follows from (the proof of) Lemma 4.1 that there are at most N_1^{0+} terms in the sum. Hence, we have

$$(6.14) \lesssim T^{\theta - \frac{\delta}{2}} N_1^{s - \alpha + \varepsilon +} N_2^{-s} N_3^{-s} \leq T^{\theta'} N_1^{0-} N_2^{0-} N_3^{0-}$$

for $\alpha > s \geq 0$.

• **Case (e)** : (Case (f) is basically the same.)

In this case, we have $N_2^{2-4\alpha+} \gg |\mu| = |2(n_2 - n_1)(n_2 - n_3)|$. This implies that $|n_1|, |n| \lesssim N_2^\beta$ for some $\beta > 0$. Now, let

$$F_n(\omega) := \sum_{A_n} a_1(n_1) \frac{\overline{g_{n_2}(\omega)} g_{n_3}(\omega)}{|n_2|^\alpha |n_3|^\alpha},$$

By the observation above, this is a finite sum. Then, by hypercontractivity property related to the product of two independent Gaussians (recall $n_2 \neq n_3$), we have $\|F_n\|_{L^p(\Omega)} \leq p \|F_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, it follows from Lemma 4.5 in [27] that $\mathbb{P}(|F_n(\omega)| \geq \lambda) \leq \exp(-c\|F_n\|_{L^2(\Omega)}^{-1} \lambda)$. By choosing $\lambda = T^{-\delta} N_2^\varepsilon \|F_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$\mathbb{P}(|F_n(\omega)| \geq T^{-\delta} N_2^\varepsilon \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{cN_2^\varepsilon}{T^\delta}}.$$

It follows from the proof of Lemma 4.1 and $|n_1| \lesssim N_2^\beta$ that, for fixed n_1 , there are at most $O(N_2^{0+})$ many choices for n_2 and n_3 . Thus, we have

$$(6.13) \lesssim T^{\theta-\delta} N_2^{-\alpha+\varepsilon+} N_3^{-\alpha} \left(\sum_{|n| \lesssim N_2^\beta} \sum_{A_n} |a_1(n_1)|^2 \right)^{\frac{1}{2}} \lesssim T^{\theta-\delta} N_2^{-\alpha+\varepsilon+} N_3^{-\alpha} \leq T^{\theta'} N_2^{0-} N_3^{0-}$$

for $\alpha > 0$ outside an exceptional set of measure $\lesssim N_2^\beta e^{-\frac{cN_2^\varepsilon}{T^\delta}}$.

• **Case (g)** : (Case (h) is basically the same.)

Let $F_n(\omega) := \sum_{B_n} \frac{g_{n_1}(\omega) \overline{g_{n_2}(\omega)}}{|n_1|^\alpha |n_2|^\alpha} a_3(n_3)$. As in Case (e), we use the hypercontractivity property related to the product of two independent Gaussians (recall $n_1 \neq n_2$.) Then, we have

$$\mathbb{P}(|F_n(\omega)| \geq T^{-\delta} N_1^\varepsilon \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{cN_1^\varepsilon}{T^\delta}}.$$

By summing over n_1 for fixed n_3 and then over n_3 , it follows from (the proof of) Lemma 4.1 that

$$(6.14) \lesssim T^{\theta-\delta} N_1^{s-\alpha+\varepsilon+} N_2^{-\alpha} \left(\sum_{|n| \lesssim N_1} \sum_{B_n} |a_3(n_3)|^2 \right)^{\frac{1}{2}} \\ \lesssim T^{\theta-\delta} N_1^{s-\alpha+\varepsilon+} N_2^{-\alpha} N_3^{-s} \leq T^{\theta'} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > s \geq 0$ outside an exceptional set of measure $< N_1^{0-} e^{-\frac{c'}{T^\delta}}$.

• **Cases (i), (j)** : The contribution for $n_1 \neq n_3$ in Case (i) (and in Case (j)) can be estimated as in Case (h) (and as in Case (g), respectively) using the hypercontractivity. Now, assume $n_1 = n_3$. It follows from (7.1) that there are at most two choices of (n_1, n_2, n_3) for each fixed n . Thus, $\sum_{|n| \lesssim N_1} \left| \sum_{B_n, n_1=n_3} |a_2(n_2)| \right|^2 \sim \sum_{|n| \lesssim N_1} \sum_{B_n, n_1=n_3} |a_2(n_2)|^2$. By Lemma 4.3 and by summing over n_2 , we have

$$(6.14) \lesssim T^{\theta-\delta} N_1^{s-2\alpha+2\varepsilon+} \left(\sum_{|n| \lesssim N_1} \sum_{\substack{B_n \\ n_1=n_3}} |a_2(n_2)|^2 \right)^{\frac{1}{2}} \lesssim T^{\theta-\delta} N_1^{s-2\alpha+2\varepsilon+} N_2^{-s} \leq T^{\theta'} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > \frac{s}{2} \geq 0$ outside an exceptional set of measure $< e^{-\frac{c}{T^\delta}}$.

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