

ALMOST SURE GLOBAL SOLUTIONS OF THE PERIODIC CUBIC NONLINEAR SCHRÖDINGER EQUATION BELOW L^2

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ABSTRACT. We continue our study of the well-posedness of the periodic cubic nonlinear Schrödinger equation with random initial data below L^2 . In this paper, we prove that it is globally well-posed almost surely for the initial data in the support of the Gaussian measures on $H^s(\mathbb{T})$ for each $s > -\frac{1}{8}$.

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1. INTRODUCTION

In this paper, we continue our study on the well-posedness problem of the periodic nonlinear Schrödinger equation (NLS) with cubic nonlinearity:

$$(1.1) \quad \begin{cases} iu_t - u_{xx} \pm u|u|^2 = 0 \\ u|_{t=0} = u_0 \end{cases}$$

for random initial data in the negative Sobolev spaces. In particular, we take the initial data u_0 to be of the form

$$(1.2) \quad u_0(x) = u_0^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1 + |n|^{2\alpha}}} e^{inx}$$

for $x \in \mathbb{T} = [0, 2\pi)$, where $\{g_n\}_{n \in \mathbb{Z}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that a function with Fourier coefficient $\frac{1}{1+|n|^\alpha}$ is in H^s for $s < \alpha - \frac{1}{2}$ but not in $H^{\alpha - \frac{1}{2}}$. This implies that u_0 in (1.2) is in $\bigcap_{s < \alpha - \frac{1}{2}} H^s \setminus H^{\alpha - \frac{1}{2}}$, since there is almost surely no smoothing in differentiability under the Gaussian randomization on the Fourier coefficients (c.f. Burq-Tzvetkov [9].) Recall that Bourgain proved the global well-posedness (GWP) result in

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$L^2(\mathbb{T})$ in [2]. Hence, we assume that $\alpha \leq \frac{1}{2}$ in the following so that u_0 lies strictly in the negative Sobolev spaces.

Note that $\mu = \int |u|^2 dx = \frac{1}{2\pi} \int |u|^2 dx$ is conserved under the flow of (1.1) as long as it makes sense. By letting $u \rightarrow e^{\pm 2i\mu t} u$, we see that (1.1) is equivalent to

$$(1.3) \quad \begin{cases} iu_t - u_{xx} \pm (u|u|^2 - 2u \int |u|^2 dx) = 0 \\ u|_{t=0} = u_0 \end{cases}$$

at least for $u_0 \in L^2(\mathbb{T})$. In the following, we study the Cauchy problem (1.3) instead of (1.1). (c.f. Bourgain [5], Christ [11].) Bourgain [5] refers to (1.3) as the Wick ordered cubic NLS since it may also be obtained from the Wick ordered Hamiltonian. In [13], we proved that (1.3) is locally well-posed (LWP) a.s. in ω for any $\alpha > \frac{1}{6}$. i.e. a.s. in H^s for each $s > -\frac{1}{3}$. Our main goal here is to show that (1.3) is globally well-posed a.s. for each fixed $\alpha \in (\alpha_0, \frac{1}{2}]$ with some $\alpha_0 \geq \frac{1}{6}$.

First recall that we can regard u_0 in (1.2) as a typical element in the support of the Gaussian measure:

$$(1.4) \quad d\rho_\alpha = Z^{-1} \exp\left(-\frac{1}{2} \int |u|^2 dx - \frac{1}{2} \int |D^\alpha u|^2 dx\right) \prod_{x \in \mathbb{T}} du(x),$$

where $D = \sqrt{-\partial_x^2}$. From the basic theory of Gaussian measures on Hilbert/Banach spaces (c.f. Kuo [16] and Zhidkov [24]), it follows that $u_0 \in H^s(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$ a.s. for $s < \alpha - \frac{1}{2}$. Also, note that ρ_α corresponds to (the leading term of) the Gibbs measure and the white noise for $\alpha = 1$ and $\alpha = 0$, respectively.

So far, there is basically only one method known for proving the a.s. global well-posedness of PDEs with random initial data of type (1.2). In [3], Bourgain proved the invariance of the Gibbs measures for NLS. In dealing with the super-cubic nonlinearity, (where only the local well-posedness result was available), he used a probabilistic argument and the approximating finite dimensional ODEs (with the invariant finite dimensional Gibbs measures) to extend the local solutions to the global ones almost surely on the statistical ensemble and showed the invariance of the Gibbs measures. We point out that this method can be applied in a general setting, provided that the local well-posedness is obtained with a ‘‘good’’ estimate on the solutions (e.g. via the fixed point argument) and that we have a formally invariant measure such as the Gibbs measure or the white noise (where the leading term corresponds to (1.4) for $\alpha = 1$ and $\alpha = 0$.) See Bourgain [4, 5], Burq-Tzvetkov [8, 10], Oh [18, 19], and Tzvetkov [21, 22].

From [13], we have the local solutions in the support of the Gaussian measure ρ_α in (1.4) for $\alpha \in (\frac{1}{6}, \frac{1}{2}]$, which we would like to extend to the global ones. Since the values of α is strictly between 0 and 1, the initial condition u_0 in (1.2) is not in the support of an invariant measure for (1.3) (i.e. (the leading term of) the Gibbs measure or the white noise.) Therefore, Bourgain’s probabilistic argument [3] is not applicable here.

Now, we briefly review the idea behind the argument in [13]. By writing (1.3) as an integral equation, we have

$$(1.5) \quad u(t) = \Gamma u(t) := S(t)u_0 \pm i \int_0^t S(t-t') \mathcal{N}(u)(t') dt'$$

where $S(t) = e^{-i\partial_x^2 t}$, u_0 is as in (1.2), and $\mathcal{N}(u) = u|u|^2 - 2u \int |u|^2$. Note that $S(t)u_0$ has the same regularity as u_0 for each fixed $t \in \mathbb{R}$. i.e. $S(t)u_0 \in H^{\alpha-\frac{1}{2}}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$ a.s. Hence, $S(t)u_0$ is a.s. strictly in the negative Sobolev space for $\alpha \leq \frac{1}{2}$.

However, it turned out that the nonlinear part $\int_0^t S(t-t')\mathcal{N}(u)(t')dt'$ lies almost surely in a smoother space $L^2(\mathbb{T})$ even for $\alpha \leq \frac{1}{2}$. (Also, see [5], [9].) We indeed showed that for each $\delta > 0$ there exists Ω_δ with complementary measure $< e^{-\frac{1}{\delta^\epsilon}}$ such that Γ defined in (1.5) is a contraction on $S(t)u_0^\omega + B$ for $\omega \in \Omega_\delta$ on the time interval $[0, \delta]$, where B denotes the ball of radius 1 in the Bourgain space $Z^{0, \frac{1}{2}, \delta}$ defined in (2.1) below.

This observation led us to consider the Bourgain's high-low method [6] since this kind of *nonlinear smoothing* is the crucial ingredient for the method. Moreover, as you see below, the implementation of the Bourgain's high-low method naturally lets us apply our probabilistic local theory iteratively since the data for the difference equations have random Fourier coefficients at each step.

Let $s = \alpha - \frac{1}{2} -$ with $\alpha \leq \frac{1}{2}$. i.e. $s < 0$. By the large deviation estimate, we have

$$(1.6) \quad \mathbb{P}(\|u_0(\omega)\|_{H^s} \geq K) \leq e^{-cK^2}.$$

In the following, we restrict ourselves on $\Omega_K = \{\omega \in \Omega : \|u_0(\omega)\|_{H^s} \leq K\}$. By writing $u_0 = \phi_0 + \psi_0$, where $\phi_0 := \mathbb{P}_{\leq N} u_0 = \sum_{|n| \leq N} \widehat{u}_0(n) e^{inx}$, the low-frequency part ϕ_0 is in $L^2(\mathbb{T})$, and it satisfies $\|\phi_0\|_{L^2} \leq N^{-s} \|\phi_0\|_{H^s} \leq N^{-s} K$.

Let u^1 denote the solution of (1.3) with the initial data ϕ_0 on some time interval $[0, \delta]$, where δ is the time of local existence, i.e. $\delta = \delta(N^{-s}K) \lesssim \delta(\|\phi_0\|_{L^2})$. Then, we have

$$(1.7) \quad \begin{cases} i\partial_t u^1 - \partial_x^2 u^1 \pm \mathcal{N}(u^1) = 0 \\ u^1|_{t=0} = \phi_0. \end{cases}$$

From the L^2 well-posedness theory of Bourgain [2], (1.7) is globally well-posed with the L^2 -conservation: $\|u^1(t)\|_{L^2} = \|\phi_0\|_{L^2} \lesssim N^{-s}K$ for any $t \in \mathbb{R}$. Moreover, from the local theory, we have

$$(1.8) \quad \|u^1\|_{Z^{0, \frac{1}{2}}[0, \delta]} \lesssim \|\phi_0\|_{L^2} \lesssim N^{-s}K.$$

Now, let v^1 be a solution of the following difference equation on $[0, \delta]$:

$$(1.9) \quad \begin{cases} i\partial_t v^1 - \partial_x^2 v^1 \pm (\mathcal{N}(u^1 + v^1) - \mathcal{N}(u^1)) = 0 \\ v^1|_{t=0} = \psi_0 = \sum_{|n| > N} \frac{g_n(\omega)}{1+|n|^\alpha} e^{inx}. \end{cases}$$

i.e. we have $u(t) = u^1(t) + v^1(t)$ as long as the solution v^1 of (1.9) exists. Note that ψ_0 has Gaussian-randomized Fourier coefficients. Hence, we can use our probabilistic local theory in [13] to study (1.9).

Suppose that, by the local theory in [13], we can show that (1.9) is locally well-posed on the time interval $[0, \delta]$ except on a set of measure $e^{-\frac{1}{\delta^\epsilon}}$. i.e. we have $v^1(t) = S(t)\psi_0 + w^1(t)$, where the nonlinear part $w^1(t)$ is smoother and is in $L^2(\mathbb{T})$ for all $t \in [0, \delta]$. Note that due to the appearance of the external function u^1 in (1.9) with the large $Z^{0, \frac{1}{2}, \delta}$ -norm, we need to refine our argument from [13] in order to have a good estimate on $\|w^1(t)\|_{L^2}$.

At time $t = \delta$, we redistribute the data. i.e. write $u(\delta) = \phi_1 + \psi_1$, where $\phi_1 := u^1(\delta) + w^1(\delta)$ and $\psi_1 := S(\delta)\psi_0$. Let u^2 denote the solution of (1.3) with the initial data ϕ_1 starting at time $t = \delta$. i.e.

$$(1.10) \quad \begin{cases} i\partial_t u^2 - \partial_x^2 u^2 \pm \mathcal{N}(u^2) = 0 \\ u^2|_{t=\delta} = \phi_1 = u^1(\delta) + w^1(\delta) \in L^2(\mathbb{T}). \end{cases}$$

Then, (1.10) is globally well-posed. Also, from the local theory, we have

$$(1.11) \quad \|u^2\|_{Z^{0, \frac{1}{2}}[\delta, 2\delta]} \lesssim \|\phi_1\|_{L^2} \leq \|u^1(\delta)\|_{L^2} + \|w^1(\delta)\|_{L^2} \lesssim N^{-s}K + \|w^1(\delta)\|_{L^2} \lesssim N^{-s}K$$

as long as

$$(1.12) \quad \|w^1(\delta)\|_{L^2} \lesssim N^{-s}K.$$

Now, let v^2 be the solution of the difference equation on $[\delta, 2\delta]$:

$$(1.13) \quad \begin{cases} i\partial_t v^2 - \partial_x^2 v^2 \pm (\mathcal{N}(u^2 + v^2) - \mathcal{N}(u^2)) = 0 \\ v^2|_{t=\delta} = \psi_1 = \sum_{|n|>N} \frac{g_n(\omega)e^{i\delta n^2}}{1+|n|^\alpha} e^{inx}. \end{cases}$$

Once again, ψ_1 has Gaussian-randomized Fourier coefficients. Hence, we can use our local theory in [13] to study (1.13).

In this way, we iterate the deterministic local theory to the “low-frequency” part u^j and the probabilistic local theory to the high-frequency part v^j to prove that (1.3) is well-posed on $[0, T]$ for arbitrary $T > 0$. For details, see Section 3.

Theorem 1. *Let $\alpha \in (\frac{3}{8}, \frac{1}{2}]$. Then, the periodic (Wick ordered) cubic NLS (1.3) is globally well-posed almost surely in $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$. More precisely, for almost every $\omega \in \Omega$ there exists a unique solution u of (1.3) in*

$$e^{-i\partial_x^2 t} u_0 + C(\mathbb{R}; L^2(\mathbb{T})) \subset C(\mathbb{R}; H^{\alpha-\frac{1}{2}-}(\mathbb{T}))$$

with the initial condition u_0^ω given by (1.2).

In particular, we have a.s. global well-posedness with respect to the Gaussian measure (1.4) supported in $H^s(\mathbb{T})$ for each $s > -\frac{1}{8}$.

Let us make several remarks.

Remark 1.1. A linear part of a solution constructed in Theorem 1 indeed lies in $C([-T, T]; B(\mathbb{T}))$ for any Banach space $B(\mathbb{T}) \supset H^\alpha(\mathbb{T})$ such that $(H^\alpha, B, \rho_\alpha)$ is an abstract Wiener space (roughly speaking, any Banach space B containing H^α where the Gaussian measure ρ_α makes sense as a countable additive probability measure.) In this case, a solution u to (1.3) lies in

$$u = e^{-i\partial_x^2 t} u_0 + (-i\partial_t + \partial_x^2)^{-1} u \in C(\mathbb{R}; B(\mathbb{T})) + C(\mathbb{R}; L^2(\mathbb{T})).$$

As examples of B , we can take the Sobolev spaces $W^{s,p}$ with $s < \alpha - \frac{1}{2}$, and the Fourier-Lebesgue spaces $\mathcal{FL}^{s,p}$ with $s < \alpha - \frac{1}{p}$, where $\mathcal{FL}^{s,p}$ is defined via the norm $\|f\|_{\mathcal{FL}^{s,p}} = \|\langle n \rangle^s \widehat{f}(n)\|_{L_n^p}$. See Bényi-Oh [1] for regularity of ρ_α (and u_0 in (1.2)) in different function spaces. In [1], we study the regularity of ρ_α for $\alpha = 1$ but it can be easily adjusted for any α . We can also take the Besov space $B_{2,\infty}^{\alpha-\frac{1}{2}}$ for example. For global-in-time argument, it is important that the large deviation estimate (1.6) still holds for these spaces.

Remark 1.2. On the one hand, uniqueness for “low-frequency” part u^j in the j th step holds in $C([(j-1)\delta, j\delta], L^2(\mathbb{T})) \cap Z^{0, \frac{1}{2}}[(j-1)\delta, j\delta]$ as usual. On the other hand, uniqueness for the high-frequency part v^j in the j th step holds only in the ball centered at $S(t)\psi_{j-1}$ of small radius in $Z^{0, \frac{1}{2}}[(j-1)\delta, j\delta]$. Also, continuous dependence for v^j holds in $L^2(\mathbb{T})$ in the following sense. For simplicity, take $j = 1$. Suppose the solution v^1 of (1.9) with initial condition $\psi_0 = \psi_0^\omega$ exists on $[0, \delta]$. Let \tilde{v}_0 be a function on \mathbb{T} such that $\|\psi_0 - \tilde{v}_0\|_{L^2} < C(N, \alpha)$. Then, we have $\|v^1(t) - \tilde{v}(t)\|_{L^2} \leq c\|u_0 - \tilde{u}_0\|_{L^2}$ for $|t| \leq \delta$, where \tilde{v} is the solution of (1.9) with the initial condition \tilde{v}_0 . See [5], [13].

Remark 1.3. The periodic cubic NLS (1.1) is known to be ill-posed in $H^s(\mathbb{T})$ for $s < 0$. See Molinet [17] for the most recent work and the references therein. As for the Wick ordered cubic NLS (1.3), note that $u_{N,a}(x,t) = ae^{i(Nx+N^2t\mp|a|^2t)}$ solves (1.3) for $a \in \mathbb{C}$ and $N \in \mathbb{N}$. Then, we can follow the argument by Burq-Gérard-Tzvetkov [7] to show the ill-posedness of (1.3) below $L^2(\mathbb{T})$. Also, see Christ-Colliander-Tao [12].

Remark 1.4. Recall that the white noise corresponds to $\alpha = 0$ in (1.4) (up to constants). We would like to treat the case $\alpha = 0$. Our result in [13] and Theorem 1 are partial results in this direction.

Given a function $f = \sum_{n \in \mathbb{Z}} \widehat{f}_n e^{inx} \in H^\sigma(\mathbb{T})$, define $u_0^\omega = \sum_{n \in \mathbb{Z}} g_n(\omega) \widehat{f}_n e^{inx}$. Then, one can consider the local well-posedness problem of (1.3) with these randomized data u_0^ω . See Burq-Tzvetkov [9] Thomann [20]. However, we do not pursue this issue here.

Remark 1.5. Note that u_0^ω in (1.2) can also be expressed as $u_0^\omega = \sum \widetilde{g}_n e_n$ where e_n is another orthonormal basis in $H^\alpha(\mathbb{T})$ and $\{\widetilde{g}_n\}$ is another family of independent standard complex-valued Gaussians. (Strictly speaking, we need to replace $1 + |n|$ by $(1 + |n|)^\alpha$ in (1.2).) i.e. One should regard Theorem 1 as a global well-posedness result for a typical element in $H^{\alpha-\frac{1}{2}}(\mathbb{T})$ (independent of choice of a basis.)

This paper is organized as follows. In Section 2, we introduce the basic function spaces and notations. In Section 3, we present the proof of Theorem 1, assuming Proposition 3.2. In Section 4, we set up the basic strategy for proving Proposition 3.2. In Section 5, we list some deterministic and probabilistic lemmata. Then, we prove the estimates of the nonlinear term in Sections 6–8.

2. NOTATION

First, recall the Bourgain space $X^{s,b}(\mathbb{T} \times \mathbb{R})$, c.f. [2], whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^2 \rangle^b \widehat{u}(n, \tau)\|_{l_n^2 l_\tau^2}.$$

Since the $X^{s,\frac{1}{2}}$ norm fails to control $L_t^\infty H_x^s$ norm, we use a smaller space $Z^{s,b}(\mathbb{T} \times \mathbb{R})$ whose norm is given by

$$(2.1) \quad \|u\|_{Z^{s,b}(\mathbb{T} \times \mathbb{R})} = \|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} + \|u\|_{Y^{s,b-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})}$$

where $\langle \cdot \rangle = 1 + |\cdot|$ and $\|u\|_{Y^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^2 \rangle^b \widehat{u}(n, \tau)\|_{l_n^2 l_\tau^1}$. We define the local-in-time version $Z^{s,b,\delta}$ on $\mathbb{T} \times [-\delta, \delta]$, by

$$\|u\|_{Z^{s,b,\delta}} = \inf \{ \|\widetilde{u}\|_{Z^{s,b}(\mathbb{T} \times \mathbb{R})} : \widetilde{u}|_{[-\delta,\delta]} = u \}.$$

We also define the local-in-time version $Z_I^{s,b} = Z^{s,b}[a, b]$ on an interval $I = [a, b]$. The local-in-time versions of other function spaces are defined analogously.

If a function depends on both x and t , we use $\widehat{\cdot}^x$ (and $\widehat{\cdot}^t$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $\widehat{\cdot}$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context. For simplicity, we often drop 2π in dealing with the Fourier transforms. If a function f is random, we may use the superscript f^ω to show its dependence on ω .

Lastly, let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function supported on $[-2, 2]$ with $\eta \equiv 1$ on $[-1, 1]$ and let $\eta_\delta(t) = \eta(\delta^{-1}t)$. We use c, C to denote various constants, usually depending only on α . If a constant depends on other quantities, we will make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote

$A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. We also use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

3. PROOF OF MAIN RESULT

First, we prove Theorem 1, assuming the following proposition.

Proposition 3.1. *Let $\alpha \in (\frac{3}{8}, \frac{1}{2}]$. Give $T > 0$ and $\varepsilon > 0$, there exists $\Omega_{T,\varepsilon} \in \mathcal{F}$ with the following properties:*

- (i) $\mathbb{P}(\Omega_{T,\varepsilon}^c) = \rho_\alpha \circ u_0(\Omega_{T,\varepsilon}^c) < \varepsilon$, where $u_0 : \Omega \rightarrow H^{\alpha - \frac{1}{2}^-}(\mathbb{T})$.
- (ii) For each $\omega \in \Omega_{T,\varepsilon}$ there exists a unique solution u of (1.3) in

$$e^{-i\partial_x^2 t} u_0 + C([-T, T]; L^2(\mathbb{T})) \subset C([-T, T]; H^{\alpha - \frac{1}{2}^-}(\mathbb{T}))$$

with the initial condition u_0^ω given by (1.2).

Proof of Theorem 1. For fixed $\varepsilon > 0$, let $T_j = 2^j$ and $\varepsilon_j = 2^{-j}\varepsilon$. Apply Proposition 3.1 and construct $\Omega_{T_j, \varepsilon_j}$. Then, let $\Omega_\varepsilon = \bigcap_{j=1}^\infty \Omega_{T_j, \varepsilon_j}$. Note that (1.5) is globally well-posed on Ω_ε with $\mathbb{P}(\Omega_\varepsilon^c) < \varepsilon$. Now, let $\tilde{\Omega} = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$. Then, (1.5) is globally well-posed on $\tilde{\Omega}$ and $\mathbb{P}(\tilde{\Omega}^c) = 0$. \square

Now, we present the proof of Proposition 3.1.

Proof of Proposition 3.1. First, recall the following argument which relates the time of local existence δ and the size of the initial condition. Consider (1.3). From the deterministic local theory (especially Bourgain's L^4 Strichartz [2] and Lemma 5.2 below), we have

$$(3.1) \quad \begin{aligned} \|u\|_{Z^{0, \frac{1}{2}, \delta}} &\leq \|u_0\|_{L^2} + C_1 \delta^{0-} \left\| \int_0^t S(t-t') \mathcal{N}(t') dt' \right\|_{Z^{0, -\frac{1}{2}, \delta}} \\ &\leq \|u_0\|_{L^2} + C_2 \delta^{\frac{1}{2}-} \|u\|_{Z^{0, \frac{1}{2}, \delta}}^3. \end{aligned}$$

In proving LWP via the fixed point argument, we require

$$(3.2) \quad \delta^{\frac{1}{2}-} \|u\|_{Z^{0, \frac{1}{2}, \delta}}^2 \lesssim 1$$

on the ball $\{u : \|u\|_{Z^{0, \frac{1}{2}, \delta}} \leq 2\|u_0\|_{L^2}\}$. Hence, we can choose $\delta \sim \|u_0\|_{L^2}^{-4-}$.

Let $T > 0$ and $\varepsilon > 0$ be given, and we continue the argument from Section 1. First, in view of (1.6), choose $K \sim (\log \frac{1}{\varepsilon})^{\frac{1}{2}}$ so that $\mathbb{P}(\|u_0(\omega)\|_{H^s} \geq K) \leq \frac{1}{2}\varepsilon$. In the following, we assume $\|u_0\|_{H^s} \leq K$. Now, fix $\delta \sim N^{4s-} K^{-4-}$.

Before proceeding further, we present an important proposition whose proof is given in the remaining sections of the paper.

Proposition 3.2. *Let $s = \alpha - \frac{1}{2}-$ with $\alpha \in (\frac{3}{8}, \frac{1}{2}]$. Given $T > 0$ and $K > 0$, there exists N sufficiently large with $\delta \sim N^{4s-} K^{-4-}$ such that the following holds. Suppose that*

$$(3.3) \quad \|u^j(t)\|_{Z^{0, \frac{1}{2}}[(j-1)\delta, j\delta]} \leq CN^{-s}K$$

such that $\delta^{\frac{1}{2}-} \|u^j(t)\|_{Z^{0, \frac{1}{2}}[(j-1)\delta, j\delta]}^2 \lesssim 1$ (see (3.2)) for $j = 1, \dots, [\frac{T}{\delta}]$. Write the solution v^j of the following difference equation:

$$(3.4) \quad \begin{cases} i\partial_t v^j - \partial_x^2 v^j \pm (\mathcal{N}(u^j + v^j) - \mathcal{N}(u^j)) = 0 \\ v^j|_{t=(j-1)\delta} = \psi_{j-1} = \sum_{|n| > N} \frac{g_n(\omega) e^{i(j-1)\delta n^2}}{1 + |n|^\alpha} e^{inx} \end{cases}$$

as $v^j(t) = S(t - (j-1)\delta)\psi_{j-1} + w^j(t)$. Then, (3.4) is locally well-posed on the time interval $[(j-1)\delta, j\delta]$ except on a set of measure $e^{-\frac{1}{\delta^c}}$ for each $j = 1, \dots, [\frac{T}{\delta}]$. Moreover, we have the following bound on the nonlinear terms:

$$(3.5) \quad \sum_{j=1}^{[T/\delta]} \|w^j(j\delta)\|_{L^2} \lesssim N^{-s}K.$$

Now, we continue the proof of Proposition 3.1. Our choice of δ guarantees that (1.7) is well-posed on $[0, \delta]$ with the bound (1.8). Then, by Proposition 3.2, (1.9) is well-posed on $[0, \delta]$ except on a set of measure $e^{-\frac{1}{\delta^c}}$ with the bound (1.12), which in turn shows that (1.10) is well-posed on $[\delta, 2\delta]$ with the bound (1.11).

Write the solution v^2 of (1.13) as $v^2(t) = S(t - \delta)\psi_1 + w^2(t)$. It follows from (1.11) and Proposition 3.2 that (1.13) is well-posed on the time interval $[\delta, 2\delta]$ except on a set of measure $e^{-\frac{1}{\delta^c}}$. Moreover, we have

$$(3.6) \quad \sum_{j=1}^2 \|w^j(j\delta)\|_{L^2} \lesssim N^{-s}K.$$

At time $t = 2\delta$, write $u(2\delta) = \phi_2 + \psi_2$, where $\phi_2 := u^2(2\delta) + w^2(2\delta)$ and $\psi_2 := S(\delta)\psi_1 = S(2\delta)\psi_0$. Then, (3.6) guarantees that the solution u^3 to

$$(3.7) \quad \begin{cases} i\partial_t u^j - \partial_x^2 u^j \pm \mathcal{N}(u^j) = 0 \\ u^j|_{t=(j-1)\delta} = \phi_{j-1} \end{cases}$$

with $j = 3$ satisfies

$$(3.8) \quad \|u^3\|_{Z^{0, \frac{1}{2}}[2\delta, 3\delta]} \leq \|\phi_0\|_{L^2} + \sum_{j=1}^2 \|w^j(j\delta)\|_{L^2} \lesssim N^{-s}K.$$

Clearly, we can iterate this argument to show that (1.3) is well-posed on $[0, T]$, assuming (3.5). Lastly, note that the measure of the exceptional sets can be estimated by

$$\left[\frac{T}{\delta}\right] e^{-\frac{1}{\delta^c}} \leq e^{\ln \frac{T}{\delta} - \frac{1}{\delta^c}} \leq e^{-\frac{1}{2} \frac{1}{\delta^c}} < \frac{1}{2} \varepsilon$$

for sufficiently small $\delta > 0$, i.e. for sufficiently large $N = N(T, \varepsilon)$. This completes the proof of Proposition 3.1. \square

4. BASIC SETUP

In the remaining sections of the paper, we prove Proposition 3.2. In the following, fix $T > 0$ and $K > 0$, and let $s = \alpha - \frac{1}{2}$ and

$$(4.1) \quad \delta \sim N^{4s-} K^{-4-},$$

where $N = N(T, K)$ to be determined later.

First, consider the following difference equation:

$$(4.2) \quad \begin{cases} i\partial_t v - \partial_x^2 v \pm (\mathcal{N}(u^0 + v) - \mathcal{N}(u^0)) = 0 \\ v|_{t=0} = \psi = \sum_{|n| > N} \frac{c_n g_n(\omega)}{1 + |n|^\alpha} e^{inx} \end{cases}$$

where $|c_n| = 1$ for all $n \in \mathbb{Z}$ and u^0 is a given function such that

$$(4.3) \quad \|u^0(t)\|_{Z^{0, \frac{1}{2}, \delta}} \leq CN^{-s}K$$

satisfying (3.2). Let w denote the nonlinear part of the solution v of (4.2). i.e. it is given by

$$(4.4) \quad w(t) := w(t; v, \psi, u^0) = \pm i \int_0^t S(t-t') (\mathcal{N}(u^0 + v) - \mathcal{N}(u^0))(t') dt'$$

for $t \in [0, \delta]$. From the linear estimate [2], [14], we have

$$(4.5) \quad \|w(\delta)\|_{L^2} \lesssim \|\eta_\delta(t)w\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim \delta^{0-} \|\mathcal{N}(u^0 + v) - \mathcal{N}(u^0)\|_{Z^{0, -\frac{1}{2}, \delta}},$$

where η_δ is a smooth cutoff on $[-2\delta, 2\delta]$. Suppose that we have

$$(4.6) \quad \|\mathcal{N}(u^0 + v) - \mathcal{N}(u^0)\|_{Z^{0, -\frac{1}{2}, \delta}} \lesssim N^{3s-\gamma-}$$

for some small $\gamma > 0$ except on a set of measure $e^{-\frac{1}{\delta^c}}$. Then, it follows that the mapping Γ defined by

$$(4.7) \quad \Gamma v(t) := S(t)\psi + w(t; v, \psi, u^0)$$

is a contraction on $S(t)\psi^\omega + B$ on the time interval $[0, \delta]$ except on a set of measure $e^{-\frac{1}{\delta^c}}$, where B denotes the ball of radius $\sim N^{3s-\gamma}$ in $Z^{0, \frac{1}{2}, \delta}$. Moreover, from (4.5) and (4.6), we have

$$(4.8) \quad \frac{T}{\delta} \|w(\delta)\|_{L^2} \lesssim T\delta^{-1} N^{3s-\gamma} \lesssim TK^{4+} N^{-s} N^{-\gamma+} \lesssim N^{-s} K$$

for sufficiently large $N = N(T, K)$. Note that (4.5) and (4.6) imply only the boundedness of the map Γ from $S(t)\psi^\omega + B$ into itself. In establishing the contraction property, one needs to consider $\Gamma v_1 - \Gamma v_2$ for $v_1, v_2 \in S(t)\psi^\omega + B$. We omit the details of this part since the argument is standard. For details, see [2], [5].

Finally, note that the bound (3.3) on u^j is uniform in j in Proposition 3.2. Hence, the above LWP result can be applied to (3.4) on $[(j-1)\delta, j\delta]$ for $j = 1, \dots, [\frac{T}{\delta}]$, and moreover (3.5) follows from (4.8). Therefore, it remains to prove (4.6) for $\alpha \in (\frac{3}{8}, \frac{1}{2}]$ (and for large N .)

By letting

$$(4.9) \quad \begin{cases} \mathcal{N}_1(u_1, u_2, u_3)(x) = \sum_{n_2 \neq n_1, n_3} \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_2)} \widehat{u}_3(n_3) e^{i(n_1 - n_2 + n_3)x} \\ \mathcal{N}_2(u_1, u_2, u_3)(x) = \sum_n \widehat{u}_1(n) \widehat{u}_2(n) \widehat{u}_3(n) e^{inx}, \end{cases}$$

we have $\mathcal{N}(u) = u|u|^2 - 2u f|u|^2 dx = \mathcal{N}_1(u, u, u) - \mathcal{N}_2(u, u, u)$. Then, (4.6) follows, once we prove

$$(4.10) \quad \|\mathcal{N}_j(u_1, u_2, u_3)\|_{Z^{0, -\frac{1}{2}, \delta}} \lesssim N^{3s-\gamma-}, \quad j = 1, 2,$$

except on a set of measure $e^{-\frac{1}{\delta^c}}$, where u_j is either of type

(I) random, less regular:

$$u_j(x, t) = S(t)\psi = \sum_{|n| > N} \frac{c_n g_n(\omega)}{1 + |n|^\alpha} e^{i(nx + n^2 t)} \text{ with } |c_n| = 1, \text{ or}$$

(II) deterministic, smoother:

$$(4.11) \quad u_j = w \text{ with } \|w\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim N^{3s-\gamma}, \text{ or } u^0 \text{ with } \|u^0\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim N^{-s} K \text{ satisfying (3.2),}$$

except for the case $u_j = u^0$ for all $j = 1, 2, 3$. We may insert the smooth cutoff function η_δ supported on $[-2\delta, 2\delta]$ if necessary.

Note that u^0 has a larger norm than w since $s < 0$. Thus, we assume that $u_j = u^0$ if u_j is of type (II), unless u_j is of type (II) for all $j = 1, 2, 3$. In the latter case, we may assume that two of u_j 's are u^0 and the remaining u_j is w , and it suffices to prove

$$(4.12) \quad \|\mathcal{N}_j(u^0, u^0, w)\|_{Z^{0, -\frac{1}{2}, \delta}} \lesssim N^{3s-\gamma}, \quad j = 1, 2,$$

in view of (3.1) and (3.2). In the following sections, we prove (4.10) by separately estimating the contributions from \mathcal{N}_1 and \mathcal{N}_2 .

5. DETERMINISTIC AND PROBABILISTIC LEMMATA

In this section, we state several useful lemmata. See [13] for the proofs. First, recall the following algebraic identity related to the cubic NLS:

$$(5.1) \quad n^2 - (n_1^2 - n_2^2 + n_3^2) = 2(n_2 - n_1)(n_2 - n_3)$$

for $n = n_1 - n_2 + n_3$. Let N^1, N^2, N^3 be the decreasing ordering of N_1, N_2, N_3 , where $|n_j| \sim N_j$, and let n^j denote the corresponding frequency.

Lemma 5.1. *Let*

$$S_\mu = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \sim N_j, n_2 \neq n_1, n_3, \text{ and } 2(n_2 - n_1)(n_2 - n_3) = \mu\}.$$

Then, we have

$$(5.2) \quad \#S_\mu \lesssim (N^1)^{0+} N^3.$$

Next, recall that by restricting the Bourgain spaces onto a small time interval $[-\delta, \delta]$, we can gain a small power of δ (at a slight loss of regularity on $\langle \tau - n^2 \rangle$.)

Lemma 5.2. *For $b < \frac{1}{2}$, we have*

$$(5.3) \quad \|u\|_{X^{s,b,\delta}} = \|\eta_\delta u\|_{X^{s,b,\delta}} \lesssim \delta^{\frac{1}{2}-b-} \|u\|_{X^{s,\frac{1}{2},\delta}}.$$

Recall that the proof basically follows from

$$(5.4) \quad \|\widehat{\eta}_\delta\|_{L^q_\tau} \sim \delta^{\frac{q-1}{q}} \|\widehat{\eta}\|_{L^q_\tau} \sim \delta^{\frac{q-1}{q}},$$

where $\widehat{\eta}_\delta(\tau) = \delta \widehat{\eta}(\delta\tau)$, and interpolation. Lastly, we present several probabilistic lemmata related to the Gaussians.

Lemma 5.3. *Let $\varepsilon, \beta > 0$. Then, for $\delta > 0$, we have*

$$(5.5) \quad |g_n(\omega)| \leq C \delta^{-\frac{\beta}{2}} \langle n \rangle^\varepsilon$$

for all $n \in \mathbb{Z}$ for ω outside an exceptional set of measure $< e^{-\frac{1}{\delta^\varepsilon}}$.

Lemma 5.4. *Let $f^\omega(x, t) = \sum c_n g_n(\omega) e^{i(nx+n^2t)}$, where $\{g_n\}$ is a family of complex valued standard i.i.d. Gaussian random variables. Then, for $p \geq 2$, there exists $\delta, \delta_0 > 0$ such that*

$$\mathbb{P}(\|f^\omega\|_{L^p(\mathbb{T} \times [-\delta, \delta])} > C \|c_n\|_{l^2_n}) < e^{-\frac{1}{\delta^\varepsilon}}$$

for $\delta \leq \delta_0$.

Note that we have $\delta_0 \sim e^{-p \ln p}$ from the proof of this lemma in [13].

In Section 8, we also use the hypercontractivity properties related to the Gaussians. See Sections 3 and 4 in Tzvetkov [23].

6. ESTIMATE ON \mathcal{N}_2

In this section, we prove the estimate (4.10) for $\mathcal{N}_2(u_1, u_2, u_3)$ defined in (4.9). By Hölder inequality with p large ($\frac{1}{2} = \frac{1}{2+} + \frac{1}{p}$), we have

$$\|\mathcal{N}_2(u_1, u_2, u_3)\|_{Z^{0, -\frac{1}{2}, \delta}} \lesssim \left\| \frac{1}{\langle \tau - n^2 \rangle^{\frac{1}{2}-}} \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{u}_1(n, \tau_1) \overline{\widehat{u}_2(n, \tau_2)} \widehat{u}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_{n, \tau}^2}$$

By Hölder inequality with p large ($\frac{1}{2} = \frac{1}{2+} + \frac{1}{p}$),

$$(6.1) \quad \lesssim \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_{\tau}^{2+}} \left\| \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{u}_1(n, \tau_1) \overline{\widehat{u}_2(n, \tau_2)} \widehat{u}_3(n, \tau_3) d\tau_1 d\tau_2 \right\|_{L_n^2 L_{\tau}^p}.$$

In the following, we omit details if the computation is basically the same as in Sections 5 of [13].

• **Case (a):** u_j of type (II), $j = 1, \dots, 3$.

In this case, we prove (4.12). By Young and Hölder inequalities in τ , followed by Hölder in n , $L_n^2 \subset L_n^6$, and Lemma 5.2, we have

$$(6.1) \lesssim \prod_{j=1}^3 \|\langle \tau - n^2 \rangle^{\frac{1}{6}+} \widehat{u}_j(n, \tau)\|_{L_n^6 L_{\tau}^2} \leq \delta^{1-} \|u^0\|_{Z^{0, \frac{1}{2}, \delta}}^2 \|w\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim \delta^{\frac{1}{2}-} N^{3s-\gamma} \lesssim N^{3s-\gamma}$$

for $s \leq 0$.

• **Case (b):** u_j of type (I), $j = 1, \dots, 3$.

By Lemma 5.3, we have $|g_n(\omega)| \leq C\delta^{-\frac{\beta}{2}} \langle n \rangle^{\varepsilon}$ for ω outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$. Then, by Young's inequality,

$$(6.1) \lesssim \delta^{1-} \|\langle n \rangle^{-3\alpha} |g_n(\omega)|^3\|_{L_{|n|>N}^2} \lesssim \delta^{1-\frac{3}{2}\beta-} \|\langle n \rangle^{-3\alpha+3\varepsilon}\|_{L_{|n|>N}^2} \lesssim \delta^{1-\frac{3}{2}\beta-} N^{-3\alpha+\frac{1}{2}+3\varepsilon} \lesssim N^{3s-2\alpha+} K^{-4-} \lesssim N^{3s-\gamma-}$$

for $\alpha > \frac{1}{2}\gamma > 0$.

• **Case (c):** Exactly two u_j 's of type (I). Say $u_1(\text{I})$, $u_2(\text{I})$, and $u_3(\text{II})$.

By Young's inequality and Lemmata 5.2 and 5.3, we have

$$(6.1) \lesssim \delta^{\frac{1}{2}-} \left(\sup_{|n|>N} \langle n \rangle^{-2\alpha} |g_n|^2 \right) \|\widehat{u^0}(n, \tau)\|_{L_{n, \tau}^2} \lesssim \delta^{1-\beta-} N^{-2\alpha+2\varepsilon} \|u^0\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim N^{3s-2\alpha+} K^{-3-} \lesssim N^{3s-\gamma-}$$

for $\alpha > \frac{1}{2}\gamma > 0$ outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$.

• **Case (d):** Exactly one u_j of type (I). Say $u_1(\text{I})$, $u_2(\text{II})$, and $u_3(\text{II})$.

By Young's inequality, followed by Hölder inequality in n ($\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$) and in τ ($\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$) and Lemmata 5.2 and 5.3, we have

$$(6.1) \lesssim \delta^{\frac{1}{2}-} \left(\sup_{|n|>N} \langle n \rangle^{-\alpha} |g_n| \right) \left\| \|\widehat{u^0}(n, \tau)\|_{L_{\tau}^{\frac{4}{3}}} \right\|_{L_n^2}^2 \lesssim \delta^{\frac{1}{2}-\frac{\beta}{2}-} N^{-\alpha+\varepsilon} \sup_n \|\langle \tau - n^2 \rangle^{-\frac{1}{4}-}\|_{L_{\tau}^4}^2 \|\langle \tau - n^2 \rangle^{\frac{1}{4}+} \widehat{u^0}(n, \tau)\|_{L_n^4 L_{\tau}^2}^2 \lesssim \delta^{1-\frac{\beta}{2}-} N^{-\alpha+\varepsilon} \|\langle \tau - n^2 \rangle^{\frac{1}{2}} \widehat{v}_j(n, \tau)\|_{L_n^4 L_{\tau}^2}^2 \lesssim N^{2s-\alpha+} K^{-2-} \lesssim N^{3s-\gamma-}$$

for ω outside an exceptional set of measure $< e^{-\frac{1}{\delta^\varepsilon}}$ as long as $\alpha > \frac{1}{4} + \frac{1}{2}\gamma > \frac{1}{4}$.

7. ESTIMATE ON \mathcal{N}_1 : BASIC SETUP

In the next two sections, we prove the main part of the estimate (4.10):

$$(7.1) \quad \|\mathcal{N}_1(u_1, u_2, u_3)\|_{Z^{0, -\frac{1}{2}, \delta}} \lesssim N^{3s-\gamma-}$$

for some small $\gamma > 0$, where

$$\mathcal{N}_1(u_1, u_2, u_3) = \sum_{n_2 \neq n_1, n_3} \widehat{u}_1(n_1, t) \overline{\widehat{u}_2(n_2, t)} \widehat{u}_3(n_3, t) e^{i(n_1 - n_2 + n_3)x}$$

with u_j of type (I) or (II). Once again, we omit details in the following when the computation basically follows from Sections 6 and 7 of [13].

As in [5] and [13], let N^1, N^2, N^3 be the decreasing ordering of N_1, N_2, N_3 and u^1, u^2, u^3 be the corresponding u_j -factors. Also, let $\sigma^1, \sigma^2, \sigma^3$ denote the corresponding $\sigma_j := \langle \tau_j - n_j^2 \rangle$. In the following, we use superscripts to imply that the functions (or variables) are arranged in the decreasing order of the spatial frequencies N_1, N_2, N_3 .

In the remaining of this section, we consider the basic cases. Recall Bourgain's periodic L^4 Strichartz [2]:

$$(7.2) \quad \|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0, \frac{3}{8}}}.$$

Interpolating this with $\|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}}$, we have

$$(7.3) \quad \|u\|_{L^{3+}_{x,t}} \lesssim \|u\|_{X^{0, \frac{1}{4}+}}, \text{ and } \|u\|_{L^{2+}_{x,t}} \lesssim \|u\|_{X^{0,0+}}.$$

By Hölder inequality, we have $\|w_1\|_{Z^{0, -\frac{1}{2}, T}} \lesssim \|w_1\|_{X^{0, -\frac{1}{2}+, T}}$. Then, using duality, we can estimate (7.1) by

$$(7.4) \quad \int u^1 u^2 u^3 \cdot v \, dx dt$$

where $\|v\|_{X^{0, \frac{1}{2}-, T}} \leq 1$ (with the complex conjugate on an appropriate u^j).

• **Case (A):** u^1 and u^2 are of type (II).

Suppose that u^3 is of type (II). In this case, we prove (4.12) instead of (7.1). By Hölder inequality, (7.2), Lemma 5.2, and (4.11), we have

$$(7.4) \leq \|u_1\|_{L^4_{x,t}} \|u_2\|_{L^4_{x,t}} \|u_3\|_{L^4_{x,t}} \|v\|_{L^4_{x,t}} \lesssim \delta^{\frac{1}{2}-} \|u^0\|_{Z^{0, \frac{1}{2}, \delta}}^2 \|w\|_{Z^{0, \frac{1}{2}, \delta}} \lesssim N^{3s-\gamma}.$$

If u^3 is of type (I) i.e. $u^3 = S(t)\psi$, then apply dyadic decompositions on N^2 and N^3 . Then, by Hölder inequality with p large, (7.3), and Lemmata 5.4 and 5.2, we have

$$(7.4) \leq \|u^1\|_{L^{3+}} \|u^2\|_{L^{3+}} \|u^3\|_{L^p} \|v\|_{L^{3+}} \lesssim (N^3)^{\frac{1}{2}-\alpha+} \|u^0\|_{Z^{0, \frac{1}{4}+, \delta}}^2 \|v\|_{Z^{0, \frac{1}{4}+, \delta}}$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^\varepsilon}}$. If $\langle \tau_j - n_j^2 \rangle^{\frac{1}{4}-} \gtrsim (N^3)^{\frac{1}{2}-\alpha+} N^{-3s+\gamma+\varepsilon}$ for u_j of type (II), or if $\langle \tau - n^2 \rangle^{\frac{1}{4}-} \gtrsim (N^3)^{\frac{1}{2}-\alpha+} N^{-3s+\gamma+\varepsilon}$, then it follows from (4.1), (4.11), and Lemma 5.2 that

$$(7.4) \lesssim \delta^{\frac{1}{2}-} N^{-2s} K^2 N^{3s-\gamma-\varepsilon} \lesssim N^{3s-\gamma-}$$

for N sufficiently large. Recall $N^3 > N$, $s = \alpha - \frac{1}{2}-$, and $\gamma = 0+$. Hence, in the following, we may assume

$$(7.5) \quad \langle \tau - n^2 \rangle \ll (N^3)^{8-16\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^3)^{8-16\alpha+} \text{ if } u_j \text{ of type (II)}.$$

- **Case (B):** u^1 of type (II), and u^2 of type (I).

Dyadically decompose the spatial frequencies for N^2 and N^3 .

- Subcase (B.1): u^3 is of type (II). By Hölder inequality with p large, (7.3), and Lemma 5.4, we have

$$(7.4) \leq \|u^1\|_{L^{3+}} \|u^2\|_{L^p} \|u^3\|_{L^{3+}} \|v\|_{L^{3+}} \lesssim (N^2)^{\frac{1}{2}-\alpha+} \|u^0\|_{X^{0, \frac{1}{4}+, \delta}}^2 \|v\|_{X^{0, \frac{1}{4}+, \delta}}$$

outside an exceptional set of size $< e^{-\frac{1}{\delta^c}}$. If $\langle \tau_j - n_j^2 \rangle^{\frac{1}{4}-} \gtrsim (N^2)^{\frac{1}{2}-\alpha+} N^{-3s+\gamma+}$ for u_j of type (II), or if $\langle \tau - n^2 \rangle^{\frac{1}{4}-} \gtrsim (N^2)^{\frac{1}{2}-\alpha+} N^{-3s+\gamma+}$, then (7.1) follows as in Case (A). Hence, in the following, we may assume

$$(7.6) \quad \langle \tau - n^2 \rangle \ll (N^2)^{8-16\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^2)^{8-16\alpha+} \text{ if } u_j \text{ of type (II).}$$

- Subcase (B.2): u^3 is of type (I). Again, by Hölder inequality with p large, (7.3), and Lemma 5.4, we have

$$(7.4) \leq \|u^1\|_{L^{2+}} \|u^2\|_{L^p} \|u^3\|_{L^p} \|v\|_{L^{2+}} \lesssim (N^2)^{1-2\alpha+} \|u^0\|_{X^{0,0+, \delta}} \|v\|_{X^{0,0+, \delta}}.$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$. If $(\sigma^1)^{\frac{1}{2}-} \gtrsim (N^2)^{1-2\alpha+} N^{-2s+\gamma+}$ or if $\langle \tau - n^2 \rangle^{\frac{1}{2}-} \gtrsim (N^2)^{1-2\alpha+} N^{-2s+\gamma+}$, then (7.1) follows from (4.1), (4.11), and Lemma 5.2 for N sufficiently large. Hence, in the following, we can assume

$$(7.7) \quad \langle \tau - n^2 \rangle \ll (N^2)^{4-8\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^2)^{4-8\alpha+} \text{ if } u_j \text{ of type (II).}$$

- **Case (C):** u^1 of type (I), and u^2, u^3 of type (II).

Dyadically decompose all the spatial frequencies. Suppose $\langle \tau - n^2 \rangle \gg \sigma^2, \sigma^3$. By Hölder inequality with p large, Lemmata 5.4 and 5.2, and (4.11), we have

$$(7.4) \leq \|u^1\|_{L^p} \|u^2\|_{L^4} \|u^3\|_{L^4} \|v\|_{L^{2+}} \lesssim (N^1)^{\frac{1}{2}-\alpha+} \|u^0\|_{X^{0, \frac{3}{8}, \delta}}^2 \|v\|_{X^{0,0+, \delta}} \\ \lesssim \delta^{\frac{1}{4}-} (N^1)^{\frac{1}{2}-\alpha+} N^{-2s} K^2 \|v\|_{X^{0,0+}}$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$. Hence, as before, (7.1) follows for large N as long as $\langle \tau - n^2 \rangle^{\frac{1}{2}-} \gtrsim (N^1)^{\frac{1}{2}-\alpha+} N^{-4s+\gamma+}$. Similar results hold if $\sigma^2 \gg \sigma^3, \langle \tau - n^2 \rangle$ or $\sigma^3 \gtrsim \sigma^2, \langle \tau - n^2 \rangle$. Hence, we assume

$$(7.8) \quad \langle \tau - n^2 \rangle \ll (N^1)^{5-10\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^1)^{5-10\alpha+} \text{ if } u_j \text{ of type (II).}$$

- **Case (D):** u^1 of type (I), and either $u^2(\text{I}), u^3(\text{II})$ or $u^2(\text{II}), u^3(\text{I})$.

Suppose that u^2 is of type (I) and that u^3 is of type (II). Moreover, suppose $\langle \tau - n^2 \rangle \gg \sigma^3$. By Hölder inequality with p large, Lemmata 5.4 and 5.2, and (4.11), we have

$$(7.4) \leq \|u^1\|_{L^p} \|u^2\|_{L^p} \|u^3\|_{L^{2+}} \|v\|_{L^2} \lesssim (N^1)^{1-2\alpha+} \|u^0\|_{X^{0,0+}} \|v\|_{X^{0,0}} \\ \lesssim \delta^{\frac{1}{2}-} (N^1)^{1-2\alpha+} N^{-s} K \|v\|_{X^{0,0}}$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$. Hence, (7.1) follows as long as $\langle \tau - n^2 \rangle^{\frac{1}{2}-} \gtrsim (N^1)^{1-2\alpha+} N^{-2s+\gamma+}$. Similar results hold if $\sigma^3 \gtrsim \langle \tau - n^2 \rangle$, (or u^2 is of type (II) and u^3 is of type (I).) Hence, we assume

$$(7.9) \quad \langle \tau - n^2 \rangle \ll (N^1)^{4-8\alpha+}, \text{ and } \langle \tau_j - n_j^2 \rangle \ll (N^1)^{4-8\alpha+} \text{ if } u_j \text{ of type (II).}$$

Summary: Given a function $v(x, t)$, we can write v as

$$(7.10) \quad v(x, t) = \int \langle \lambda \rangle^{-\frac{1}{2}} \left(\sum_n \langle \lambda \rangle |\widehat{v}(n, n^2 + \lambda)|^2 \right)^{\frac{1}{2}} \left\{ e^{i\lambda t} \sum_n a_\lambda(n) e^{i(nx+n^2t)} \right\} d\lambda$$

where $a_\lambda(n) = \frac{\widehat{v}(n, n^2 + \lambda)}{(\sum_m |\widehat{v}(m, m^2 + \lambda)|^2)^{\frac{1}{2}}}$. Note that $\sum_n |a_\lambda(n)|^2 = 1$. For $\|v\|_{X^{0, \frac{1}{2}}} \leq C$, we have

$$(7.11) \quad \int_{|\lambda| < K} \langle \lambda \rangle^{-\frac{1}{2}} \left(\sum_n \langle \lambda \rangle |\widehat{v}(n, n^2 + \lambda)|^2 \right)^{\frac{1}{2}} d\lambda \lesssim C(\log K)^{\frac{1}{2}}$$

by Hölder inequality. See [5, (22) and (23)]. Note that (7.10) is a standard representation for functions in $X^{s, b}$ for $b > \frac{1}{2}$. e.g. Klainerman-Selberg [15]. We have a logarithmic loss in (7.10) since $b = \frac{1}{2}$ in our case.

Now, consider (7.1) on a small time interval $[-\delta, \delta]$. First, replace $\widehat{\mathcal{N}}_1$ with $\widehat{\eta}_\delta * \widehat{\mathcal{N}}_1$. Then, by Hölder and Young's inequalities, we have, for each $n \in \mathbb{Z}$,

$$\|\langle \tau - n^2 \rangle^{-\frac{1}{2}+} \widehat{\eta}_\delta * \widehat{\mathcal{N}}_1(n, \cdot)\|_{L_\tau^2} \leq \|\langle \tau - n^2 \rangle^{-\frac{1}{2}+}\|_{L_\tau^{2+}} \|\widehat{\eta}_\delta\|_{L_\tau^{2-}} \|\widehat{\mathcal{N}}_1(n, \cdot)\|_{L_\tau^2} \lesssim \delta^{\frac{1}{2}-} \|\widehat{\mathcal{N}}_1(n, \cdot)\|_{L_\tau^2}.$$

Then, letting $* = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3, n_2 \neq n_1, n_3\}$ and $** = \{(\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 : \tau = \tau_1 - \tau_2 + \tau_3\}$, we have

$$\text{LHS of (7.1)} \lesssim \delta^{\frac{1}{2}-} \|\mathcal{N}_1\|_{X^{s, 0, \delta}} \leq \delta^{\frac{1}{2}-} \left\| \sum_* \left\| \int_{**} \prod_{j=1}^3 \widehat{u}_j(n_j, \tau_j) d\tau_1 d\tau_2 \right\|_{L_\tau^2} \right\|_{L_n^2}$$

where $\widehat{u}_j(n_j, \tau_j) = \frac{g_{n_j}(\omega) \delta_0(\tau_j - n_j^2)}{1 + |n_j|^\alpha}$ or

$$\widehat{u}_j(n_j, \tau_j) = \int_{\{|\lambda_j| < K\}} \langle \lambda_j \rangle^{-\frac{1}{2}} c_j(\lambda_j) a_{\lambda_j}(n_j) \delta_0(\tau_j - n_j^2 - \lambda_j) d\lambda_j$$

with $\sum_{n_j} \langle n_j \rangle^{2s} |a_{\lambda_j}(n_j)|^2 \leq 1$ and $c_j(\lambda_j) = \left(\sum_{n_j} \langle \lambda_j \rangle |\widehat{u}_j(n_j, n_j^2 + \lambda_j)|^2 \right)^{\frac{1}{2}}$, where δ_0 is the Dirac delta function supported at 0. Therefore, we can reduce the estimate into the following two cases.

• u^1 is of type (II): By (7.10) and (7.11), we can bound (7.1) as follows:

$$(7.12) \quad (7.1) \lesssim \delta^{\frac{1}{2}-} M(N, N^2, N^3) \left(\sum_n \left| \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3 \\ n^2=n_1^2-n_2^2+n_3^2+\mu}} a_1(n_1) \overline{a_2(n_2)} a_3(n_3) \right|^2 \right)^{\frac{1}{2}},$$

where $\sum_n |a^1(n)|^2 \leq 1$, $a^2(n) = \frac{g_{n^2}(\omega)}{1 + |n^2|^\alpha}$, $a^3(n) = \frac{g_{n^3}(\omega)}{1 + |n^3|^\alpha}$ or $\sum_{|n| \sim N^3} |a^3(n)|^2 \leq 1$, and

Case (A): $M(N, N^2, N^3) = (N^3)^{0+} N^{-2s}$ and $|\mu| \ll (N^3)^{8-16\alpha+}$

Subcase (B.1): $M(N, N^2, N^3) = (N^2)^{0+} N^{-2s}$ and $|\mu| \ll (N^2)^{8-16\alpha+}$

Subcase (B.2): $M(N, N^2, N^3) = (N^2)^{0+} N^{-s}$ and $|\mu| \ll (N^2)^{4-8\alpha+}$.

Note that we did not apply dyadic decomposition on N^1 .

• u^1 is of type (I): By (7.10) and (7.11), we can bound (7.1) as follows:

$$(7.13) \quad (7.1) \lesssim \delta^{\frac{1}{2}-} (N^1)^{0+} M(N) \left(\sum_{|n| \lesssim N^1} \left| \sum_{\substack{n=n_1-n_2+n_3 \\ n_2 \neq n_1, n_3 \\ n^2=n_1^2-n_2^2+n_3^2+\mu}} a_1(n_1) \overline{a_2(n_2)} a_3(n_3) \right|^2 \right)^{\frac{1}{2}},$$

where $a^1(n) = \frac{g_{n_1}(\omega)}{1+|n_1|^\alpha}$, $a^j(n) = \frac{g_{n_j}(\omega)}{1+|n_j|^\alpha}$ or $\sum_{|n| \sim N^j} |a^j(n)|^2 \leq 1$ for $j = 2, 3$, and

$$\text{Case (C): } M(N) = N^{-2s} \text{ and } |\mu| \ll (N^1)^{5-10\alpha+}$$

$$\text{Case (D): } M(N) = N^{-s} \text{ and } |\mu| \ll (N^1)^{4-8\alpha+}$$

$$\text{All type (I): } M(N) = 1 \text{ and } |\mu| \lesssim (N^1)^2.$$

Note that all the spatial frequencies are dyadically decomposed.

By symmetry between u_1 and u_3 , we assume $|n_1| \sim N^1$ or $|n_2| \sim N^1$ in the following. Moreover, in Subcase (B.2) and Case (D), we may assume that $|n_1| \sim N^1$. If not, say we have $|n_2| > 10(|n_1| + |n_3|)$. Then, $|\mu| \sim |(n_2 - n_1)(n_2 - n_3)| \sim |n_2|^2 \sim (N^1)^2$ by (5.1). In these two cases, we have $|\mu| \ll (N^j)^{4-8\alpha+} \ll (N^1)^2$ as long as $\alpha > \frac{1}{4}$. i.e. we would have a contradiction.

Lastly, we list all the different cases following [5] and [13]. We consider these cases in details in the next section.

• $n_1 = N^1$:

$$\text{Case (a): } n_1 = N^1(\text{II}), n_2 = N^2(\text{I}), n_3 = N^3(\text{II}) \text{ or } n_2 = N^3(\text{I}), n_3 = N^2(\text{II}).$$

$$\text{Case (b): } n_1 = N^1(\text{II}), n_2 = N^3(\text{II}), n_3 = N^2(\text{I}) \text{ or } n_2 = N^2(\text{II}), n_3 = N^3(\text{I}).$$

$$\text{Case (c): } n_1 = N^1(\text{I}), n_2 = N^2(\text{II}), n_3 = N^3(\text{II}).$$

$$\text{Case (d): } n_1 = N^1(\text{I}), n_2 = N^3(\text{II}), n_3 = N^2(\text{II}).$$

$$\text{Case (e): } n_1 = N^1(\text{II}), n_2 = N^2(\text{I}), n_3 = N^3(\text{I}).$$

$$\text{Case (f): } n_1 = N^1(\text{II}), n_2 = N^3(\text{I}), n_3 = N^2(\text{I}).$$

$$\text{Case (g): } n_1 = N^1(\text{I}), n_2 = N^2(\text{I}), n_3 = N^3(\text{II}).$$

$$\text{Case (h): } n_1 = N^1(\text{I}), n_2 = N^3(\text{I}), n_3 = N^2(\text{II}).$$

$$\text{Case (i): } n_1 = N^1(\text{I}), n_2 = N^2(\text{II}), n_3 = N^3(\text{I}).$$

$$\text{Case (j): } n_1 = N^1(\text{I}), n_2 = N^3(\text{II}), n_3 = N^2(\text{I}).$$

$$\text{Case (k): All type (I).}$$

• $n_2 = N^1$:

$$\text{Case (a)': } n_2 = N^1(\text{II}), n_1 = N^2(\text{I}), n_3 = N^3(\text{II}) \text{ or } n_1 = N^3(\text{I}), n_3 = N^2(\text{II}).$$

$$\text{Case (b)': } n_2 = N^1(\text{II}), n_1 = N^3(\text{II}), n_3 = N^2(\text{I}) \text{ or } n_1 = N^2(\text{II}), n_3 = N^3(\text{I}).$$

$$\text{Case (c)': } n_2 = N^1(\text{I}), n_1 = N^2(\text{II}), n_3 = N^3(\text{II}).$$

$$\text{Case (d)': } n_2 = N^1(\text{I}), n_1 = N^3(\text{II}), n_3 = N^2(\text{II}).$$

$$\text{Case (k)': All type (I).}$$

8. ESTIMATE ON \mathcal{N}_1 : ALL DIFFERENT CASES

For notational simplicity, we use $|n|^\alpha$ for $1 + |n|^\alpha$. We may drop the complex conjugate on u_2 when it plays no significant role. Now, let

$$A_n = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3, |n_j| \sim N_j, j = 2, 3, \\ n_2 \neq n_1, n_3, \text{ and } n^2 = n_1^2 - n_2^2 + n_3^2 + \mu\}$$

and $B_n = A_n \cap \{|n_1| \sim N_1\}$. Also, recall

$$(8.1) \quad \mu = 2(n_2 - n_1)(n_2 - n_3) = 2(n - n_1)(n - n_3),$$

$\delta \sim N^{4s-K-4-}$ from (4.1), $s = \alpha - \frac{1}{2}-$, and $N_j > N$ if u_j is of type (I).

• **Cases (k), (k')**: u_1, u_2, u_3 of type (I). In this case, we have

$$(8.2) \quad (7.13) \lesssim \delta^{\frac{1}{2}-} (N^1)^{0+} \left(\sum_{|n| \lesssim N^1} \left| \sum_{B_n} \frac{g_{n_1}}{|n_1|^\alpha} \frac{\overline{g_{n_2}}}{|n_2|^\alpha} \frac{g_{n_3}}{|n_3|^\alpha} \right|^2 \right)^{\frac{1}{2}}.$$

First, we consider the contribution from $n_1 \neq n_3$. Let

$$F_n(\omega) := \sum_{C_n} \frac{g_{n_1}(\omega)}{|n_1|^\alpha} \frac{\overline{g_{n_2}(\omega)}}{|n_2|^\alpha} \frac{g_{n_3}(\omega)}{|n_3|^\alpha},$$

where $C_n = B_n \cap \{n_1 \neq n_3\}$. Then, by hypercontractivity property related to the product of Gaussians (cf. [23, Propositions 3.1 and 3.3]), we have $\|F_n\|_{L^p(\Omega)} \leq p^{\frac{3}{2}} \|F_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, it follows from Lemma 4.5 in [23] that $\mathbb{P}(|F_n(\omega)| \geq \lambda) \leq \exp(-c' \|F_n\|_{L^2(\Omega)}^{-\frac{2}{3}} \lambda^{\frac{2}{3}})$. By choosing $\lambda = \delta^{-\frac{3}{2}c} (N^1)^{\frac{3}{2}\varepsilon} \|F_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$(8.3) \quad \mathbb{P}(|F_n(\omega)| \geq \delta^{-\frac{3}{2}c} (N^1)^{\frac{3}{2}\varepsilon} \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{c'(N^1)^\varepsilon}{\delta^c}}.$$

In this case, we have $N_1, N_2, N_3 > N$. Then, by Lemma 5.1, we have

$$\begin{aligned} \text{RHS of (8.2)} &\lesssim \delta^{\frac{1}{2}-\frac{3}{2}c-} (N^1)^{\frac{3}{2}\varepsilon+} \left(\sum_{|n| \lesssim N^1} \sum_{C_n} \frac{1}{|n_1|^{2\alpha} |n_2|^{2\alpha} |n_3|^{2\alpha}} \right)^{\frac{1}{2}} \\ &\lesssim \delta^{\frac{1}{2}-\frac{3}{2}c-} (N^1)^{-\alpha+\frac{3}{2}\varepsilon+} (N^2)^{-\alpha} (N^3)^{-\alpha+\frac{1}{2}} \lesssim N^{2s-3\alpha+\frac{1}{2}+} \leq N^{3s-\gamma-} \prod_{j=1}^3 N_j^{0-} \end{aligned}$$

for $\alpha > \frac{1}{4} + \frac{1}{4}\gamma > \frac{1}{4}$ and sufficiently large N outside an exceptional set of measure

$$< \sum_{|n| \lesssim N^1} e^{-\frac{c'(N^1)^\varepsilon}{\delta^c}} \lesssim N^1 e^{-\frac{c'(N^1)^\varepsilon}{\delta^c}} \leq (N^1)^{0-} e^{-\frac{c'}{\delta^c} (N^1)^\varepsilon + (1+)\log(N^1)} < (N^1)^{0-} e^{-\frac{1}{\delta^c r}}.$$

Note that in this case we need to make sure that the measures of these exceptional sets corresponding to different dyadic blocks are indeed summable and bounded by $e^{-\frac{1}{\delta^c}}$. We may not be explicit about this point in other cases. e.g. Cases (A)–(D) in Section 7. We do not encounter this issue in using Lemma 5.3 since it gives one exceptional set of measure $< e^{-\frac{1}{\delta^c}}$ for all the frequencies.

Now, consider the contribution from $n_1 = n_3$. It follows from (8.1) that there is at most one choice of (n_1, n_2, n_3) for each fixed n . Thus, $\sum_{|n| \lesssim N^1} \left| \sum_{B_n, n_1=n_3} 1 \right|^2 = \sum_{|n| \lesssim N^1} \sum_{B_n, n_1=n_3} 1$. Hence, by Lemmata 5.1 and 5.3, we have

$$\text{RHS of (8.2)} \lesssim \delta^{\frac{1}{2}-\frac{3}{2}\beta-} (N^1)^{0+} N_1^{-2\alpha+2\varepsilon} N_2^{-\alpha+\varepsilon} (N^3)^{\frac{1}{2}} \leq N^{3s-\gamma-} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > \frac{1}{4} + \frac{1}{4}\gamma > \frac{1}{4}$ and sufficiently large N outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$.

• **Case (a)**: (Cases (b), (a'), and (b')) can be treated in a similar way by replacing n_2 with n_3 , n_2 with n_1 , and (n_1, n_2, n_3) with (n_2, n_3, n_1) , respectively.)

In this case, we have $\mu = 2(n_2 - n_1)(n_2 - n_3) = o((N_2)^{8-16\alpha+})$. This implies that $|n|, |n_1|, |n_3| \lesssim N_2^q$ for some $q > 0$ since $n_2 \neq n_1, n_3$. Now, let

$$G_n(\omega) := \sum_{A_n} a_1(n_1) \frac{\overline{g_{n_2}(\omega)}}{|n_2|^\alpha} a_3(n_3),$$

Note that the above sum is over a finite set. Letting $A_n = \cup_{|n_2| \sim N_2} A_{n, n_2}$ (i.e. a disjoint union over distinct n_2), write

$$G_n(\omega) = \sum_{n_2} c_{n, n_2} \overline{g_{n_2}}, \quad \text{where } c_{n, n_2} = \sum_{A_{n, n_2}} |n_2|^{-\alpha} a_1(n_1) a_3(n_3).$$

Then, by hypercontractivity of the Gaussians, we have $\|G_n\|_{L^p(\Omega)} \leq p^{\frac{1}{2}} \|G_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, by Lemma 4.5 in [23], we have $\mathbb{P}(|G_n(\omega)| \geq \lambda) \leq \exp(-c' \|G_n\|_{L^2(\Omega)}^{-2} \lambda^2)$. By choosing $\lambda = \delta^{-\frac{1}{2}c} (N^2)^\varepsilon \|G_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$(8.4) \quad \mathbb{P}(|G_n(\omega)| \geq \delta^{-\frac{1}{2}c} (N^2)^\varepsilon \|G_n\|_{L^2(\Omega)}) \leq e^{-\frac{c'(N^2)^{2\varepsilon}}{\delta^c}}.$$

Note that $(N^2)^\varepsilon \lesssim N_2^{q\varepsilon}$. By Hölder inequality (note that $\#A_{n, n_2} = O(N_2^{0+})$), we have

$$(7.12) \lesssim \delta^{\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{-\alpha+q\varepsilon} N^{-2s} \left(\sum_n \sum_{A_n} |a_1(n_1)|^2 |a_3(n_3)|^2 \right)^{\frac{1}{2}} \\ \leq \delta^{\frac{1}{2}-\frac{\varepsilon}{2}} N_2^{-\alpha+q\varepsilon} N^{-2s} \lesssim N^{3s-\gamma-} N_2^{0-} N_3^{0-}$$

for $\alpha > \frac{3}{8} + \frac{1}{4}\gamma > \frac{3}{8}$ and sufficiently large N outside an exceptional set of measure $\lesssim N_2^q e^{-\frac{c'(N^2)^{2\varepsilon}}{\delta^c}}$. Note that the factor N_2^q appears since we need to sum up the measures of the exceptional sets for each n .

- **Case (c)** : (Cases (d), (c'), and (d') are basically the same.)

By Lemma 5.3 and Hölder inequality on n_3 in the inner sum,

$$(7.13) \lesssim \delta^{\frac{1}{2}-\frac{\beta}{2}} N_1^{-\alpha+\varepsilon} N^{-2s} \left(\sum_{|n| \lesssim N^1} \sum_{B_n} |a_2(n_2)|^2 \right)^{\frac{1}{2}}$$

outside an exceptional set of measure $< e^{-\frac{1}{\delta^c}}$. For fixed n_2 , it follows from (the proof of) Lemma 5.1 that there are at most N_1^{0+} terms in the sum. Hence, we have

$$(7.13) \lesssim \delta^{\frac{1}{2}-\frac{\beta}{2}} N_1^{-\alpha+\varepsilon} N^{-2s} \lesssim N^{3s-\gamma-} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > \frac{3}{8} + \frac{1}{4}\gamma > \frac{3}{8}$ and sufficiently large N .

- **Case (e)** : (Case (f) is basically the same.)

In this case, we have $N_2^{4-8\alpha+} \gg |\mu| = |2(n_2 - n_1)(n_2 - n_3)|$. This implies that $|n_1|, |n| \lesssim N_2^q$ for some $q > 0$. Now, let

$$F_n(\omega) := \sum_{A_n} a_1(n_1) \frac{\overline{g_{n_2}(\omega)}}{|n_2|^\alpha} \frac{g_{n_3}(\omega)}{|n_3|^\alpha},$$

By the observation above, this is a finite sum. Then, by hypercontractivity property related to the product of two independent Gaussians (recall $n_2 \neq n_3$), we have $\|F_n\|_{L^p(\Omega)} \leq p \|F_n\|_{L^2(\Omega)}$ for all $p \geq 2$. Hence, it follows from Lemma 4.5 in [23] that

$\mathbb{P}(|F_n(\omega)| \geq \lambda) \leq \exp(-c' \|F_n\|_{L^2(\Omega)}^{-1} \lambda)$. By choosing $\lambda = \delta^{-c} N_2^\varepsilon \|F_n\|_{L^2(\Omega)}$ with $\varepsilon = 0+$, we have

$$\mathbb{P}(|F_n(\omega)| \geq \delta^{-c} N_2^\varepsilon \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{c' N_2^\varepsilon}{\delta^{c'}}}.$$

It follows from the proof of Lemma 5.1 and $|n_1| \lesssim N_2^q$ that, for fixed n_1 , there are at most $O(N_2^{0+})$ many choices for n_2 and n_3 . Thus, we have

$$(7.12) \lesssim \delta^{\frac{1}{2}-c-} N_2^{-\alpha+\varepsilon+} N_3^{-\alpha} N^{-s} \left(\sum_{|n| \lesssim N_2^q} \sum_{A_n} |a_1(n_1)|^2 \right)^{\frac{1}{2}} \\ \lesssim \delta^{\frac{1}{2}-c-} N_2^{-\alpha+\varepsilon+} N_3^{-\alpha+} N^{-s} \leq N^{3s-\gamma-} N_2^{0-} N_3^{0-}$$

for $\alpha > \frac{1}{4} + \frac{1}{4}\gamma > \frac{1}{4}$ and sufficiently large N outside an exceptional set of measure

$$\lesssim N_2^q e^{-\frac{c' N_2^\varepsilon}{\delta^{c'}}} \leq N_2^{0-} e^{-\frac{c'}{\delta^{c'}} N_2^\varepsilon - (q+) \log(N_2)} < N_2^{0-} e^{-\frac{1}{\delta^{c'}}}.$$

• **Case (g)** : (Case (h) is basically the same.)

Let $F_n(\omega) := \sum_{B_n} \frac{g_{n_1}(\omega)}{|n_1|^\alpha} \frac{g_{n_2}(\omega)}{|n_2|^\alpha} a_3(n_3)$. As in Case (e), we use the hypercontractivity property related to the product of two independent Gaussians (recall $n_1 \neq n_2$.) Then, we have

$$\mathbb{P}(|F_n(\omega)| \geq \delta^{-c} N_1^\varepsilon \|F_n\|_{L^2(\Omega)}) \leq e^{-\frac{c' N_1^\varepsilon}{\delta^{c'}}}.$$

By summing over n_1 for fixed n_3 and then over n_3 , it follows from (the proof of) Lemma 5.1 that

$$(7.13) \lesssim \delta^{\frac{1}{2}-c-} N_1^{-\alpha+\varepsilon+} N_2^{-\alpha} N^{-s} \left(\sum_{|n| \lesssim N_1} \sum_{B_n} |a_3(n_3)|^2 \right)^{\frac{1}{2}} \\ \lesssim \delta^{\frac{1}{2}-c-} N_1^{-\alpha+\varepsilon+} N_2^{-\alpha} N^{-s} \lesssim N^{3s-\gamma-} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > \frac{1}{4} + \frac{1}{4}\gamma > \frac{1}{4}$ and sufficiently large N outside an exceptional set of measure $< N_1^{0-} e^{-\frac{1}{\delta^{c'}}$.

• **Cases (i), (j)** : The contribution for $n_1 \neq n_3$ in Case (i) (and in Case (j)) can be estimated as in Case (h) (and as in Case (g), respectively) using the hypercontractivity. Now, assume $n_1 = n_3$. It follows from (8.1) that there are at most two choices of (n_1, n_2, n_3) for each fixed n . Thus, $\sum_{|n| \lesssim N_1} \left| \sum_{B_n, n_1=n_3} |a_2(n_2)| \right|^2 \sim \sum_{|n| \lesssim N_1} \sum_{B_n, n_1=n_3} |a_2(n_2)|^2$. By Lemma 5.3 and by summing over n_2 , we have

$$(7.13) \lesssim \delta^{\frac{1}{2}-c-} N_1^{-2\alpha+2\varepsilon+} N^{-s} \left(\sum_{|n| \lesssim N_1} \sum_{\substack{B_n \\ n_1=n_3}} |a_2(n_2)|^2 \right)^{\frac{1}{2}} \\ \lesssim \delta^{\frac{1}{2}-c-} N_1^{-2\alpha+2\varepsilon+} N^{-s} \leq N^{3s-\gamma-} \prod_{j=1}^3 N_j^{0-}$$

for $\alpha > \frac{1}{4} + \frac{1}{4}\gamma > \frac{1}{4}$ and sufficiently large N outside an exceptional set of measure $< e^{-\frac{1}{\delta^{c'}}$.

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