MODULATION SPACES, WIENER AMALGAM SPACES, AND BROWNIAN MOTIONS

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Abstract. We study the local-in-time regularity of the Brownian motion with respect to the modulation spaces $M_s^{p,q}$ and Wiener amalgam spaces $W_s^{p,q}$. We show that the Brownian motion belongs locally in time to $M_s^{p,q}$ and $W_s^{p,q}$ for $(s-1)q < -1$, and the condition on the indices is optimal. Moreover, with the Wiener measure μ on $\mathbb{T}$, we show that $(M_s^{p,q}, \mu)$ and $(W_s^{p,q}, \mu)$ form abstract Wiener spaces for the same range of indices, yielding large deviation estimates. We also establish the endpoint regularity of the Brownian motion with respect to a Besov-type space $\widehat{b}^s_{p,\infty}$. Specifically, we prove that the Brownian motion belongs to $\widehat{b}^s_{p,\infty}$ for $(s-1)p = -1$, and it obeys a large deviation estimate. Finally, we revisit the regularity of Brownian motion on usual Besov spaces $B^s_{p,q}$, and indicate the endpoint large deviation estimates.

Contents

1. Introduction

Modulation spaces were born during the early eighties in pioneering work of H. Feichtinger. In subsequent fruitful collaborations with K. Gröchenig [10], [11] they established the basic theory of these function spaces, in particular their invariance, continuity, embeddings, and convolution properties. In contrast with the Besov spaces, which are defined by a dyadic decomposition of the frequency space, modulation spaces (and Wiener amalgam spaces) arise from a uniform partition of the frequency space. Their appeal is due

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to the fact that they can effectively capture the time-frequency concentration of a distribution, and, lately, modulation spaces have established themselves as the "right" function spaces in time-frequency analysis. Several authors have contributed their share to this enterprise. Indeed, the list is extensive, and we cannot hope to acknowledge here all those who made the theory of modulation spaces such a successful story. We will simply contend ourselves to mention that it is nowadays common to see these spaces appearing in investigations that concern problems in signal analysis, physics, boundedness of Fourier multipliers and pseudodifferential operators, localization operators, well-posedness of nonlinear partial differential equations, Strichartz estimates, Fourier integral operators, almost diagonal estimates, changes of variables, and so on.

The main purpose of this article is to unravel a new bridge from the spaces of time-frequency analysis (modulation, Wiener amalgam) to the interface between probability theory and partial differential equations. Specifically, we will establish the local regularity of the Brownian motion on appropriate modulation spaces and Wiener amalgam spaces, and prove that it obeys so called large deviation estimates. For the "end-point" results we appeal to a Besov-type space introduced by the second author in a series of works that were concerned with the invariance of the white noise for the KdV equation [21] and the stochastic KdV equation with additive space-time white noise [22]. We also revisit the local regularity of Brownian motion on the usual Besov spaces. A common thread throughout this work is the use of random Fourier series.

1.1. **Function spaces of time-frequency analysis.** Let us now recall some basic definitions regarding these function spaces; see, for example, Gröchenig's book [13]. Given a (fixed, non-zero) window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the short-time Fourier transform (STFT) $V_\phi f$ of a tempered distribution u is

$$V_\phi u(x,\xi) = \langle u, M_\xi T_x \phi\rangle = \mathcal{F}(u\overline{T_x\phi})(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot y}\overline{\phi(y-x)}u(y)dy.$$

Here, $\mathcal{F}g = \widehat{g}$ denotes the Fourier transform of a distribution g, while, for $x, \xi \in \mathbb{R}^d$, M_ξ, T_x denote the modulation and translation operators, respectively. For $s \in \mathbb{R}$, we let $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$.

The (continuous weighted) *modulation space* $M^{p,q}_s(\mathbb{R}^d), 1 \leq p, q \leq \infty$, consists of all tempered distributions $u \in \mathcal{S}(\mathbb{R}^d)$ such that

$$V_\phi u(x,\xi)\langle\xi\rangle^s \in L^q_\xi L^p_x,$$

and we equip the space $M^{p,q}_s(\mathbb{R}^d)$ with the norm

$$\|u\|_{M^{p,q}_s(\mathbb{R}^d)} = \big\|\|V_\phi u(x,\xi)\langle\xi\rangle^s\|_{L^p_x(\mathbb{R}^d)}\big\|_{L^q_\xi(\mathbb{R}^d)}.$$

It is not hard to see that these spaces are Banach spaces, two different windows yield equivalent norms, $M^{2,2}_s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d)$, $(M^{p,q}_s(\mathbb{R}^d))' = M^{p',q'}_{-s}(\mathbb{R}^d)$, $M^{p_1,q_1}_s(\mathbb{R}^d) \subset M^{p_2,q_2}_s(\mathbb{R}^d)$ for $p_1 \leq p_2$ and $q_1 \leq q_2$, and $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}_s(\mathbb{R}^d)$.

Closely related to these spaces are the (continuous weighted) *Wiener amalgam spaces* $W^{p,q}_s(\mathbb{R}^d)$ which are now equipped with the norm

$$\|u\|_{W^{p,q}_s(\mathbb{R}^d)} = \big\|\|V_\phi u(x,\xi)\langle\xi\rangle^s\|_{L^q_\xi(\mathbb{R}^d)}\big\|_{L^p_x(\mathbb{R}^d)}.$$

It is clear then that, in fact, $M^{p,q}_0(\mathbb{R}^d) = \mathcal{F}W^{q,p}_0(\mathbb{R}^d)$. As such, completely analogous comments to the one following the definition of modulation spaces hold for Wiener amalgam spaces as well. For example, $W^{p_1,q_1}_s(\mathbb{R}^d) \subset W^{p_2,q_2}_s(\mathbb{R}^d)$ for $p_1 \leq p_2$ and $q_1 \leq q_2$, $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{p,q}_s(\mathbb{R}^d)$, and so on. Moreover, $M^{p,q}_s(\mathbb{R}^d) \subset W^{p,q}_s(\mathbb{R}^d)$ for $q \le p$, while the reverse inclusion holds if $p \le q$.

The (weighted) *Fourier-Lebesgue spaces* $\mathcal{F}L^{s,p}(\mathbb{R}^d)$ are defined via the norm

$$\|u\|_{\mathcal{F}L^{s,p}(\mathbb{R}^d)} = \|\langle\xi\rangle^s \widehat{u}(\xi)\|_{L^p_\xi(\mathbb{R}^d)} \tag{1.1}$$

For our purposes, we will use the following equivalent definitions for the norms of modulation, respectively Wiener amalgam spaces. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\psi \subset [-1,1]^d$ and $\sum_{n\in\mathbb{Z}^d} \psi(\xi - n) \equiv 1$. Then,

$$\|u\|_{M^{p,q}_s(\mathbb{R}^d)} = \big\|\langle n\rangle^s \|\psi(D-n)u\|_{L^p_x(\mathbb{R}^d)}\big\|_{l^q_n(\mathbb{Z}^d)} \tag{1.2}$$

and

$$\|u\|_{W^{p,q}_s(\mathbb{R}^d)} = \big\|\|\langle n\rangle^s \psi(D-n)u\|_{l^q_n(\mathbb{Z}^d)}\big\|_{L^p_x(\mathbb{R}^d)}. \tag{1.3}$$

One should contrast these definitions with the one of *Besov spaces*. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi_0 \subset \{|\xi| \le 2\}$, $\operatorname{supp}\varphi \subset \{\frac{1}{2} \le |\xi| \le 2\}$, and $\varphi_0(\xi) + \sum_{j=1}^\infty \varphi(2^{-j}\xi) \equiv 1$. With $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, we define the usual Besov space $B^s_{p,q}$ via the norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^d)} = \big\|2^{jsq}\|\varphi_j(D)u\|_{L^p(\mathbb{R}^d)}\big\|_{l^q_j(\mathbb{Z})}. \tag{1.4}$$

There are several known embeddings between Besov, Sobolev and modulation spaces; see, for example, the works by Okoudjou [24] and Toft [31].

Our goal is to investigate the local-in-time regularity of the Brownian motion. As such, we can restrict ourselves, without any loss of generality, to the study of the periodic Brownian motion; see Subsection 2.2. There are several results dealing with the local regularity of Brownian motion on Sobolev and Besov spaces, which we summarize in Subsection 2.1. In Subsection 2.2, we will establish similar results for modulation and Wiener amalgam spaces. In fact, we will do much more (in Section 3) by showing a nice connection to abstract Wiener spaces, and, in particular, their large deviation estimates. Moreover, in Subsection 2.3 we reprove some known results for Besov spaces by working, as for the other spaces of time-frequency analysis, on the Fourier side (compared to previous proofs which were done on the physical side). From this perspective, the use of Fourier-Wiener series (see (2.2) below) can be viewed as a unifying theme of this article. Since we are only concerned with the study of periodic Brownian motions, we need a "localized" version of the time-frequency spaces defined above.

The modulation and Wiener amalgam spaces on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ were first introduced and studied by Ruzhansky, Sugimoto, Toft, and Tomita in [27]. For some ψ with compact support in the discrete topology of $\mathbb{Z}^d$, we let

$$\|u\|_{M^{p,q}_s(\mathbb{T}^d)} = \big\|\langle n\rangle^s \|\psi(D-n)u\|_{L^p_x(\mathbb{T}^d)}\big\|_{l^q_n(\mathbb{Z}^d)} \tag{1.5}$$

and

$$\|u\|_{W^{p,q}_s(\mathbb{T}^d)} = \big\|\|\langle n\rangle^s \psi(D-n)u\|_{l^q_n(\mathbb{Z}^d)}\big\|_{L^p_x(\mathbb{T}^d)}. \tag{1.6}$$

It is not hard to see that for $1 \le p, q \le \infty$, the following equalities of these "localized" spaces hold:

$$M^{p,q}_s(\mathbb{T}^d) = W^{p,q}_s(\mathbb{T}^d) = \mathcal{F}L^{s,q}(\mathbb{T}^d) = \mathcal{F}l^q_s.$$

For the remainder of the paper, whenever we refer to modulation spaces, Wiener amalgam spaces, or Fourier Lebesgue spaces we tacitly assume that they are defined on the torus, and we will simply write $M_s^{p,q}$, $W_s^{p,q}$, or $\mathcal{F}L^{s,q}$. We will also use the Besov-type space $\widehat{b}^s_{p,q}(\mathbb{T})$ defined by the norm:

$$\|u\|_{\widehat{b}^s_{p,q}(\mathbb{T})} = \big\| \|\langle n\rangle^s \widehat{u}(n)\|_{L^p_{|n|\sim 2^j}} \big\|_{l^q_j}. \tag{1.7}$$

In particular, in our end-point regularity analysis we will need the space $\widehat{b}^s_{p,\infty}(\mathbb{T})$ defined via

$$\|u\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} = \sup_j \|\langle n\rangle^s \widehat{u}(n)\|_{L^p_{|n|\sim 2^j}} = \sup_j \bigg(\sum_{|n|\sim 2^j} \langle n\rangle^{sp} |\widehat{u}(n)|^p \bigg)^{1/p}, \tag{1.8}$$

where we used $|n| \sim 2^j$ to denote $2^{j-1} < |n| \leq 2^j$. This space is well-suited for the analysis of the white noise [21], since, in particular, for $sp \leq -1$ it contains its full support. The Hausdorff-Young inequality immediately implies that $B^s_{p',\infty} \subset \widehat{b}^s_{p,\infty}$ for $p > 2$, and the equality holds for $p = 2$, where $B^s_{p',\infty}$ is the usual Besov space and $p' = p/(p-1)$ denotes the dual exponent of p. Note also that $\mathcal{F}L^{s,p} = \widehat{b}^s_{p,p}$ for $p \geq 1$.

1.2. **Brownian motion.** Let $T > 0$ and (Ω, F, Pr) be a probability space. We define the *d-dimensional Brownian motion* (or Wiener process) as a stochastic process $\beta : [0,T]\times\Omega \to \mathbb{R}^d$ that satisfies the following properties:

(i) $\beta(0) = 0$ almost surely (a.s. $\omega \in \Omega$)
(ii) $\beta(t)$ has independent increments, and $\beta(t) - \beta(t')$ has the normal distribution with mean 0 and variance $t - t'$ (for $0 \leq t' \leq t$).

Here, we abused the notation and wrote, for a given $t \in [0,T]$, $\beta(t) : \Omega \to \mathbb{R}^d$, $\beta(t)(\omega) = \beta(t,\omega)$. This further induces a mapping Φ_β from Ω into the collection of functions from $[0,T]$ to $\mathbb{R}^d$ by $(\Phi_\beta(\omega))(t) = \beta(t)(\omega)$. The law of the Brownian motion, that is, the pushforward measure $(\Phi_\beta)_*(Pr)$ on Borel sets in $(\mathbb{R}^d)^{[0,T]}$, is nothing but the classical Wiener measure. Replacing $\mathbb{R}^d$ with some other Banach space will take us into the realm of abstract Wiener spaces. It is worth noting that, while there are several technicalities involved in properly defining a Wiener measure in this general context, the same point of view as in the classical $\mathbb{R}^d$ case is employed.

2. Regularity of Brownian motion

In what follows, we first discuss some known results about the regularity of Brownian motion on function spaces that are often used in PDEs, that is, Sobolev and Besov spaces. All spaces are considered local in time.

2.1. **Modulus of continuity.** It is well-known that $\beta(t)$ is almost surely (a.s.) continuous. Moreover, we have Lévy's theorem for the modulus of continuity:

$$\limsup_{\substack{t-t'=\varepsilon\downarrow 0\\ 0\leq t'<t\leq 1}} \frac{|\beta(t)-\beta(t')|}{\sqrt{-2\varepsilon\log\varepsilon}} = 1, \text{ a.s.} \tag{2.1}$$

It follows from (2.1) that the Brownian motion is a.s. locally Hölder continuous of order s for every $s < 1/2$, and is a.s. nowhere locally Hölder continuous of order s for $s \geq 1/2$; see Revuz and Yor's book [25].

Regularity on Sobolev spaces $H^s(0,1)$: By definition of the Brownian motion, we have

$$\mathbb{E}\big[|\beta(t)-\beta(t')|^2\big] = \mathbb{E}\big[|\beta(t-t')|^2\big] = |t-t'|,$$

where $\mathbb{E}$ denotes expectation. Thus, we have

$$\mathbb{E}\big[\|\beta(t)\|^2_{\dot{H}^s(0,1)}\big] = \mathbb{E}\bigg[\int_0^1\int_0^1 \frac{|\beta(t)-\beta(t')|^2}{|t-t'|^{1+2s}}dt'dt\bigg] = \int_0^1\int_0^1 \frac{1}{|t-t'|^{2s}}dt'dt$$
$$= 2\int_0^1\int_0^t \frac{1}{(t-t')^{2s}}dt'dt < \infty,$$

if and only if $s < \frac{1}{2}$. Hence, the Brownian motion $\beta(t)$ belongs almost surely to H^s_{loc} for $s < \frac{1}{2}$. Indeed, one can also show that $\beta(t) \notin H^s_{\text{loc}}$ a.s. for $s \geq \frac{1}{2}$.

Regularity on Sobolev spaces $\mathcal{W}^{s,p}(0,1)$:

Recall that we have, for $p > 0$,

$$\mathbb{E}\big[|\beta(t)-\beta(t')|^p\big] = C_p|t-t'|^{\frac{p}{2}},$$

since $\beta(t)-\beta(t')$ is a mean-zero Gaussian with variance $|t-t'|$. Then, by the characterization of $\dot{\mathcal{W}}^{s,p}$ via the L^p modulus of continuity as in Tartar's book [30], we have

$$\mathbb{E}\big[\|\beta(t)\|^p_{\dot{\mathcal{W}}^{s,p}(0,1)}\big] = \mathbb{E}\bigg[\int_0^1\int_0^1 \frac{|\beta(t)-\beta(t')|^p}{|t-t'|^{1+sp}}dt'dt\bigg] \sim \int_0^1\int_0^1 |t-t'|^{-1-sp+\frac{p}{2}}dt'dt$$
$$= 2\int_0^1\int_0^t |t-t'|^{-1-sp+\frac{p}{2}}dt'dt < \infty,$$

if and only if $s < \frac{1}{2}$. Hence, the Brownian motion $\beta(t)$ belongs almost surely to $\mathcal{W}^{s,p}_{\text{loc}}$ for $s < \frac{1}{2}$.

Regularity on Besov spaces $B^s_{p,q}(0,1)$:

Ciesielski [8], [9] and Roynette [26] proved that if $s < \frac{1}{2}$, then the Brownian motion $\beta(t)$ belongs a.s. to $B^s_{p,q}(0,1)$ for all $p, q \geq 1$, and that if $s > \frac{1}{2}$, then $\beta(t) \notin B^s_{p,q}(0,1)$ a.s. for any $p, q \geq 1$. Regarding the endpoint regularity of the Brownian motion, it was also shown that for $s = \frac{1}{2}$, the Brownian motion $\beta(t) \in B^{\frac{1}{2}}_{p,q}(0,1)$ a.s. if and only if $1 \leq p < \infty$ and $q = \infty$. Moreover, if the latter holds, there exists $c_p > 0$ such that $\|\beta(t)\|_{B^{\frac{1}{2}}_{p,\infty}(0,1)} \geq c_p$ a.s. The proof is based on the Schauder basis representation (or Franklin-Wiener series) of the Brownian motion; see Kahane's book [16].

In Subsection 2.3, we will present an alternate proof of these results which is of interest in its own, by using random Fourier series (also referred to as Fourier-Wiener series). For now, however, we turn our attention to the regularity of the Brownian motion on modulation spaces.

2.2. **Fourier analytic representation.** We are interested in the local-in-time regularity of the periodic Brownian motion on modulation spaces (defined on the torus). Without loss of generality, it suffices to consider the mean-zero complex-valued Brownian loop, that is, satisfying both $\beta(0) = \beta(2\pi)$ and $\int_0^{2\pi}\beta(t)dt = 0$. The case of a Brownian loop with non-zero mean follows easily from the mean-zero case through a translation by the mean of the Brownian motion. Furthermore, if we let $b(t)$ denote any (non-periodic) Brownian motion, and set $\beta(t) = b(t) - tb(2\pi)/2\pi$, then $\beta(t)$ is periodic, and $b(t)$ has the same regularity as $\beta(t)$.

Since for almost every $\omega \in \Omega$ the Brownian motion $\beta(t)$ represents a continuous function, to simplify the notation, we will simply write u in the following to denote this function (for a fixed ω). It is known that u can be represented through a Fourier-Wiener series[1] as

$$u(t) = u(t;\omega) = \sum_{n\neq 0} \frac{g_n(\omega)}{n} e^{int}, \tag{2.2}$$

where $\{g_n\}$ is a family of independent standard complex-valued Gaussian random variables, that is, $\operatorname{Re} g_n$ and $\operatorname{Im} g_n$ are independent standard real-valued Gaussian random variables. Note that we are missing the linear term $g_0(\omega)t$ in the Fourier-Wiener series representation since we are only considering the (mean zero) periodic case.

With the notation above, our result can be stated as follows.

Theorem 2.1. *Let $1 \leq p, q \leq \infty$.*

(a) *If $q < \infty$, then $u \in M^{p,q}_s$ a.s. for $(s-1)q < -1$, and $u \notin M^{p,q}_s$ a.s. for $(s-1)q \geq -1$.*
(b) *If $q = \infty$, then $u \in M^{p,\infty}_s$ a.s. for $s < 1$, and $u \notin M^{p,\infty}_s$ a.s. for $s \geq 1$.*
   Moreover,
(c) *If $q < \infty$, then $u \in \widehat{b}^s_{p,q}$ a.s. for $(s-1)p < -1$, and $u \notin \widehat{b}^s_{p,q}$ a.s. for $(s-1)p \geq -1$.*
(d) *If $q = \infty$ and $p < \infty$, then $u \in \widehat{b}^s_{p,\infty}$ a.s. for $(s-1)p \leq -1$, and $u \notin \widehat{b}^s_{p,\infty}$ a.s. for $(s-1)p > -1$.*

Note that, in our statements regarding the Besov-type spaces $\widehat{b}^s_{p,q}$, the case $p = q = \infty$ is already addressed by part (b), since $\widehat{b}^s_{\infty,\infty} = \mathcal{F}L^{s,\infty}$. We think of the case in which $(s-1)p = -1$ as an "end-point" that makes the transition of regularity from $\mathcal{F}L^{s,p}$ to $\widehat{b}^s_{p,\infty}$. The case when $(s-1)p = -1$ and $p = 2$ (that is, $s = 1/2$) corresponds to the end-point case for the usual Besov spaces.

It is also worthwhile to note that the Besov-type spaces $\widehat{b}^s_{p,q}$ are more finely tuned for this analysis than the regular Besov spaces because of their sensitivity to the value of p under randomization. Roughly speaking, this is implied by the definitions of the two spaces: the $\widehat{b}^s_{p,q}$ spaces use the L^p_n norm on the Fourier side for each dyadic block, while the $B^s_{p,q}$ spaces use the L^p_t norm on the physical side for each dyadic block. Now, in the case of regular Besov spaces, the L^p_t norms are equivalent under the expectation, that is, the L^p_t spaces are equivalent under randomization due to a Khintchine type argument or Paley-Zygmund's theorem; see [16]. In other words, the L^p_t part is "insensitive" to a finite p under randomization.

Theorem 2.1 states that the Brownian motion belongs a.s. to $M^{p,q}_s$, $W^{p,q}_s$, and $\mathcal{F}L^{s,q}$ for $(s-1)q < -1$, and to $\widehat{b}^s_{p,q}$ for $(s-1)p < -1$, and to $\widehat{b}^s_{p,\infty}$ for $(s-1)p \leq -1$. However, in applications, it is often very important to know how large the estimate on the norm is likely to be; see the works by Bourgain [1, 2, 3, 4], Burq and Tzvetkov [5, 7], and the second author [20, 21, 23]. The following theorem provides us with the desirable "large deviation estimates".

Theorem 2.2. *There exists $c > 0$ such that for (sufficiently large) $K > 0$, the following holds:*

(i) *If $(s-1)q < -1$, then $Pr\big(\|u(\omega)\|_{M^{p,q}_s} > K\big) < e^{-cK^2}$.*
(ii) *If $(s-1)p < -1$, then $Pr\big(\|u(\omega)\|_{\widehat{b}^s_{p,q}} > K\big) < e^{-cK^2}$.*
(iii) *If $(s-1)p = -1$ (and $q = \infty$), then $Pr\big(\|u(\omega)\|_{\widehat{b}^s_{p,\infty}} > K\big) < e^{-cK^2}$.*

[1]Henceforth, we drop a factor of 2π when it plays no role.

The proofs of parts (i), (ii) in Theorem 2.2 rely on the theory of abstract Wiener spaces and Fernique's theorem. A detailed discussion of these proofs and the afferent technicalities is given in Section 3.

Proof of Theorem 2.1. We begin by showing statements (a) and (b). Recall that on the torus we have $M^{p,q}_s = W^{p,q}_s = \mathcal{F}L^{s,q}$, where we defined the latter space via the norm:

$$\|u\|_{\mathcal{F}L^{s,q}} = \|\langle n\rangle^s \widehat{u}(n)\|_{L^q_n}.$$

We easily see that

$$\mathbb{E}\big[\|u\|^q_{\mathcal{F}L^{s,q}}\big] = \sum_{n\neq 0}\langle n\rangle^{sq}|n|^{-q}\mathbb{E}\big[|g_n|^q\big] \sim \sum_{n\neq 0}\langle n\rangle^{(s-1)q} < \infty$$

if and only if $(s-1)q < -1$.

Let us now define

$$X^{(q)}_j(\omega) := 2^{-j}\sum_{|n|\sim 2^j}|g_n(\omega)|^q. \tag{2.3}$$

Then, $X^{(q)}_j \to c_q := \mathbb{E}|g_1|^q$ almost surely, since $Y^{(q)}_j := 2^{-j}\sum_{1\leq|n|\leq 2^{j-1}}|g_n|^q \to c_q$ by the strong law of large numbers, and $X^{(q)}_j = 2Y^{(q)}_{j+1} - Y^{(q)}_j$.

Suppose now that $(s-1)q \geq -1$. Then, we have

$$\begin{aligned}\|u\|^q_{\mathcal{F}L^{s,q}} &= \sum_{n\neq 0}\langle n\rangle^{sq}|n|^{-q}|g_n(\omega)|^q \sim \sum_{j=0}^\infty\sum_{|n|\sim 2^j}\langle n\rangle^{(s-1)q}|g_n(\omega)|^q \\ &\geq \sum_{j=0}^\infty\sum_{|n|\sim 2^j}\langle n\rangle^{-1}|g_n(\omega)|^q \sim \sum_{j=0}^\infty X^{(q)}_j(\omega) = \infty, \text{ a.s.}\end{aligned} \tag{2.4}$$

Hence, when $q < \infty$, $u \in \mathcal{F}L^{s,q}$ a.s. for $(s-1)q < -1$, and $u \notin \mathcal{F}L^{s,q}$ a.s. for $(s-1)q \geq -1$.

Now, let $q = \infty$. Then, we have

$$\|u\|_{\mathcal{F}L^{s,\infty}} = \sup_{n\neq 0}\langle n\rangle^s|n|^{-1}|g_n(\omega)| \sim \sup_{n\neq 0}\langle n\rangle^{s-1}|g_n(\omega)| < \infty,$$

a.s. for $s < 1$ (i.e. "$(s-1)\cdot\infty < -1$"), since $\lim_{n\to\infty} n^{-\varepsilon}|g_n(\omega)| = 0$ for any $\varepsilon > 0$. When $s \geq 1$, the continuity from below of the probability measure gives

$$Pr(\|u\|_{\mathcal{F}L^{s,\infty}} < \infty) \leq Pr(\sup_n|g_n(\omega)| < \infty) = \lim_{K\to\infty} Pr(\sup_n|g_n(\omega)| < K) = 0.$$

Hence, $u \in \mathcal{F}L^{s,\infty}$ a.s. for $s < 1$, and $u \notin \mathcal{F}L^{s,\infty}$ a.s. for $s \geq 1$.

Regarding (c) and (d), recall that we defined $\widehat{b}^s_{p,q}$ via the norm in (1.7). First, suppose $(s-1)p < -1$. Then, for $1 \leq p < \infty$ and $q \geq 1$, we have

$$\begin{aligned}\mathbb{E}\big[\|u\|^p_{\widehat{b}^s_{p,q}}\big] &\leq \mathbb{E}\big[\|u\|^p_{\widehat{b}^s_{p,1}}\big] \leq \bigg(\sum_{j=0}^\infty\Big(\sum_{|n|\sim 2^j}\langle n\rangle^{sp}|n|^{-p}\mathbb{E}[|g_n|^p]\Big)^{\frac{1}{p}}\bigg)^p \\ &\sim \bigg(\sum_{j=0}^\infty\Big(\sum_{|n|\sim 2^j}\langle n\rangle^{(s-1)p}\Big)^{\frac{1}{p}}\bigg)^p \sim \Big(\sum_{j=0}^\infty 2^{\frac{(s-1)p+1}{p}j}\Big)^p < \infty.\end{aligned} \tag{2.5}$$

Also, when $p, q < \infty$, $\|u\|_{\widehat{b}^s_{p,q}} = \infty$ a.s. for $(s-1)p \geq -1$ by modifying the argument in (2.4). When $p = \infty$ and $s < 1$, we have $\|u\|_{\widehat{b}^s_{\infty,q}} \leq \|u\|_{\widehat{b}^s_{r,q}}$ for $r < \infty$. Moreover, we can take r to be sufficiently large such that $(s-1)r < -1$. Then, it follows from (2.5) that $\|u\|_{\widehat{b}^s_{\infty,q}} < \infty$ a.s. For $s \geq 1$, we have

$$\|u\|_{\widehat{b}^s_{\infty,q}} \sim \big\| \sup_{|n|\sim 2^j} \langle n\rangle^{s-1}|g_n|\big\|_{l^q_j} \geq \Big\|2^{-j} \sum_{|n|\sim 2^j} |g_n|\Big\|_{l^q_j} = \infty, \text{ a.s.},$$

as in (2.4). Hence, $u \in \widehat{b}^s_{p,q}$ a.s. for $(s-1)p < -1$, while $u \notin \widehat{b}^s_{p,q}$ a.s. for $(s-1)p \geq -1$ and $q < \infty$.

Finally, we consider the case $q = \infty$ and $p < \infty$. In the endpoint case $(s-1)p = -1$, we have

$$\begin{aligned}\text{(2.6)} \qquad \|u\|^p_{\widehat{b}^s_{p,\infty}} &\sim \sup_j \sum_{|n|\sim 2^j} \langle n\rangle^{(s-1)p}|g_n(\omega)|^p \\ &\sim \sup_j 2^{-j} \sum_{|n|\sim 2^j} |g_n(\omega)|^p = \sup_j X^{(p)}_j(\omega) < \infty, \text{ a.s.},\end{aligned}$$

where $X^{(p)}_j$ is defined in (2.3). When $(s-1)p > -1$, a similar computation along with the convergence of $X^{(p)}_j$ shows that $u \notin \widehat{b}^s_{p,\infty}$ a.s. □

2.3. **Alternate proof for the Besov spaces.** We close this section by providing a new (alternate) proof, via Fourier-Wiener series (2.2), of the regularity results on Besov spaces that we exposed at the end of Subsection 2.1. We decided to include this proof because random Fourier series are a unifying theme of this paper. Furthermore, our proof, which is done on the Fourier side, seems to complement nicely the existing one using Franklin-Wiener series on the physical side [26].

We begin by recalling the general Gaussian bound

$$\text{(2.7)} \qquad \Big\| \sum_n c_n g_n(\omega)\Big\|_{L^p(\Omega)} \leq C\sqrt{p}\|c_n\|_{l^2_n};$$

see the works of Burq and Tzvetkov [6], and Tzvetkov [32]. Also, see Lemma 3.10 below. Then, we have, for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, (recall $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$),

$$\begin{aligned}\mathbb{E}\big[\|u\|_{B^s_{p,q}}\big] &= \mathbb{E}\Big\|\Big\| \sum_{|n|\sim 2^j} \langle n\rangle^s|n|^{-1}g_n e^{int}\Big\|_{L^p_t}\Big\|_{l^q_j} \leq \Big\|\Big\|\big\| \sum_{|n|\sim 2^j} \langle n\rangle^s|n|^{-1}g_n(\omega) e^{int}\big\|_{L^p(\Omega)}\Big\|_{L^p_t}\Big\|_{l^1_j} \\ &\lesssim \sum_{j=0}^\infty \Big(\sum_{|n|\sim 2^j} \langle n\rangle^{2(s-1)}\Big)^{\frac{1}{2}} < \infty\end{aligned}$$

for $2(s-1) < -1$, i.e. $s < \frac{1}{2}$. When $s < \frac{1}{2}$ and $p = \infty$, Sobolev's inequality gives $\|\langle\partial_t\rangle^s u\|_{L^\infty} \lesssim \|\langle\partial_t\rangle^{s+\varepsilon}u\|_{L^r}$, for small $\varepsilon > 0$ and large r such that $s+\varepsilon < \frac{1}{2}$ and $\varepsilon r > 1$. Then, the above computation shows that $\|u\|_{B^s_{\infty,q}} \lesssim \|u\|_{B^{s+\varepsilon}_{r,q}} < \infty$ a.s. for $s < \frac{1}{2}$.

Suppose $2 \leq p \leq \infty$. Then, for $q < \infty$ and $s \geq \frac{1}{2}$, we have

$$\text{(2.8)} \qquad \|u\|_{B^s_{p,q}} \geq \|u\|_{B^s_{2,q}} \sim \Big\|\Big(\sum_{|n|\sim 2^j} 2^{2(s-1)j}|g_n|^2\Big)^{\frac{1}{2}}\Big\|_{l^q_j} \sim \big\|\big(2^{(2s-1)j}X^{(2)}_j\big)^{\frac{1}{2}}\big\|_{l^q_j} = \infty, \text{ a.s.}$$

since $X_j^{(2)}$ defined in (2.3) converges to $c_2 > 0$ a.s. It also follows from (2.8) that $\|u\|_{B^s_{p,\infty}} = \infty$ a.s. for $s > \frac{1}{2}$ when $q = \infty$. Now, let $1 \leq p < 2$, $q < \infty$, and $s \geq \frac{1}{2}$. Then, we have

$$\|u\|^q_{B^s_{p,q}} \geq \|u\|^q_{B^{\frac{1}{2}}_{1,q}} \sim \sum_j Z_j^{(q)}(\omega),$$

where $Z_j^{(q)}(\omega) = \|\widetilde{X}_j(t,\omega)\|^q_{L^1_t}$ and

$$\widetilde{X}_j(t,\omega) := 2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n(\omega)e^{int}. \tag{2.9}$$

Note that for each $t \in \mathbb{T}$, $\widetilde{X}_j(t,\omega)$ is a standard complex Gaussian random variable. Thus, we have $\mathbb{E}\big[\|\widetilde{X}_j(t,\omega)\|_{L^1_t}\big] = \big\|\mathbb{E}\,|\widetilde{X}_j(t,\omega)|\big\|_{L^1_t} = 2\pi c_1 > 0$. In particular, we have

$$\big(\mathbb{E}\big[Z_j^{(q)}\big]\big)^{\frac{1}{q}} \geq \mathbb{E}\big[\|\widetilde{X}_j(t,\omega)\|_{L^1_t}\big] = 2\pi c_1 \tag{2.10}$$

Also, by (2.7), we have $\mathbb{E}\big[|Z_j^{(q)}|^2\big] \leq \big\|\|\widetilde{X}_j(t,\omega)\|_{L^{2q}_\omega}\big\|^{2q}_{L^1_t} \leq C(2q)^q < \infty$. Hence, by Kolmogorov's strong law of large numbers, we have $\frac{S_m - \mathbb{E}[S_m]}{m+1} \to 0$ a.s., where $S_m = \sum_{j=0}^m Z_j^{(q)}$. It follows from (2.10) that $\frac{\mathbb{E}[S_m]}{m+1} \geq (2\pi c_1)^q$. This implies that $S_m(\omega) \to \infty$ a.s. $\omega \in \Omega$. Hence, we have

$$\|u\|_{B^s_{p,q}} \geq \|u\|_{B^{\frac{1}{2}}_{1,q}} = \Big(\lim_{m\to\infty} S_m\Big)^{\frac{1}{q}} = \infty, \text{ a.s.}$$

For $q = \infty$ and $s > \frac{1}{2}$, we have

$$\|u\|_{B^s_{p,\infty}} \geq \|u\|_{B^s_{1,\infty}} \sim \sup_j 2^{(s-\frac{1}{2})j} Z_j^{(1)}(\omega), \tag{2.11}$$

where $Z_j^{(1)}(\omega) = \|\widetilde{X}_j(t,\omega)\|_{L^1_t}$. Note that $\mathbb{E}\big[Z_j^{(1)}\big] = 2\pi c_1$ and $\mathbb{E}\big[|Z_j^{(1)}|^2\big] \leq C < \infty$ for all j. This implies that there exist $\delta, \varepsilon > 0$ and Ω_j, $j = 0, 1, \dots$, such that $Z_j^{(1)}(\omega) > \delta$ for $\omega \in \Omega_j$ and $Pr(\Omega_j) > \varepsilon$. In particular, we have $\sum_j Pr(\Omega_j) = \infty$. Then, by the Borel zero-one law,

$$Pr([Z_j^{(1)}(\omega) > \delta, \text{ infinitely often}]) = 1. \tag{2.12}$$

From (2.11) and (2.12), $\|u\|_{B^s_{p,\infty}} = \infty$ a.s. for $s > \frac{1}{2}$.

Now, consider $p = q = \infty$ and $s = \frac{1}{2}$. We have

$$\|u\|_{B^{\frac{1}{2}}_{\infty,\infty}} = \sup_j \|\widetilde{X}_j(t,\omega)\|_{L^\infty_t} \geq \sup_j |\widetilde{X}_j(t_j^*,\omega)|,$$

where $\widetilde{X}_j$ is defined in (2.9) and t_j^*'s are points in $\mathbb{T}$. Recall that $\{\widetilde{X}_j(t_j^*)\}_{j=0}^\infty$ is a family of independent standard complex-valued Gaussian random variables. Hence, $\sup_j |\widetilde{X}_j(t_j^*,\omega)| = \infty$ a.s. and thus $\|u\|_{B^{\frac{1}{2}}_{\infty,\infty}} = \infty$ a.s.

Finally, we consider the case $p < \infty$, $q = \infty$ and $s = \frac{1}{2}$. First, assume $p \leq 2$. Then, we have

$$\|u\|^2_{B^{\frac{1}{2}}_{p,\infty}} \leq \|u\|^2_{B^{\frac{1}{2}}_{2,\infty}} = \sup_j X_j^{(2)} < \infty, \text{ a.s.}$$

since $X_j^{(2)}$ defined in (2.3) converges to c_2 a.s. In the following, we consider

$$\|u\|^p_{B^{\frac{1}{2}}_{p,\infty}} \sim \sup_j \Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|^p_{L^p_t} \tag{2.13}$$

only for $p = 2k$ with $k = 2, 3, \dots$ since $\|u\|_{B^{\frac{1}{2}}_{p,\infty}} \leq \|u\|_{B^{\frac{1}{2}}_{2k,\infty}}$ for $p \leq 2k$.

When $p = 4$, we have

$$\begin{aligned} \Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|^4_{L^4_t} &= 2\cdot 2^{-2j} \sum_{|n_1|,|n_2|\sim 2^j} |g_{n_1}|^2 |g_{n_2}|^2 + 2^{-2j} \sum_{\substack{|n_\alpha|,|m_\beta|\sim 2^j\\ n_\alpha\neq m_\beta\\ n_1+n_2=m_1+m_2}} g_{n_1} g_{n_2} \overline{g}_{m_1} \overline{g}_{m_2} \\ &\quad - 2^{-2j} \sum_{|n|\sim 2^j} |g_n|^4 =: \mathrm{I}_j^{(2)} + \mathrm{II}_j^{(2)} + \mathrm{III}_j^{(2)}. \end{aligned} \tag{2.14}$$

Note that $\mathrm{I}_j^{(2)} = \big(X_j^{(2)}\big)^2 \to c_2^2$ and $\mathrm{III}_j^{(2)} = 2^{-j} X_j^{(4)} \to 0$ a.s. by the strong law of large numbers, where $X_j^{(p)}$ is defined in (2.3). Hence, it suffices to prove that $\sup_j |\mathrm{II}_j^{(2)}| < \infty$ a.s. By independence of $\mathrm{II}_j^{(2)}$ and the Borel zero-one law, it suffices to show that

$$\sum_j Pr(|\mathrm{II}_j^{(2)}| > K) < \infty \tag{2.15}$$

for some $K > 0$. By Chebyshev's inequality, we have

$$Pr(|\mathrm{II}_j^{(2)}| > K) \leq K^{-2}\mathbb{E}\big[|\mathrm{II}_j^{(2)}|^2\big] \leq C_2 K^{-2} 2^{-j}. \tag{2.16}$$

Hence, (2.15) follows, and thus we have $\|u\|_{B^{\frac{1}{2}}_{4,\infty}} < \infty$ a.s.

In order to estimate (2.13) for the general case $p = 2k$, we use an induction argument and assume the existence of estimates for $p = 2, \dots, 2(k-1)$. We have

$$\begin{aligned} \Big\|2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int}\Big\|^{2k}_{L^{2k}_t} &= k!\cdot 2^{-kj} \sum_{|n_\alpha|\sim 2^j} \prod_{j=1}^k |g_{n_\alpha}|^2 + 2^{-kj} \sum_{*} \prod_{\alpha=1}^k g_{n_\alpha} \prod_{\beta=1}^k \overline{g}_{m_\beta} \\ &\quad + \text{ error terms} =: \mathrm{I}_j^{(k)} + \mathrm{II}_j^{(k)} + \text{ error terms}, \end{aligned} \tag{2.17}$$

where $* = \{n_\alpha, m_\beta : \alpha, \beta = 1, \dots, k,\ |n_\alpha|, |m_\beta| \sim 2^j,\ n_\alpha \neq m_\beta\ \sum n_\alpha = \sum m_\beta\}$. Note that $\mathrm{I}_j^{(k)}$ consists of the terms for which n_α's and m_β's form exactly k pairs (including higher multiplicity) and that $\mathrm{II}_j^{(k)}$ consists of the terms with no pair. There are two types of error terms, which we call of type (i) and type (ii):

(i) $\text{error}_j^{(k)}$(i): $\{n_\alpha\}_{\alpha=1}^k = \{m_\beta\}_{\beta=1}^k$, and there exists at least one pair α, $\widetilde{\alpha}$ with $\alpha \neq \widetilde{\alpha}$ such that $n_\alpha = n_{\widetilde{\alpha}}$, i.e., n_α's and m_β's form exactly k pairs, but there exists at least four (or higher order) of a kind: $n_{\alpha_1} = n_{\alpha_2} = m_{\beta_1} = m_{\beta_2}$ with $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$.

(ii) $\text{error}_j^{(k)}$(ii): $\{n_\alpha\}_{\alpha=1}^k \neq \{m_\beta\}_{\beta=1}^k$, but there exists at least one pair α, β such that $n_\alpha = m_\beta$. i.e. n_α's and m_β's form exactly l pairs for some $1 \leq l \leq k-1$.

As before, by the strong law of large numbers, $\mathrm{I}_j^{(k)} = \big(X_j^{(2)}\big)^k \to c_2^k$ a.s. and the error terms of type (i) go to 0 a.s. since each of them can be written as

$$2^{-(k-L)j} \prod_{l\,:\,k=\sum_{l=1}^L k_l} X_j^{(2k_l)}, \quad L < k,$$

where each $X_j^{(2k_l)} \to c_{2k_l}$ a.s. Hence, we have $Pr(\sup_j |\text{error}_j^{(k)}(\text{i})| < \infty) = 1$. As for the error terms of type (ii), the worst ones can be written as

$$2^{-kj} \sum_n |g_n|^2 \sum_{**} \prod_{\alpha=1}^{k-1} g_{n_\alpha} \prod_{\beta=1}^{k-1} \overline{g}_{m_\beta}$$

where $** = \{n_\alpha, m_\beta : \alpha, \beta = 1, \dots, k-1, \, |n_\alpha|, |m_\beta| \sim 2^j, \; n_\alpha \neq m_\beta \; \sum n_\alpha = \sum m_\beta\}$. It is basically a product of $X_j^{(2)}$ (which converges to c_2 a.s.) and $\mathrm{II}_j^{(k-1)}$, i.e., the $(k-1)$-fold products over frequencies $\{n_\alpha\}_{\alpha=1}^{k-1}$ and $\{m_\beta\}_{\beta=1}^{k-1}$ containing no pair, which appeared at the $k-1$ inductive step. All the other error terms of type (ii) can be basically written as $\text{error}_j^{(l)}(\text{i}) \cdot \mathrm{II}_j^{(k-l)}$ for some $l = 1, \dots, k-1$. Hence, we have $Pr(\sup_j |\text{error}_j^{(k)}(\text{ii})| < \infty) = 1$.

Now, it remains to estimate $\mathrm{II}_j^{(k)}$. As before, it suffices to show that

$$\sum_j Pr(|\mathrm{II}_j^{(k)}| > K) < \infty \tag{2.18}$$

for some $K > 0$. By Chebyshev's inequality, we have

$$Pr(|\mathrm{II}_j^{(k)}| > K) \leq K^{-2}\mathbb{E}\big[|\mathrm{II}_j^{(k)}|^2\big] \leq C_k K^{-2} 2^{-j}.$$

Hence, (2.18) follows, and thus we have $\|u\|_{B^{\frac{1}{2}}_{2k,\infty}} < \infty$ a.s. This completes the proof.

3. Large deviation estimates

3.1. Abstract Wiener spaces and Fernique's theorem. Let B denote any of the spaces $M_s^{p,q}$, $W_s^{p,q}$, $\mathcal{F}L^{s,q}$, or $\widehat{b}^s_{p,q}$, and, as before, let u be the mean zero complex-valued Brownian loop on $\mathbb{T}$. While the previous section was concerned with the question of B-regularity, i.e., whether or not u is in B, this section will be concerned with the complementary topic of large deviation estimates on B. Specifically, we will establish estimates of the form

$$Pr\big(\|u(\omega)\|_B > K\big) < e^{-cK^2}, \tag{3.1}$$

for large $K > 0$, and some constant $c = c(B) > 0$. As we shall see, the theory of abstract Wiener spaces and Fernique's theorem play a crucial role in establishing estimates such as (3.1) on all non-endpoint cases, see Proposition 3.5 below. In Subsection 3.2, we prove that the large deviation estimate still holds for $\widehat{b}^s_{p,\infty}$ in the endpoint case $(s-1)p = -1$ even though Fernique's theorem is not applicable. We also discuss, in Subsection 3.3, the issue of deviation estimates in the endpoint case of regular Besov spaces $B^{\frac{1}{2}}_{p,\infty}, 1 \leq p < \infty$. For non-endpoint deviation estimates on $B^s_{p,q}$, the reader is referred to Roynette's work [26].

Recall that if u is the mean zero complex-valued Brownian loop on $\mathbb{T}$, then we can expand it in its Fourier-Wiener series as

$$u(x, \omega) = \sum_{n \neq 0} \frac{g_n(\omega)}{n} e^{inx}, \quad x \in \mathbb{T}, \tag{3.2}$$

where $\{g_n(\omega)\}_{n\neq 0}$ is a family of independent standard complex-valued Gaussian random variables. This induces a probability measure on the periodic functions on $\mathbb{T}$, namely the mean zero *Wiener measure* on $\mathbb{T}$, which can be formally written as

$$d\mu = Z^{-1} \exp\bigg(-\frac{1}{2} \int_{\mathbb{T}} |u_x|^2 dx \bigg) \prod_{x\in\mathbb{T}} du(x), \quad u \text{ mean } 0. \tag{3.3}$$

In the following, we use the theory of abstract Wiener spaces to provide the precise meaning of expression (3.3). Let $u(x) = \sum_{n\neq 0} \widehat{u}_n e^{inx}$ denote any periodic function on $\mathbb{T}$ with mean 0. We define the Gaussian measure μ_N on $\mathbb{C}^{2N}$ with the density

$$d\mu_N = Z_N^{-1} \exp\bigg(-\frac{1}{2} \sum_{0<|n|\leq N} |n|^2 |\widehat{u}_n|^2 \bigg) \prod_{0<|n|\leq N} d\widehat{u}_n, \tag{3.4}$$

where $d\widehat{u}_n$ denotes the complex Lebesgue measure on $\mathbb{C}$ and

$$Z_N = \int_{\mathbb{C}^{2N}} \exp\bigg(-\frac{1}{2} \sum_{0<|n|\leq N} |n|^2 |\widehat{u}_n|^2 \bigg) \prod_{0<|n|\leq N} d\widehat{u}_n.$$

In our definition above, we have abused the notation and we denoted *any* generic periodic function on $\mathbb{T}$ (not just the Brownian motion) by the letter u. The context, however, will make it clear when we refer specifically to Brownian motion.

Note that the measure μ_N is the induced probability measure on $\mathbb{C}^{2N}$ (that is, the $2N$ dimensional complex Gaussian or $4N$ dimensional real Gaussian measure) under the map $\omega \mapsto \{g_n(\omega)/|n|\}_{0<|n|\leq N}$. Indeed, if we replace $\widehat{u}_n$ by $g_n/|n|$ in (3.4), we have

$$d\mu_N = \widetilde{Z_N}^{-1} \prod_{0<|n|\leq N} \exp\bigg(-\frac{1}{2} |g_n|^2 \bigg) dg_n,$$

where

$$\widetilde{Z_N} = \int_{\mathbb{C}^{2N}} \exp\bigg(-\frac{1}{2} |g_n|^2 \bigg) dg_n = (2\pi)^{2N}.$$

We would like to define the mean zero Wiener measure in (3.3) as a limit of the finite dimensional Gaussian measures μ_N as $N \to \infty$, i.e., we would like to define the Wiener measure μ in (3.3) by

$$d\mu = Z^{-1} \exp\bigg(-\frac{1}{2} \sum_{n\neq 0} |n|^2 |\widehat{u}_n|^2 \bigg) \prod_{n\neq 0} d\widehat{u}_n, \tag{3.5}$$

where

$$Z = \int \exp\bigg(-\frac{1}{2} \sum_{n\neq 0} |n|^2 |\widehat{u}_n|^2 \bigg) \prod_{n\neq 0} d\widehat{u}_n.$$

Note that the expression in the exponent in (3.5) can be written as

$$-\frac{1}{2} \sum_{n\neq 0} |n|^2 |\widehat{u}_n|^2 = -\frac{1}{2} \|u\|_{\dot{H}^1} = -\frac{1}{2} \Big\langle |\partial_x|^{2-2s} |\partial_x|^s u, |\partial_x|^s u \Big\rangle_{L^2} = -\frac{1}{2} \langle B_s^{-1} u, u \rangle_{\dot{H}^s},$$

where $B_s = |\partial_x|^{2s-2}$.

It follows from the theory of Gaussian measures on Hilbert spaces that (3.5) defines a countably additive measure on $\dot{H}^s$ if and only if B_s is of trace class, i.e., if $\sum_{n\neq 0} |n|^{2s-2} < \infty$, which is equivalent to $s < \frac{1}{2}$; see Zhidkov's work [33]. This makes the Sobolev space H^s, $s < \frac{1}{2}$, a strong and natural candidate for the study of Brownian motion. Unfortunately, the spaces under consideration are not Hilbert spaces in general. To deal with this issue, the concept of abstract Wiener space comes to the rescue, since, roughly speaking, it provides us with a larger (Hilbert or Banach) space, as an extension of $\dot{H}^1$, on which μ can be realized as a countably additive probability measure.

In the following, we recall first some basic definitions from Kuo's monograph [17]. Given a real separable Hilbert space H with norm $\|\cdot\|_H$, let $\mathcal{F}$ denote the set of finite dimensional orthogonal projections $\mathbb{P}$ of H. Then, a *cylinder set* E is defined by $E = \{u \in H : \mathbb{P}u \in F\}$ where $\mathbb{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbb{P}H$. We let $\mathcal{R}$ denote the collection of all such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. Then, the Gaussian measure μ on H is defined by

$$\mu(E) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{1}{2}\|u\|_H^2} du$$

for $E \in \mathcal{R}$, where $n = \dim \mathbb{P}H$ and du is the Lebesgue measure on $\mathbb{P}H$. It is known that μ is finitely additive but not countably additive on $\mathcal{R}$.

A seminorm $|||\cdot|||$ in H is called *measurable* if, for every $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{F}$ such that

$$\mu(|||\mathbb{P}u||| > \varepsilon) < \varepsilon \tag{3.6}$$

for $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_\varepsilon$. Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite dimensional. Let B be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into B. The triple (i, H, B) is called an *abstract Wiener space*. (The pair (B, μ) is often called an abstract Wiener space as well.)

Now, regarding $v \in B^*$ as an element of $H^* \equiv H$ by restriction, we embed B^* in H. Define the extension of μ onto B (which we still denote by μ) as follows. For a Borel set $F \subset \mathbb{R}^n$, set

$$\mu(\{u \in B : ((u, v_1), \cdots, (u, v_n)) \in F\}) := \mu(\{u \in H : (\langle u, v_1\rangle_H, \cdots, \langle u, v_n\rangle_H) \in F\}),$$

where v_j's are in B^* and $(\cdot, \cdot)$ denote the dual pairing between B and B^*. Let $\mathcal{R}_B$ denotes the collection of cylinder sets $\{u \in B : ((u, v_1), \cdots, (u, v_n)) \in F\}$ in B.

Proposition 3.1 (Gross [14])**.** *μ is countably additive in the σ-field generated by $\mathcal{R}_B$.*

In the context of our paper, let $H = \dot{H}^1(\mathbb{T})$. Then, we have

Theorem 3.2. *The seminorms $\|\cdot\|_{M_s^{p,q}}$, $\|\cdot\|_{W_s^{p,q}}$, and $\|\cdot\|_{\mathcal{F}L^{s,q}}$ are measurable for $(s-1)q < -1$. Also, the seminorm $\|\cdot\|_{\widehat{b}^s_{p,q}}$ is measurable for $(s-1)p < -1$.*

The proof of Theorem 3.2, which we present in detail at the end of this subsection, employs some of the ideas in [21, Proposition 3.4].

Corollary 3.3. *Let μ be the mean zero Wiener measure on $\mathbb{T}$. Then, $(M_s^{p,q}, \mu)$, $(W_s^{p,q}, \mu)$, and $(\mathcal{F}L^{s,q}, \mu)$ are abstract Wiener spaces for $(s-1)q < -1$. Also, $(\widehat{b}^s_{p,q}, \mu)$ is an abstract Wiener space for $(s-1)p < -1$.*

Remark 3.4. As we shall see later, condition (3.6) is not satisfied for the endpoint case $\widehat{b}^s_{p,\infty}$ with $(s-1)p = -1$. Nevertheless, we can still establish a large deviation estimate using a different approach.

Given an abstract Wiener space (B, μ), we have the following integrability result due to Fernique [12].

Proposition 3.5 (Theorem 3.1 in [17]). *Let (B, μ) be an abstract Wiener space. Then, there exists $c > 0$ such that $\int_B e^{c\|u\|_B^2} \mu(du) < \infty$. In particular, this implies the following large deviation estimate: there exists $c' > 0$ such that*

$$\mu(\|u\|_B \geq K) \leq e^{-c'K^2}, \tag{3.7}$$

for sufficiently large $K > 0$.

From Theorem 3.2 and Proposition 3.5, we obtain the following corollary.

Corollary 3.6. *Let μ be the mean zero Wiener measure on $\mathbb{T}$. Then, the large deviation estimate* (3.7) *holds for $B = M_s^{p,q}$, $W_s^{p,q}$, and $\mathcal{F}L^{s,q}$ with $(s-1)q < -1$, Also,* (3.7) *holds for $B = \widehat{b}^s_{p,q}$ with $(s-1)p < -1$.*

While Proposition 3.5 is not applicable to the endpoint case $\widehat{b}^s_{p,\infty}$ with $(s-1)p = -1$ (see Remark 3.4), we can still prove the following result.

Theorem 3.7. *Let $(s-1)p = -1$. Then,*

$$\mu(\|u\|_{\widehat{b}^s_{p,\infty}} \geq K) \leq e^{-cK^2}, \tag{3.8}$$

for sufficiently large $K > 0$.

We prove Theorem 3.7 in Subsection 3.2. Theorem 3.7 also holds for the endpoint case of the usual Besov spaces $B^s_{p,\infty}$, with $s = \frac{1}{2}$ and $p < \infty$. However, the proof becomes rather cumbersome for large values of p. Therefore, we will only map the proof of the large deviation estimates for the Besov spaces with $p \leq 4$; see Subsection 3.3.

For the proof of Theorem 3.2, we will need the following lemma from [23], which we now recall.

Lemma 3.8 (Lemma 4.7 in [23]). *Let $\{g_n\}$ be a sequence of independent standard complex-valued Gaussian random variables. Then, for M dyadic and $\delta < \frac{1}{2}$, we have*

$$\lim_{M\to\infty} M^{2\delta} \frac{\max_{|n|\sim M} |g_n|^2}{\sum_{|n|\sim M} |g_n|^2} = 0 \ \ a.s.$$

With these preliminaries, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, note that $\|u\|_{M_s^{p,q}} = \|u\|_{W_s^{p,q}} = \|u\|_{\mathcal{F}L^{s,q}} = \|u\|_{\widehat{b}^s_{q,q}}$. Hence, it suffices to prove the result for $\widehat{b}^s_{p,q}$ with $(s-1)p < -1$ and any $q \in [1, \infty]$. In view of (3.6), it suffices to show that for given $\varepsilon > 0$, there exists large M_0 such that

$$\mu\big(\|\mathbb{P}_{>M_0} u\|_{\widehat{b}^s_{p,q}} > \varepsilon\big) < \varepsilon, \tag{3.9}$$

where $\mathbb{P}_{>M_0}$is the projection onto the frequencies $|n| > M_0$.

Since $\widehat{b}^s_{p,1} \subset \widehat{b}^s_{p,q}$, it suffices to prove (3.9) for $q = 1$. If $p < 2$ with $(s-1)p < -1$, then by Hölder inequality, we have

$$\begin{aligned} \|\langle n\rangle^s \widehat{u}(n)\|_{L^p_{|n|\sim 2^j}} &\leq \|\langle n\rangle^{-\frac{2-p}{2p}}\|_{L^{\frac{2p}{2-p}}_{|n|\sim 2^j}} \|\langle n\rangle^{s+\frac{2-p}{2p}} \widehat{u}(n)\|_{L^2_{|n|\sim 2^j}} \\ &\sim \|\langle n\rangle^{s+\frac{2-p}{2p}} \widehat{u}(n)\|_{L^2_{|n|\sim 2^j}}, \end{aligned}$$

where $(s + \frac{2-p}{2p} - 1) \cdot 2 < -1$. Hence, it suffices to prove (3.9) for $2 \leq p \leq \infty$ and $q = 1$ with $(s-1)p < -1$.

Now, we consider the case $2 \le p < \infty$. In the following, we assume that u is of the form (3.2). Fix $K > 1$ and $\delta \in (0, \frac{1}{2})$ (to be chosen later.) Then, by Lemma 3.8 and Egoroff's theorem, there exists a set E such that $\mu(E^c) < \frac{1}{2}\varepsilon$ and the convergence in Lemma 3.8 is uniform on E, i.e. we can choose dyadic M_0 large enough such that

$$\frac{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^\infty_n}}{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^2_n}} \le M^{-\delta}, \tag{3.10}$$

for all $\omega \in E$ and dyadic $M > M_0$. In the following, we will work only on E and drop '$\cap E$' for notational simplicity. However, it should be understood that all the events are under the intersection with E so that (3.10) holds.

Let $\{\sigma_j\}_{j\ge 1}$ be a sequence of positive numbers such that $\sum \sigma_j = 1$, and let $M_j = M_0 2^j$ dyadic. Note that $\sigma_j = C2^{-\lambda j} = CM_0^\lambda M_j^{-\lambda}$ for some small $\lambda > 0$ (to be determined later.) Then, from (3.2), we have

$$\mu\big(\|\mathbb{P}_{>M_0} u(\omega)\|_{\widehat{b}^s_{p,1}} > \varepsilon\big) \le \sum_{j=1}^\infty \mu\big(\|\{\langle n\rangle^s n^{-1} g_n(\omega)\}_{|n|\sim M_j}\|_{L^p_n} > \sigma_j \varepsilon\big). \tag{3.11}$$

By interpolation and (3.10), we have

$$\begin{aligned}
\|\{\langle n\rangle^s n^{-1} g_n\}_{|n|\sim M_j}\|_{L^p_n} &\sim M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{L^p_n} \le M_j^{s-1}\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}^{\frac{2}{p}} \|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}^{\frac{p-2}{p}}\\
&\le M_j^{s-1}\|\{g_n\}_{|n|\sim M}\|_{L^2_n}\left(\frac{\|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}}{\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}}\right)^{\frac{p-2}{p}} \le M_j^{s-1-\delta\frac{p-2}{p}}\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}.
\end{aligned}$$

Thus, if we have $\|\{\langle n\rangle^s n^{-1} g_n\}_{|n|\sim M_j}\|_{L^p_n} > \sigma_j\varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j$ where $R_j := \sigma_j \varepsilon M_j^\alpha$ with $\alpha := -s+1+\delta\frac{p-2}{p}$. With $p = 2+\theta$, we have $\alpha = \frac{-(s-1)p+\delta\theta}{2+\theta} > \frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$ since $-(s-1)p > 1$. Then, by taking $\lambda > 0$ sufficiently small, $R_j = \sigma_j\varepsilon M_j^\alpha = C\varepsilon M_0^\lambda M_j^{\alpha-\lambda} \gtrsim C\varepsilon M_0^\lambda M_j^{\frac{1}{2}+}$. By a direct computation in the polar coordinates, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \sim \int_{B^c(0,R_j)} e^{-\frac{1}{2}|g_n|^2}\prod_{|n|\sim M_j} dg_n \lesssim \int_{R_j}^\infty e^{-\frac{1}{2}r^2} r^{2\cdot\#\{|n|\sim M_j\}-1}dr.$$

Note that, in the inequality, we have dropped the implicit constant $\sigma(S^{2\cdot\#\{|n|\sim M_j\}-1})$, a surface measure of the $2\cdot\#\{|n|\sim M_j\}-1$ dimensional unit sphere, since $\sigma(S^n) = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}) \lesssim 1$. By the change of variable $t = M_j^{-\frac{1}{2}}r$, we have $r^{2\cdot\#\{|n|\sim M_j\}-2} \lesssim r^{4M_j} \sim M_j^{2M_j}t^{4M_j}$. Since $t > M_j^{-\frac{1}{2}}R_j = C\varepsilon M_0^\lambda M_j^{0+}$, we have $M_j^{2M_j} = e^{2M_j\ln M_j} < e^{\frac{1}{8}M_j t^2}$ and $t^{4M_j} < e^{\frac{1}{8}M_j t^2}$ for M_0 sufficiently large. Thus, we have $r^{2\cdot\#\{|n|\sim M_j\}-2} < e^{\frac{1}{4}M_j t^2} = e^{\frac{1}{4}r^2}$ for $r > R$. Hence, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \le C\int_{R_j}^\infty e^{-\frac{1}{4}r^2} r\,dr \le e^{-cR_j^2} = e^{-cC^2M_0^{2\lambda}M_j^{1+}\varepsilon^2}. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\mu\big(\|\mathbb{P}_{>M_0}u\|_{\widehat{b}^s_{p,1}} > \varepsilon\big) \le \sum_{j=1}^\infty e^{-cC^2M_0^{1+2\lambda+}(2^j)^{1+}\varepsilon^2} \le \tfrac{1}{2}\varepsilon$$

by choosing M_0 sufficiently large as long as $(s-1)p < -1$.

When $p = \infty$, we have $s < 1$. By repeating the computation with (3.10), we see that if we have $\|\{\langle n\rangle^s n^{-1} g_n\}_{|n|\sim M_j}\|_{L^\infty_n} > \sigma_j\varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j$ where $R_j := \sigma_j \varepsilon M_j^\alpha$ with $\alpha := -s+1+\delta$. Since $-s+1 > 0$, we have $\alpha > \frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$. The rest follows exactly as before. □

3.2. **Large deviation estimates for $\widehat{b}^s_{p,\infty}$ at the endpoint** $(s-1)p = -1$**.** Now, we show that the condition (3.6) actually fails for $\widehat{b}^s_{p,\infty}$ for the endpoint case $(s-1)p = -1$; see Remark 3.4. By the strong law of large numbers, $X_j^{(p)}$ defined in (2.3) converges a.s. to $c_p > 0$. Then, by Egoroff's theorem, there exists E with $Pr(E) \leq \frac{1}{2}$ such that X_j converges uniformly to c_p on E. Thus, given $\delta > 0$, there exists $J_0 \in \mathbb{N}$ such that $Pr(\{\omega : \sup_{j\geq J} X_j^{(p)}(\omega) > c_p - \delta) \geq \frac{1}{2}$ for any $J \geq J_0$. In view of (2.6), this shows that the condition (3.6) does not hold once $\varepsilon < c_p$. In particular, Proposition 3.5 does not hold automatically.

The remainder of this subsection is dedicated to the proof of Theorem 3.7 via a direct approach that bypasses the assumption of abstract Wiener space. Specifically, we establish that, for some $c = c(p)$ and all sufficiently large $K \geq K_p$, the large deviation estimate

$$Pr(\|u\|_{\widehat{b}^s_{p,\infty}} > K) < e^{-cK^2}, \tag{3.13}$$

also holds in the endpoint case $(s-1)p = -1$.

Let us first consider the case $p \leq 2$. By Hölder inequality with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$, we have

$$\big\|2^{-\frac{j}{p}}|g_n|\big\|_{L^p_{|n|\sim 2^j}} \leq \|2^{-\frac{j}{r}}\|_{L^r_{|n|\sim 2^j}} \big\|2^{-\frac{j}{2}}|g_n|\big\|_{L^2_{|n|\sim 2^j}} \sim \big\|2^{-\frac{j}{2}}|g_n|\big\|_{L^2_{|n|\sim 2^j}}. \tag{3.14}$$

i.e., we have $\|u\|_{\widehat{b}^s_{p,\infty}} \leq \|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}}$. Hence, we only need to consider the case $p = 2$ and $s = \frac{1}{2}$. By definition of the norm, we have

$$Pr(\|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}} > K) \leq \sum_{j=0}^\infty Pr\big(2^{-j} \sum_{|n|\sim 2^j} |g_n|^2 > K^2\big). \tag{3.15}$$

Let us now recall the so called *Cramér condition*: a sequence $\{\xi_n\}$ of independent identically distributed (i.i.d.) random variables is said to satisfy Cramér's condition if there exists $\lambda > 0$ such that

$$\varphi(\lambda) = \mathbb{E}\big[e^{\lambda|\xi_1|}\big] < \infty.$$

If the condition holds, then we can define *the Cramér transform*

$$H(a) = \sup_{\lambda>0}\{a\lambda - \psi(\lambda)\},$$

with $\psi(\lambda) = \ln\varphi(\lambda)$, and, for $a > \mathbb{E}\big[\xi_1\big]$, we have

$$Pr\Big(\frac{1}{N}\sum_{n=1}^N \xi_n \geq a\Big) \leq e^{-nH(a)}; \tag{3.16}$$

see Shiryaev's book [29]. With $\xi_n = |g_n|^p$ (note that g_n is complex-valued), we see that Cramér's condition is satisfied for $p = 2$. Indeed, when $p = 2$, we have

$$\mathbb{E}\big[e^{\lambda|g_n|^2}\big] = \frac{1}{2\pi}\int e^{(\lambda-\frac{1}{2})|g_n|^2} dg_n = \frac{1}{2\pi(1-2\lambda)}$$

for $\lambda < \frac{1}{2}$. Then, $H(a) = \sup_{\lambda>0}\{a\lambda + \ln(1-2\lambda) + \ln(2\pi)\}$ has the maximum value $\frac{a-2}{2} + \ln\frac{2}{a} + \ln(2\pi)$ at $\lambda = \frac{a-2}{2a}$. Then, from (3.16), we have

$$Pr\big(2^{-j}\sum_{|n|\sim 2^j}|g_n|^2 > K^2\big) < e^{-c2^jK^2}. \tag{3.17}$$

Hence, we have $Pr(\|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}} > K) < e^{-cK^2}$ in view of (3.15). This proves (3.13) for $p \leq 2$.

Note that the Cramér condition no longer holds for $p > 2$. Thus, we need another approach. There are known large deviation results even when the Cramér's condition fails; see, for example, Saulis and Nakas's work [28]. However, they do not seem to be directly applicable to obtain (3.13). Instead, we use the hypercontractivity of the Ornstein-Uhlenbeck semigroup related to products of Gaussian random variables. For the following discussion, see the works of Kuo [18], Ledoux-Talagrand [19], and Janson [15]. A nice summary is given by Tzvetkov in [32, Section 3].

In our discussion we will use the Hermite polynomials $H_n(x)$. They are defined by

$$e^{tx-\frac{1}{2}t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n.$$

The first three Hermite polynomials are: $H_0(x) = 1$, $H_1(x) = x$, and $H_2(x) = x^2 - 1$.

Now, consider the Hilbert space $H = L^2(\mathbb{R}^d, \mu_d)$ with $d\mu_d = (2\pi)^{-\frac{d}{2}}\exp(-|x|^2/2)dx$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. We define a *homogeneous Wiener chaos of order* n to be an element of the form $\prod_{j=1}^d H_{n_j}(x_j)$, $n = n_1 + \cdots + n_d$. Consider the Hartree-Fock operator $L = \Delta - x\cdot\nabla$, which is the generator for the Ornstein-Uhlenbeck semigroup. Then, by the hypercontractivity of the Ornstein-Uhlenbeck semigroup $S(t) = e^{Lt}$, we have the following

Lemma 3.9. *Fix $q \geq 2$. Then, for every $u \in H$ and $t \geq \frac{1}{2}\log(q-1)$, we have*

$$\|S(t)u\|_{L^q(\mathbb{R}^d,\mu_d)} \leq \|u\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{3.18}$$

Note that (3.18) holds, independent of the dimension d. It is known that the eigenfunction of L with eigenvalue $-n$ is precisely the homogeneous Wiener chaos of order n. Thus, we have

Lemma 3.10. *Let $F(x)$ be a linear combination of homogeneous chaoses of order n. Then, for $q \geq 2$, we have*

$$\|F(x)\|_{L^q(\mathbb{R}^d,\mu_d)} \leq (q-1)^{\frac{n}{2}}\|F(x)\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{3.19}$$

The proof is basically the same as in [32, Propositions 3.3–3.5]. We only have to note that $F(x)$ is an eigenfunction of $S(t) = e^{Lt}$ with eigenvalue e^{-nt}. Then, (3.19) follows from (3.18) by evaluating (3.18) at time $t = \frac{1}{2}\log(q-1)$.

Denote now by $\mathcal{K}_n$ the collection of the homogeneous chaoses of order n. Given a homogeneous polynomial $P_n(x) = P_n(x_1, \dots, x_d)$ of degree n, we define *the Wick ordered monomial* $:P_n(x):$ to be its projection onto $\mathcal{K}_n$. In particular, we have $: x_j^n : = H_n(x_j)$ and $: \prod_{j=1}^d x_j^{n_j} : = \prod_{j=1}^d H_{n_j}(x_j)$ with $n = n_1 + \cdots + n_d$.

Since the Fourier coefficients of Brownian motion involve complex Gaussian random variables, let us consider the Wick ordering on them as well. Let g denote a standard complex-valued Gaussian random variable. Then, g can be written as $g = x + iy$, where x and y are independent standard real-valued Gaussian random variables. Note that the variance of g is $\mathrm{Var}(g) = 2$. Next, we investigate the Wick ordering on $|g|^{2n}$ for $n \in \mathbb{N}$, that is, the projection of $|g|^{2n}$ onto $\mathcal{K}_{2n}$.

When $n = 1$, $|g|^2 = x^2 + y^2$ is Wick-ordered into

$$: |g|^2 := (x^2 - 1) + (y^2 - 1) = |g|^2 - \text{Var}(g).$$

When $n = 2$, $|g|^4 = (x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$ is Wick-ordered into

$$\begin{aligned} : |g|^4 : &= (x^4 - 6x^2 + 3) + 2(x^2 - 1)(y^2 - 1) + (y^4 - 6y^2 + 3) \\ &= x^4 + 2x^2y^2 + y^4 - 8(x^2 + y^2) + 8 \\ &= |g|^4 - 4\text{Var}(g)|g|^2 + 2\text{Var}(g)^2, \end{aligned}$$

where we used $H_4(x) = x^4 - 6x^2 + 3$.

When $n = 3$, $|g|^6 = (x^2 + y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ is Wick-ordered into

$$\begin{aligned} : |g|^6 : &= (x^6 - 15x^4 + 45x^2 - 15) + 3(x^4 - 6x^2 + 3)(y^2 - 1) \\ &\quad + 3(x^2 - 1)(y^4 - 6y^2 + 3) + (y^6 - 15y^4 + 45y^2 - 15) \\ &= |g|^6 - 9\text{Var}(g)|g|^4 + 18\text{Var}(g)^2|g|^2 - 6\text{Var}(g)^3, \end{aligned}$$

where we used $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$.

In general, we have $: |g|^{2n} : \in \mathcal{K}_{2n}$. Moreover, we have

$$: |g|^{2n} : = |g|^{2n} + \sum_{j=0}^{n-1} a_j |g|^{2j} = |g|^{2n} + \sum_{j=0}^{n-1} b_j : |g|^{2j} : . \tag{3.20}$$

This follows from the fact that $|g|^{2n}$, as a polynomial in x and y only with even powers, is orthogonal to any homogeneous chaos of odd order, and it is radial, i.e., it depends only on $|g|^2 = x^2 + y^2$. Note that $: |g|^{2n} :$ can also be obtained from the Gram-Schmidt process applied to $|g|^{2k}$, $k = 0, \dots, n$ with $\mu_2 = (2\pi)^{-1} \exp(-(x^2 + y^2)/2) dxdy$.

With these preliminaries, we are ready to return to the proof of the large deviation estimate (3.13) for $p > 2$. Given $p > 2$, choose k such that $p \le 2k$. As in (3.14), by Hölder inequality with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2k}$, we have

$$\big\| 2^{-\frac{j}{p}} |g_n| \big\|_{L^p_{|n| \sim 2^j}} \lesssim \big\| 2^{-\frac{j}{2k}} |g_n| \big\|_{L^{2k}_{|n| \sim 2^j}}, \tag{3.21}$$

i.e., we have $\|u\|_{\widehat{b}^s_{p,\infty}} \le \|u\|_{\widehat{b}^{1-\frac{1}{2k}}_{2k,\infty}}$ for $(s-1)p = -1$. Hence, it suffices to prove (3.13) for $p = 2k$ and $s = 1 - \frac{1}{2k}$. Let

$$F_j(\omega) = 2^{-j} \sum_{|n| \sim 2^j} |g_n(\omega)|^{2k}. \tag{3.22}$$

Then, we have

$$Pr(\sup_j |F_j| > K^{2k}) \le \sum_{j=0}^{\infty} Pr(|F_j| > K^{2k}).$$

Hence, it suffices to prove

$$\sum_{j=0}^{\infty} Pr(|F_j| > K^{2k}) < e^{-cK^2}. \tag{3.23}$$

By (3.20), write F_j as a linear combination of homogeneous chaoses of order $2m$, $m = 0, 1, \dots, k$, i.e., we have $F_j = \sum_{m=0}^{k} F_j^{(m)}$, where $F_j^{(m)}$ is the component of F_j projected onto $\mathcal{K}_{2m}$. Then, it suffices to prove

$$\sum_{j=0}^{\infty} Pr(|F_j^{(m)}| > \tfrac{1}{k+1}K^{2k}) < e^{-cK^2} \tag{3.24}$$

for each $m = 0, 1, \dots, k$. By choosing K sufficiently large, we see that (3.24) trivially holds for $m = 0$, since $F_j^{(0)}$ is a constant independent of j and thus the left-hand side of (3.24) is 0 for large K. For $m \geq 1$, it follows from Lemma 3.10 that, for $q \geq 2$,

$$\|F_j^{(m)}\|_{L^q(\Omega)} \leq C_m q^m \|F_j^{(m)}\|_{L^2(\Omega)} = C'_m 2^{-\frac{j}{2}} q^m \tag{3.25}$$

where the constants C_m and C'_m are independent of j.

Let us now recall the following

Lemma 3.11 (Lemma 4.5 in [32]). *Suppose that we have, for all $q \geq 2$,*

$$\|F(\omega)\|_{L^q(\Omega)} \leq CN^{-\alpha} q^{\frac{n}{2}}$$

for some $\alpha, N, C > 0$ and $n \in \mathbb{N}$. Then, there exist c and C' depending on C and n but independent *of α and N such that*

$$Pr(|F(\omega)| > \lambda) \leq C' e^{-cN^{\frac{2\alpha}{n}}\lambda^{\frac{2}{n}}}.$$

Thus, from (3.25) and Lemma 3.11 with $n = 2m$, $N = 2^j$, $\alpha = \frac{1}{2}$, and $\lambda = K^{2k}$, we have

$$Pr(|F_j^{(m)}| > K^{2k}) < e^{-c2^{\frac{j}{2m}}K^{\frac{2k}{m}}} < e^{-c2^{\frac{j}{2m}}K^2}.$$

This establishes (3.24), and hence (3.23) and (3.13).

Remark 3.12. With $(s-1)p = -1$, we have

$$\mathbb{E}[\|u_j\|^p_{\widehat{b}^s_{p,\infty}}] \sim \mathbb{E}[X_j^{(p)}] = c_p,$$

where $u_j = \mathbb{P}_{|n|\sim 2^j} u$, and $X_j^{(p)}$ is defined in (2.3). Also, note that

$$\mathbb{E}[F_j] = \mathbb{E}[F_j^{(0)}] = F_j^{(0)} = c_p.$$

Hence, it follows from the above computation for $m = 1, \dots, k$ that

$$Pr\big(\big|\|u\|_{\widehat{b}^s_{p,\infty}} - c_p^{\frac{1}{p}}\big| > K\big) < e^{-cK^2} \tag{3.26}$$

$$Pr\big(\big|\|\mathbb{P}_{|n|\geq 2^N} u\|_{\widehat{b}^s_{p,\infty}} - c_p^{\frac{1}{p}}\big| > K\big) < e^{-c2^{\frac{N}{p}}K^2}. \tag{3.27}$$

In probability theory, large deviation estimates are commonly stated as in the estimates (3.26) and (3.27). However, in applications to partial differential equations, it is more common to encounter these estimate in the form (3.7); see [1, 2, 3, 4, 5, 7, 20, 21, 23].

3.3. **Large deviation estimates for $B^{\frac{1}{2}}_{p,\infty}$.** Lastly, we briefly discuss the large deviation estimates on the Besov spaces $B^s_{p,q}$ with the endpoint regularity $s = \frac{1}{2}$, $p < \infty$, and $q = \infty$:

$$Pr(\|u\|_{B^{\frac{1}{2}}_{p,\infty}} > K) < e^{-cK^2} \tag{3.28}$$

for some $c = c(p)$ and all sufficiently large $K \geq K_p$. For the non-endpoint result, the reader is referred to [26].

For $p \le 2$, (3.28) follows from (3.13) once we note that $\|u\|_{B^{\frac{1}{2}}_{p,\infty}} \le \|u\|_{B^{\frac{1}{2}}_{2,\infty}} = \|u\|_{\widehat{b}^{\frac{1}{2}}_{2,\infty}}$. When $p > 2$, (3.23) does not follow from (3.13) anymore. But, as in the proof of (3.13), it suffices to consider only the case $p = 2k$, $k \ge 2$. The proof for a general even index p involves lots of unwieldy technicalities. In the following we will sketch the argument for $p = 4$.

When $p = 4$, we have

$$\|u\|^4_{B^{\frac{1}{2}}_{4,\infty}} = \sup_j \Big\| 2^{-\frac{j}{2}} \sum_{|n|\sim 2^j} g_n e^{int} \Big\|^4_{L^4_t} = \sup_j \big(\mathrm{I}^{(2)}_j + \mathrm{II}^{(2)}_j + \mathrm{III}^{(2)}_j \big),$$

where $\mathrm{I}^{(2)}_j$, $\mathrm{II}^{(2)}_j$, and $\mathrm{III}^{(2)}_j$ are defined in (2.14). In the following, we treat them separately.

First, note that

$$\big\{\omega : \mathrm{I}^{(2)}_j(\omega) > K^4\big\} \subset \bigcup_{l=1}^{2} \Big\{\omega : 2^{-j} \sum_{|n_l|\sim 2^j} |g_{n_l}(\omega)|^2 > K^2 \Big\}.$$

Then, from (3.17), we have

$$Pr\big(\sup_j \mathrm{I}^{(2)}_j(\omega) > K^4\big) < e^{-cK^2}. \tag{3.29}$$

Next, note that $\mathrm{III}^{(2)}_j(\omega) > K^4$ if and only if $F_j(\omega) > 2^j K^4$, where $F_j(\omega)$ is defined in (3.22) with $k = 2$. Hence, from (3.23), we have

$$Pr\big(\sup_j \mathrm{III}^{(2)}_j(\omega) > K^4\big) < e^{-cK^2}. \tag{3.30}$$

Lastly, by expanding the complex Gaussians into their real and imaginary parts, it is not difficult to see that $\mathrm{II}^{(2)}_j$ is a homogeneous Wiener chaos of order 4 since each term in the sum is a product of four independent real-valued Gaussian random variables. Then, it follows from Lemma 3.10 that, for $q \ge 2$,

$$\|\mathrm{II}^{(2)}_j\|_{L^q(\Omega)} \le Cq^2 \|\mathrm{II}^{(2)}_j\|_{L^2(\Omega)} = C' 2^{-\frac{j}{2}} q^2$$

where the constants C and C' are independent of j. Thus, from Lemma 3.11, we have $Pr(|\mathrm{II}^{(2)}_j| > K^4) < e^{-c2^{\frac{j}{4}}K^2}$. This immediately implies

$$Pr\big(\sup_j \big|\mathrm{II}^{(2)}_j(\omega)\big| > K^4\big) < e^{-cK^2}. \tag{3.31}$$

The large deviation estimate (3.28) follows from (3.29), (3.30), and (3.31).

For a general even index p, one needs to repeat the above argument, using (2.17). The estimates on $\mathrm{I}^{(k)}_j$ and $\mathrm{II}^{(k)}_j$ follow easily as before. In particular, note that $\mathrm{II}^{(k)}_j$ is a homogeneous Wiener chaos of order $2k$. One can then estimate the error terms by a combination of the arguments presented above. However, the actual computation becomes lengthy, and thus we omit details.

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