1. Weak continuity of the Wick ordered cubic NLS in $L^2(T)$

Consider the Wick ordered cubic NLS:

$$\begin{cases} iu_t - u_{xx} \pm (|u|^2 - 2 \fint |u|^2)u = 0 \\ u\big|_{t=0} = u_0 \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{T} \times \mathbb{R}$. By writing as an integral equation, (1.1) is equivalent to

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')\mathcal{N}(u)(t')dt' \tag{1.2}$$

where $\mathcal{N}(u) = (|u|^2 - 2 \fint |u|^2)u$.

Theorem 1.1 (Weak continuity of (1.1))**.** *Assume that $u_{0,n}$ converges to u_0 in $L^2(\mathbb{T})$. Let u_n and u denote the unique global solutions of* (1.1) *with initial data $u_{0,n}$ and u_0, respectively. Then, given $T > 0$, we have the following.*

(a) *u_n converges weakly to u in $L^4_{T,x} := L^4([-T,T]; L^4(\mathbb{T}))$.*

(b) *For any $|t| \le T$, $u_n(t)$ converges weakly to $u(t)$ in $L^2(\mathbb{T})$. Moreover, this weak convergence is uniform for $|t| \le T$. i.e. for any $\phi \in L^2(\mathbb{T})$,*

$$\lim_{n\to\infty} \sup_{|t|\le T} |\langle u_n(t) - u(t), \phi\rangle_{L^2}| = 0.$$

Remark 1.2. We do not expect the weak continuity in the Strichartz space, i.e. in $L^6_{T,x}$. This is due to the failure of the $L^6_{x,t}$ Strichartz estimate in the periodic setting.

Define $\mathcal{N}_1(u_1,u_2,u_3)$ and $\mathcal{N}_2(u_1,u_2,u_3)$ by

$$\mathcal{N}_1(u_1,u_2,u_3) = \sum_{\substack{n=n_1-n_2+n_3 \\ (n_2-n_1)(n_2-n_3)\neq 0}} \widehat{u}_1(n_1)\overline{\widehat{u}}_2(n_2)\widehat{u}_3(n_3)e^{inx} \tag{1.3}$$

and

$$\mathcal{N}_2(u_1,u_2,u_3) = -\sum_n \widehat{u}_1(n)\overline{\widehat{u}}_2(n)\widehat{u}_3(n)e^{inx}. \tag{1.4}$$

Moreover, let $\mathcal{N}_j(u) := \mathcal{N}_j(u,u,u)$. Then, we have

$$\mathcal{N}(u) = \mathcal{N}_1(u) + \mathcal{N}_2(u).$$

• **Part 1:** First, we show that u_n converges to u as space-time distributions. In the following, we assume that the local well-posedness guarantees the existence of the solutions on the time interval $[-1,1]$ for $u_{0,n}$ and u_0. i.e. all the estimates hold on $[-1,1]$. (Otherwise, we can replace $[-1,1]$ by $[-\delta,\delta]$ for some $\delta > 0$.)

• **Linear part:** Since $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$, we have $\|u_{0,n} - u_0\|_{H^{-\varepsilon}(\mathbb{T})} \to 0$ for any $\varepsilon > 0$. Then, for any test function $\phi \in C^\infty_c(\mathbb{T}\times\mathbb{R})$, we have

$$\begin{aligned}\iint \eta(t)S(t)(u_{0,n}-u_0)\phi(x,t)dxdt &\le \|\eta(t)S(t)(u_{0,n}-u_0)\|_{X^{-\varepsilon,\frac12+}}\|\phi\|_{X^{\varepsilon,-\frac12-}} \\ &\lesssim C_\phi \|u_{0,n}-u_0\|_{H^{-\varepsilon}} \to 0.\end{aligned}$$

Hence, $\eta(t)S(t)u_{0,n}$ converges to $\eta(t)S(t)u_0$ as space-time distributions.

• **Nonlinear part:**

(i) By the uniform boundedness principle, we have $\|u_{0,n}\|_{L^2} \leq C$ for some $C > 0$. Then, by the local thoery, we have $\|u_n\|_{X_1^{0,\frac{1}{2}+}} \leq C'$ for all n. Thus, there exists a subsequence u_{n_k} converging weakly to some v in $X_1^{0,\frac{1}{2}+}$.

(ii) It essentially follows from Lemmata 2.2 and 2.3 in [Molinet, '09] that $\mathcal{N}_j$, $j = 1,2$, is weakly continuous from $X_1^{0,\frac{1}{2}+}$ into $X_1^{0,-\frac{7}{16}}$.

(iii) Recall the nonhomogeneous linear estimate:

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{X_1^{0,b}} \lesssim \|F\|_{X_1^{0,b-1}} \quad \text{for } b > \tfrac{1}{2}.$$

Now, let $\mathcal{M}(u)$ denote the Duhamel term. i.e.

$$\mathcal{M}(u)(t) := \mp i \int_0^t S(t-t')\mathcal{N}(u)(t')dt'.$$

Similarly, define $\mathcal{M}_j(u_1,u_2,u_3)$ by

$$\mathcal{M}_j(u_1,u_2,u_3)(t) := \mp i \int_0^t S(t-t')\mathcal{N}_j(u_1,u_2,u_3)(t')dt'$$

for $j = 1,2$. Also, let $\mathcal{M}_j(u) := \mathcal{M}_j(u,u,u)$.

Then, it follows from (i)∼(iii) that $\mathcal{M}(u_{n_k}) \rightharpoonup \mathcal{M}(v)$ in $X_1^{0,\frac{1}{2}+}$. In particular, $\mathcal{M}(u_{n_k})$ converges to $\mathcal{M}(v)$ as space-time distributions.

Remark 1.3. Given linear $T : X \to Y$, we have $T^* : Y^* \to X^*$. If $f_n \rightharpoonup f$ in X, then we have, for $\phi \in Y^*$, $\langle T(f_n - f), \phi\rangle = \langle f_n - f, T^*\phi\rangle \to 0$ since $T^*\phi \in X^*$. Hence, $Tf_n \rightharpoonup Tf$ in Y.

Since u_{n_k} is a solution to (1.1) with the initial data u_{0,n_k}, we have

$$u_{n_k} = S(t)u_{0,n_k} + \mathcal{M}(u_{n_k}).$$

By taking the limits of both sides, we obtain

$$v = S(t)u_0 + \mathcal{M}(v), \tag{1.5}$$

where the equality holds in the sense of space-time distributions.

From the uniquesness of the Cauchy problem (1.1) in $L^2(\mathbb{T})$, we have $v = u$ in $X_1^{0,\frac{1}{2}+}$.

Remark 1.4. From the standard local theory, the uniquesness holds only in $X_1^{0,\frac{1}{2}+}$. In fact, we can show that the unqiueness holds in $L^4_{x,t}(\mathbb{T}\times[-1,1])$ with little effort.

$$\begin{aligned}\|\eta(t)u\|_{L^4_{x,t}} &\leq \|\eta(t)S(t)u_0\|_{L^4_{x,t}} + \left\|\eta(t)\int_0^t S(t-t')|\eta u|^2\eta u(t')dt'\right\|_{L^4_{x,t}} \\ &\lesssim \|u_0\|_{L^2_x} + \left\|\int_0^t S(t-t')|\eta u|^2\eta u(t')dt'\right\|_{X^{0,\frac{3}{8}}}\end{aligned}$$

Moreover, the second term can be estimated by

$$\lesssim \||u|^2u\|_{X^{0,-\frac{3}{8}}} = \sup_{\|v\|_{X^{0,\frac{3}{8}}}} \iint v|\eta u|^2(\eta u)dxdt \leq \sup_{\|v\|_{X^{0,\frac{3}{8}}}} \|v\|_{L^4_{x,t}}\|\eta u\|^3_{L^4_{x,t}} \leq \|\eta u\|^3_{L^4_{x,t}}.$$

This shows that u is indeed a unique solution in $L^4_{x,t}$ (which is a bigger space than $X^{0,\frac{1}{2}+}$.) In the above computation, we took $|u|^2u$ as the nonlinearity, but one can proceed exactly in the same manner for $\mathcal{N}(u)$ by imposing appropriate frequency restriction.

We showed that the subsequence u_{n_k} converges weakly to u in $X_1^{0,\frac{1}{2}+}$ and $L^4_{x,t}(\mathbb{T}\times[-1,1])$.

Remark 1.5. Weak convergence in $X_1^{0,\frac{1}{2}+}$ implies that in $L^4_{x,t}(\mathbb{T}\times[-1,1])$ since

$$X_1^{0,-\frac{1}{2}-} = (X_1^{0,\frac{1}{2}+})^* \supset (L^4_{x,t}(\mathbb{T}\times[-1,1]))^* = L^{\frac{4}{3}}_{x,t}(\mathbb{T}\times[-1,1]).$$

The argument above also shows that u is the only weak limit point (in $X_1^{0,\frac{1}{2}+}$ and $L^4_{x,t}(\mathbb{T}\times[-1,1])$) of u_n. Hence, it follows from the boundedness of u_n in $X_1^{0,\frac{1}{2}+}$ and $L^4_{x,t}(\mathbb{T}\times[-1,1])$ that the whole sequence u_n converges weakly to u. This estiablishes Part (a) of Theorem 1.1 on $[-1,1]$.

Remark 1.6. If the whole sequence u_n does not converge weakly to u, then there exists $\phi \in X_1^{0,-\frac{1}{2}-} = (X_1^{0,\frac{1}{2}+})^*$ such that $\langle u_n, \phi\rangle \not\rightharpoonup \langle u, \phi\rangle$. This, in turn, implies that there exists $\varepsilon > 0$ such that for any $N\in\mathbb{N}$, there exists $n \geq N$ such that $|\langle u_n - u, \phi\rangle| > \varepsilon$.

Given $\varepsilon > 0$, we can construct a subsequence u_{n_k} with $|\langle u_{n_k} - u, \phi\rangle| > \varepsilon$ for each k. However, by repeating the previous argument (from the boundedness of u_{n_k} in $X_1^{0,\frac{1}{2}+}$), u_{n_k} has a sub-subsequence converging to u, which is a contradiction.

- **Part 2:** Now, we prove Part (b) of Theorem 1.1. Fix $\phi \in L^2(\mathbb{T})$.
- **Linear part:** Given $\varepsilon > 0$, choose $\psi \in H^1(\mathbb{T})$ such that $\|\phi-\psi\|_{L^2} < \frac{\varepsilon}{2K}$, where $K = \sup_n \|u_{0,n} - u_0\|_{L^2} < \infty$. Then, there exists N_1 such that

$$\begin{aligned}\sup_{|t|\leq 1} |\langle S(t)(u_{0,n}-u_0,\phi\rangle_{L^2}| &\leq \sup_{|t|\leq 1} |\langle S(t)(u_{0,n}-u_0,\psi\rangle_{L^2}| + \sup_{|t|\leq 1} |\langle S(t)(u_{0,n}-u_0,\phi-\psi\rangle_{L^2}| \\ &\leq \|S(t)(u_{0,n}-u_0)\|_{L^\infty([-1,1];H^{-1})}\|\psi\|_{H^1} \\ &\qquad + \|S(t)(u_{0,n}-u_0)\|_{L^\infty([-1,1];L^2)}\|\phi-\psi\|_{L^2} \\ &\lesssim \|S(t)(u_{0,n}-u_0)\|_{X_1^{-1,\frac{1}{2}+}} \tfrac{\varepsilon}{2K}\|S(t)(u_{0,n}-u_0)\|_{X_1^{0,\frac{1}{2}+}} \\ &\leq C\|u_{0,n}-u_0\|_{H^{-1}} + \tfrac{\varepsilon}{2K}\|u_{0,n}-u_0\|_{L^2} < \varepsilon\end{aligned}$$

for $n \geq N_1$ since $u_{0,n}$ converges strongly u_n in H^{-1}.

- **Nonlinear part:** Since $u_n \rightharpoonup u$ in $X_1^{0,\frac{1}{2}+}$, we see that $\mathcal{N}(u_n)$ converges strongly $\mathcal{N}(u_n)$ in $X_1^{-1,-\frac{7}{16}}$. (See the proof of Lemmata 2.2 and 2.3 in [Molinet, '09].) This, in turn, implies that $\mathcal{M}(u_n)$ converges strongly to $\mathcal{M}(u)$ in $X_1^{-1,\frac{1}{2}+}$ in view of the nonhomogeneous linear estimate.

Given $\varepsilon > 0$, choose $\psi \in H^1(\mathbb{T})$ such that $\|\phi-\psi\|_{L^2} < \frac{\varepsilon}{2K}$, where $K = \sup_n \|\mathcal{M}(u_n) - \mathcal{M}(u_n)\|_{X_1^{0,\frac{1}{2}+}} < \infty$. Then, there exists N_2 such that

$$\begin{aligned}&\sup_{|t|\leq 1}|\langle \mathcal{M}(u_n) - \mathcal{M}(u), \phi\rangle| \\ &\quad\leq \sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \psi\rangle_{L^2}| + \sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \phi-\psi\rangle_{L^2}| \\ &\quad\lesssim \|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{X_1^{-1,\frac{1}{2}+}} + \|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{X_1^{0,\frac{1}{2}+}}\|\phi-\psi\|_{L^2} < \varepsilon\end{aligned}$$

for $n \geq N_2$.

Hence, we have

$$\lim_{n\to\infty} \sup_{|t|\leq 1} |\langle u_n(t) - u(t), \phi\rangle_{L^2}| = 0. \tag{1.6}$$

Given $[-T, T]$, we iterate Part 1 and 2 on each $[j, j+1]$ and obtain Theorem 1.1.

Remark 1.7. For fixed $\phi \in L^2(\mathbb{T})$, we can consider $\langle u_n(t), \phi\rangle_{L^2}$ as in Molinet. Then, we can apply Ascoli-Arzelá theorem to extract the uniform converging *subsequence*. However, I do not see how to show the *uniform* convergence of the whole sequence by modifying Molinet's argument.