

PERIODIC STOCHASTIC KORTEWEG-DE VRIES EQUATION WITH THE ADDITIVE SPACE-TIME WHITE NOISE

TADAHIRO OH

ABSTRACT. We prove the local well-posedness of the periodic stochastic Korteweg-de Vries equation with the additive space-time white noise. In order to treat low regularity of the white noise in space, we consider the Cauchy problem in the Besov-type space $\widehat{b}_{p,\infty}^s(\mathbb{T})$ for $s = -\frac{1}{2}+$, $p = 2+$ such that $sp < -1$. In establishing the local well-posedness, we use a variant of the Bourgain space adapted to $\widehat{b}_{p,\infty}^s(\mathbb{T})$ and establish a nonlinear estimate on the second iteration on the integral formulation. The deterministic part of the nonlinear estimate also yields the local well-posedness of the deterministic KdV in $M(\mathbb{T})$, the space of finite Borel measures on \mathbb{T} .

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1. INTRODUCTION

In this paper, we prove the local well-posedness of the periodic stochastic KdV equation (SKdV) with the additive space-time white noise:

$$(1) \quad \begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = dW \\ u(x, 0) = u_0(x) \end{cases}$$

where u is a real-valued function, $(x, t) \in \mathbb{T} \times \mathbb{R}^+$ with $\mathbb{T} = [0, 2\pi)$, and $W(t) = \frac{\partial B}{\partial x}$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. With $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$, we have $W(t) = \beta_0(t)e_0 + \sum_{n \neq 0} \frac{1}{\sqrt{2}}\beta_n(t)e_n(x)$ where $\{\beta_n\}_{n \geq 0}$ is a family of mutually independent complex-valued Brownian motions (here we take β_0 to be real-valued) in a fixed probability space (Ω, \mathcal{F}, P) associated with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and $\beta_{-n}(t) = \overline{\beta_n(t)}$ for $n \geq 1$. Note that $\text{Var}(\beta_n(1)) = 2$ for $n \geq 1$.

In [8], de Bouard-Debussche-Tsutsumi considered

$$(2) \quad \begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \phi dW \\ u(x, 0) = u_0(x), \end{cases}$$

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where ϕ is a bounded linear operator in $L^2(\mathbb{T})$. They showed that (2) is locally well-posed when ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. See [8] and the references therein for the previous works in the periodic and nonperiodic settings.

In our present work, we consider the case when ϕ is the identity operator on $L^2(\mathbb{T})$. i.e. we take the additive noise to be the space-time white noise $\frac{\partial^2 B}{\partial_t \partial_x}$, where $B(x, t)$ is a two parameter Brownian motion on $\mathbb{T} \times \mathbb{R}^+$. Note that ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

Suppose that u is the solution to (1), or equivalently to (2) with $\phi = \text{Id}$, the identity operator on $L^2(\mathbb{T})$. Let $v_1(x, t) = u(x + \alpha_0 t, t) - \alpha_0$, where α_0 is the mean of u_0 . Then, v_1 satisfies (1) with the mean 0 initial condition $u_0 - \alpha_0$. Now, let \mathbb{P}_0 be the projection onto the spatial frequency 0, and $\mathbb{P}_{n \neq 0} = \text{Id} - \mathbb{P}_0$. Note that $\mathbb{P}_0 W(t) = \beta_0(t) e_0(x) = \frac{1}{\sqrt{2\pi}} \beta_0(t)$. By letting $v_2 = v_1 - \frac{1}{\sqrt{2\pi}} \beta_0(t)$, we see that u satisfies (1) if and only if v_2 satisfies

$$\begin{cases} dv_2 + (\partial_x^3 v_2 + (v_2 + \frac{1}{\sqrt{2\pi}} \beta_0(t)) \partial_x v_2) dt = \mathbb{P}_{n \neq 0} dW \\ v_2(x, 0) = u_0(x) - \alpha_0 \end{cases}$$

almost surely since $\beta_0(0) = 0$ a.s. By setting $v_3(x, t) = v_2(x + c_\omega(t), t)$ with $c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}} \beta_0(t') dt'$, it follows that v_3 satisfies

$$\begin{cases} dv_3 + (\partial_x^3 v_3 + v_3 \partial_x v_3) dt = d\widetilde{W} \\ v_3(x, 0) = u_0(x) - \alpha_0, \end{cases}$$

where $\widetilde{W}(x, t) = \sum_{n \neq 0} \frac{1}{\sqrt{2}} \beta_n(t) e_n(x + c_\omega(t)) = \sum_{n \neq 0} \frac{1}{\sqrt{2}} \beta_n(t) e^{inc_\omega(t)} e_n(x)$. i.e. v_3 solves (2) where

$$(3) \quad \phi = \text{diag}(\phi_n; n \neq 0) \quad \text{with} \quad \phi_n(t) = e^{inc_\omega(t)} \quad \text{and} \quad c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}} \beta_0(t') dt'$$

(with respect to the basis $\{e_n\}_{n \in \mathbb{Z}}$.) Moreover, note that v_3 has the spatial mean 0 (as long as it exists) since $e_0 \notin \text{Range}(\phi)$. Therefore, in the remaining of the paper, we concentrate on studying the local well-posedness of (2) with ϕ given by (3) and the mean 0 initial condition u_0 , (which implies that u has the spatial mean 0 as long as it exists.)

Recall that u is called a (local-in-time) mild solution to (2) if u satisfies

$$(4) \quad u(t) = S(t)u_0 - \frac{1}{2} \int_0^t S(t-t') \partial_x u^2(t') dt' + \int_0^t S(t-t') \phi(t') dW(t')$$

at least for $t \in [0, T]$ for some $T > 0$, where $S(t) = e^{-t\partial_x^3}$.

Note that the first two terms in (4) also appear in the deterministic KdV theory. Thus, we briefly review recent well-posedness results of the periodic (deterministic) KdV:

$$(5) \quad \begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}.$$

In [1], Bourgain introduced a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$(6) \quad \|u\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n, \tau)\|_{L_{n,\tau}^2(\mathbb{Z} \times \mathbb{R})},$$

where $\langle \cdot \rangle = 1 + |\cdot|$. He proved the local well-posedness of (5) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding the global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2 norm. Kenig-Ponce-Vega [11] improved Bourgain's result and established the local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$(7) \quad \|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{2}}},$$

for $s \geq -\frac{1}{2}$ under the mean 0 assumption on u and v . Colliander-Keel-Staffilani-Takaoka-Tao [5] proved the corresponding global well-posedness result via the I -method.

There are also results on (5) which exploit its complete integrability. In [2], Bourgain proved the global well-posedness of (5) in the class $M(\mathbb{T})$ of measures μ , assuming that its total variation $\|\mu\|$ is sufficiently small. His proof is based on the trilinear estimate on the second iteration of the integral formulation of (5), assuming an a priori uniform bound on the Fourier coefficients of the solution u of the form

$$(8) \quad \sup_{n \in \mathbb{Z}} |\widehat{u}(n, t)| < C$$

for all $t \in \mathbb{R}$. Then, he established (8) using the complete integrability. More recently, Kappeler-Topalov [9] proved the global well-posedness of the KdV in $H^{-1}(\mathbb{T})$ via the inverse spectral method.

There are also results on the necessary conditions on the regularity with respect to smoothness or uniform continuity of the solution map $: u_0 \in H^s(\mathbb{T}) \rightarrow u(t) \in H^s(\mathbb{T})$. Bourgain [2] showed that if the solution map is C^3 , then $s \geq -\frac{1}{2}$. Christ-Colliander-Tao [4] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (Also, see Kenig-Ponce-Vega [12].) These results, in particular, imply that we can not hope to have a local-in-time solution of KdV via the fixed point argument in H^s , $s < -\frac{1}{2}$. Recall that, for each fixed t , the space-time white noise $\frac{\partial^2 B}{\partial t \partial x}$ lies in $\cap_{s < -\frac{1}{2}} H^s \setminus H^{-\frac{1}{2}}$ almost surely. Hence, these results for KdV can not be applied to study the local well-posedness of (1).

Now, let us discuss the spaces which capture the regularities of the spatial and space-time white noise. Recently, we proved the invariance of the (spatial) white noise for the (deterministic) KdV in [13] (also see [14]) by first establishing the local well-posedness in an appropriate Banach space containing the support of the (spatial) white noise. Define the Besov-type space via the norm

$$(9) \quad \|f\|_{\widehat{b}_{p,\infty}^s} := \|\widehat{f}\|_{b_{p,\infty}^s} = \sup_j \|\langle n \rangle^s \widehat{f}(n)\|_{L^p_{|n| \sim 2^j}} = \sup_j \left(\sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\widehat{f}(n)|^p \right)^{\frac{1}{p}}.$$

In [13], using the theory of abstract Wiener spaces, we showed that $\widehat{b}_{p,\infty}^s$ contains the full support of the (spatial) white noise for $sp < -1$. (The statement also holds true for $sp = -1$.)

Let's consider the stochastic convolution $\Phi(t)$ given by

$$(10) \quad \Phi(t) = \int_0^t S(t-t') \phi(t') dW(t'),$$

where ϕ is given by (3). Define a variant of the $X^{s,b}$ space adjusted to $\widehat{b}_{p,\infty}^s(\mathbb{T})$. Let $X_{p,q}^{s,b}$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm

$$(11) \quad \|u\|_{X_{p,q}^{s,b}} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n, \tau)\|_{b_{p,\infty}^0 L^q_\tau}.$$

Note that $X_{p,q}^{s,b}$ defined in (11) is the space of functions u such that $S(-t)u(\cdot, t) \in (\widehat{b}_{p,\infty}^s)_x(\mathcal{FL}^{b,q})_t$, where $\mathcal{FL}^{b,q}$ is defined via the norm

$$(12) \quad \|f\|_{\mathcal{FL}^{b,q}} := \|\langle \tau \rangle^b \widehat{f}(\tau)\|_{L^q}.$$

In [13], we also showed that the local-in-time white noise is supported on $\mathcal{FL}^{c,q}$ for $cq < -1$. This implies that the Brownian motion belongs locally in time to $\mathcal{FL}^{b,q}$ for $(b-1)q < -1$. Hence, with $b < \frac{1}{2}$ and $q = 2$, we see that the local-in-time stochastic convolution $\eta(t)\Phi(t)$ lies in $X_{p,q}^{s,b}$ almost surely, with $sp < -1$, $b < \frac{1}{2}$ and $q = 2$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

The argument by de Bouard-Debussche-Tsutsumi [8] is based on the result by Roynette [15] on the endpoint regularity of the Brownian motion. i.e. the Brownian motion $\beta(t)$ belongs to the Besov space $B_{p,q}^{1/2}$ if and only if $q = \infty$ (with $1 \leq p < \infty$.) Then, they proved a variant of the bilinear estimate (7) by Kenig-Ponce-Vega adjusted to their Besov space setting, establishing the local well-posedness via the fixed point theorem. Note that the use of a variant of the bilinear estimate (7) required a slight regularization of the noise in space via ϕ so that the smoothed noise has the spatial regularity $s > -\frac{1}{2}$. Thus, they could not treat the space-time white noise, i.e. $\phi = \text{Id}$.

Our result is based on two observations. The first one is that our l_n^p -based function spaces $\widehat{b}_{p,\infty}^s$ in (9) and $X_{p,q}^{s,b}$ in (11) capture the regularity of the spatial and space-time white noise for $sp < -1$, $b < \frac{1}{2}$ and $q = 2$. The second is that we can indeed carry out Bourgain's argument in [2], a nonlinear estimate on the second iteration, *without* assuming the a priori bound (8), if we take the initial data $u_0 \in \widehat{b}_{p,\infty}^s$ for $s > -\frac{1}{2}$ with $p > 2$. Then, we construct a solution u as a strong limit of the smooth solutions u^N (with smooth u_0^N and ϕ^N) of (2). Note that our nonlinear estimate on the second iteration in Section 5 depends on the stochastic term, whereas the bilinear estimate in [8] is entirely deterministic.

Finally, we present our main results.

Theorem 1. *Let ϕ be as in (3) and $p = 2+$. Then, let $s = -\frac{1}{2} + \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $sp < -1$. Also, let u_0 be \mathcal{F}_0 -measurable such that it has mean 0 and belongs to $\widehat{b}_{p,\infty}^s(\mathbb{T})$ almost surely. Then, there exists a stopping time $T_\omega > 0$ and a unique process $u \in C([0, T_\omega]; \widehat{b}_{p,\infty}^s(\mathbb{T}))$ satisfying (2) on $[0, T_\omega]$ almost surely.*

As a corollary, we obtain the following:

Theorem 2. *The stochastic KdV (1) with the additive space-time white noise is locally well-posed almost surely (with the prescribed mean on u_0 .)*

Remark 1.1. Our argument provides an answer to the question posed by Bourgain in [2, Remark on p.120], at least in the local-in-time setting. The deterministic part of the nonlinear estimate in Section 5 can be used to establish the local well-posedness of (5) for a finite Borel measure $u_0 = \mu \in M(\mathbb{T})$ with $\|\mu\| < \infty$ *without* the complete integrability or the smallness assumption on μ . Note that $\mu \in \widehat{b}_{p,\infty}^s$ for $sp \leq -1$ since $\sup_n |\widehat{\mu}(n)| < \|\mu\| < \infty$. Hence, it can be used to study the Cauchy problem on $M(\mathbb{T})$ for non-integrable KdV-variants. Also, see [14].

Remark 1.2. Let $\mathcal{FL}^{s,p}(\mathbb{T})$ be the space of functions on \mathbb{T} defined via the norm $\|f\|_{\mathcal{FL}^{s,p}} = \|\langle n \rangle^s \widehat{f}(n)\|_{L_n^p}$. Recall from [13] that $\mathcal{FL}^{s,p}(\mathbb{T})$ contains the support of the (spatial) white noise when $sp < -1$. Then, Theorems 1 and 2 can also be established in $\mathcal{FL}^{s,p}(\mathbb{T})$ for

$s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$. The modification is straightforward once we note that $\|f\|_{\mathcal{F}L^{s-\varepsilon,p}} \lesssim \|f\|_{\widehat{b}_{p,\infty}^s}$ for any $\varepsilon > 0$, and thus we omit the details.

This paper is organized as follows: In Section 2, we introduce some notations. In Section 3, we introduce function spaces along with their embeddings and state deterministic linear estimates from [1] and [13]. In Section 4, we study some basic properties of the stochastic convolution. In Section 5, we prove Theorem 1 by establishing the nonlinear estimate on the second iteration of the integral formulation (4).

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2. NOTATION

In the periodic setting on \mathbb{T} , the spatial Fourier domain is \mathbb{Z} . Let dn be the normalized counting measure on \mathbb{Z} . We say $f \in L^p(\mathbb{Z})$, $1 \leq p < \infty$, if

$$\|f\|_{L^p(\mathbb{Z})} = \left(\int_{\mathbb{Z}} |f(n)|^p dn \right)^{\frac{1}{p}} := \left(\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, we have the obvious definition involving the supremum. We often drop 2π for simplicity. If a function depends on both x and t , we use $\widehat{\cdot}^x$ (and $\widehat{\cdot}^t$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $\widehat{\cdot}$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context.

For a Banach space $X \subset \mathcal{S}'(\mathbb{T} \times \mathbb{R})$, we use \widehat{X} to denote the space of the Fourier transforms of the functions in X , which is a Banach space with the norm $\|f\|_{\widehat{X}} = \|\mathcal{F}_{n,\tau}^{-1} f\|_X$, where \mathcal{F}^{-1} denotes the inverse Fourier transform (in n and τ .) Also, for a space Y of functions on \mathbb{Z} , we use \widehat{Y} to denote the space of the inverse Fourier transforms of the functions in Y with the norm $\|f\|_{\widehat{Y}} = \|\mathcal{F}f\|_Y$. Now, define $\widehat{b}_{p,q}^s(\mathbb{T})$ by the norm

$$(13) \quad \|f\|_{\widehat{b}_{p,q}^s(\mathbb{T})} = \|\widehat{f}\|_{b_{p,q}^s(\mathbb{Z})} := \left\| \|\langle n \rangle^s \widehat{f}(n)\|_{L_{|n| \sim 2^j}^p} \right\|_{L_j^q} = \left(\sum_{j=0}^{\infty} \left(\sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\widehat{f}(n)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

for $q < \infty$ and by (9) when $q = \infty$.

Throughout the paper, $\eta(t)$ denotes a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$, and let $\eta_T(t) = \eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on s, p , and δ . If a constant depends on other quantities, we make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. We also use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

3. FUNCTION SPACES AND BASIC EMBEDDINGS

First, let $X^{s,b}$ denote the usual periodic Bourgain space defined in (6). We often use the shorthand notation $\|\cdot\|_{s,b}$ to denote the $X^{s,b}$ norm. Now, define $X_{p,q}^{s,b}$, the Bourgain space adapted to $\widehat{b}_{p,\infty}^s$, to be the completion of the Schwartz functions on $\mathbb{T} \times \mathbb{R}$ with respect to the norm given by

$$(14) \quad \|u\|_{X_{p,q}^{s,b}} = \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n, \tau)\|_{b_{p,\infty}^0 L_\tau^q} = \sup_j \|\langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n, \tau)\|_{L_{|n| \sim 2^j}^p L_\tau^q}.$$

In the following, we take $p = 2+$ and $s = -\frac{1}{2}+ = -\frac{1}{2} + \delta$ with $\delta < \frac{p-2}{2p}$ (and $\delta > \frac{p-2}{4p}$) such that $sp < -1$. Lastly, given $T > 0$, we define $X_{p,q}^{s,b,T}$ as a restriction of $X_{p,q}^{s,b}$ on $[0, T]$ by

$$\|u\|_{X_{p,q}^{s,b,T}} = \|u\|_{X_{p,q}^{s,b}[0,T]} = \inf \{ \|\tilde{u}\|_{X_{p,q}^{s,b}} : \tilde{u}|_{[0,T]} = u \}.$$

We define the local-in-time versions of the other function spaces analogously.

Now, we discuss the basic embeddings. For $p \geq 2$, we have $\|a_n\|_{L_n^p} \leq \|a_n\|_{L_n^2}$. Thus, we have $\|f\|_{\widehat{b}_{p,\infty}^s} \leq \|f\|_{H^s}$, and thus

$$(15) \quad \|u\|_{X_{p,2}^{s,b}} \leq \|u\|_{X^{s,b}}.$$

By Hölder inequality, we have

$$(16) \quad \begin{aligned} \|f\|_{H^{-\frac{1}{2}-\delta}} &= \left(\sum_j (2^j)^{0-} \|\langle n \rangle^{-\frac{1}{2}-\delta} \widehat{f}(n)\|_{|n| \sim 2^j}^2 \right)^{\frac{1}{2}} \\ &\leq \sup_j \|\langle n \rangle^{-2\delta} \|_{L^{\frac{2p}{p-2}}} \|\langle n \rangle^{-\frac{1}{2}+\delta} \widehat{f}(n)\|_{L_n^p} \leq \|f\|_{\widehat{b}_{p,\infty}^s} \end{aligned}$$

for $s = -\frac{1}{2} + \delta$ with $\delta > \frac{p-2}{4p}$. Hence, for $s = -\frac{1}{2} + \delta$ with $\delta > \frac{p-2}{4p}$, we have

$$(17) \quad \|u\|_{X^{-\frac{1}{2}-\delta,b}} \lesssim \|u\|_{X_{p,2}^{s,b}}.$$

Now, we briefly go over the linear estimates. Let $S(t) = e^{-t\partial_x^3}$ and $T \leq 1$ in the following. We first present the homogeneous and nonhomogeneous linear estimates. See [1], [10], [13] for details of the proofs.

Lemma 3.1. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X_{p,2}^{s,b,T}} \lesssim T^{\frac{1}{2}-b} \|u_0\|_{\widehat{b}_{p,\infty}^s}$.*

Lemma 3.2. *For any $s \in \mathbb{R}$ and $b \leq \frac{1}{2}$, we have*

$$\left\| \int_0^t S(t-t')F(x,t')dt' \right\|_{X_{p,2}^{s,b,T}} \lesssim \|F\|_{X_{p,2}^{s,b-1}} + \|F\|_{X_{p,1}^{s,-1}}.$$

Also, we have $\left\| \int_0^t S(t-t')F(x,t')dt' \right\|_{X_{p,2}^{s,b,T}} \lesssim \|F\|_{X_{p,2}^{s,b-1}}$ for $b > \frac{1}{2}$.

The next lemma is the periodic L^4 Strichartz estimate due to Bourgain [1].

Lemma 3.3. *Let u be a function on $\mathbb{T} \times \mathbb{R}$. Then, we have $\|u\|_{L_{x,t}^4} \lesssim \|u\|_{X^{0,\frac{1}{3}}}$.*

Lastly, recall that by restricting the Bourgain spaces onto a small time interval $[0, T]$, we can gain a small power of T . See Colliander-Oh [6] for the proof.

Lemma 3.4. *For $0 \leq b' < b \leq \frac{1}{2}$, we have*

$$\|u\|_{X^{s,b',T}} = \|\eta_T u\|_{X^{s,b',T}} \lesssim T^{b-b'} \|u\|_{X^{s,b}}.$$

4. STOCHASTIC CONVOLUTION

In this section, we study basic properties of the stochastic convolution $\Phi(t)$ defined in (10). In particular, we prove that $\eta\Phi$ belongs to $X_{p,2}^{s,b,T}$ and is continuous from $[0, T]$ into $\widehat{b}_{p,\infty}^s$ for $T \leq 1$ almost surely for $sp < -1$ and $(b-1) \cdot 2 < -1$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

Before stating the main results, we point out the following. Let ϕ be the identity operator on $L^2(\mathbb{T})$ or be as in (3). Then, we know that such ϕ is Hilbert-Schmidt from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. In other words, with a slight abuse of notation, define

$$(18) \quad \phi := \sum_{n \in \mathbb{Z}} \phi e_n = \sum_{n \in \mathbb{Z}} \phi_n e_n$$

in view of $\phi = \text{diag}(\phi_n; n \neq 0)$. Then, we have $\phi \in H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. Moreover, we have $\|\phi\|_{HS(L^2; H^s)} = \|\phi\|_{H^s}$, where $\|\cdot\|_{HS(L^2; H^s)}$ denotes the Hilbert-Schmidt norm from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$. For such ϕ , we also have $\phi \in \widehat{b}_{p, \infty}^s(\mathbb{T})$ if and only if $sp \leq -1$, and we can use $\|\phi\|_{\widehat{b}_{p, \infty}^s}$ to discuss the regularity of ϕ in place of the Hilbert-Schmidt norm. This is one of the reasons for using this space. (We need only $sp < -1$ for our purpose since the nonlinear estimate in Section 5 holds for $s = -\frac{1}{2}$ and $p = 2+$ with $sp < -1$.)

Proposition 4.1. *Let $0 < T \leq 1$ and $p = 2+$. Moreover, let $s = -\frac{1}{2} + \delta$ and $b = \frac{1}{2} - \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $sp < -1$ and $(b-1) \cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in (10) with ϕ as in (3), we have*

$$(19) \quad \mathbb{E}[\|\eta\Phi\|_{X_{p,2}^{s,b,T}}] \leq C(\eta, s, p) < \infty.$$

In particular, $\Phi \in X_{p,2}^{-\frac{1}{2}+\delta, \frac{1}{2}-\delta, T}$ almost surely.

Before going into the proof of Proposition 4.1, recall the following. Let β_1 and β_2 be independent real-valued Brownian motions on (Ω, \mathcal{F}, P) , and $f_1(t, \omega)$ and $f_2(t, \omega)$ be real-valued stochastic processes independent of β_1 and β_2 . Then, we can regard β_j and f_j as $\beta_j(t, \omega) = \beta_j(t, \omega_1)$ and $f_j(t, \omega) = f_j(t, \omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Thus, in taking an expectation, we can first integrate over $\omega_1 \in \Omega_1$. Then, for $m \in \mathbb{N}$, we have

$$(20) \quad \begin{aligned} & \mathbb{E} \left(\left| \int_a^b f_1(t) d\beta_1(t) + \int_a^b f_2(t) d\beta_2(t) \right|^{2m} \right) \\ &= \mathbb{E} \left(\sum_{k=0}^{2m} \binom{2m}{k} \left(\int_a^b f_1(t) d\beta_1(t) \right)^k \left(\int_a^b f_2(t) d\beta_2(t) \right)^{2m-k} \right) \\ &= \mathbb{E}_{\Omega_2} \left[\sum_{n=0}^m \binom{2m}{2n} \frac{(2n)!}{2^n n!} \|f_1(\cdot, \omega_2)\|_{L^2(a,b)}^{2n} \frac{(2(m-n))!}{2^{m-n} (m-n)!} \|f_2(\cdot, \omega_2)\|_{L^2(a,b)}^{2(m-n)} \right]. \end{aligned}$$

In the computation above, we used the fact that, for each fixed ω_2 , $\int_a^b f_j(t, \omega_2) d\beta_j(t, \omega_1)$ is a Gaussian random variable on Ω_1 with variance $\|f_j(\cdot, \omega_2)\|_{L^2(a,b)}^2$.

Proof. By Hölder inequality, we have

$$\|\langle \tau - n^3 \rangle^{\frac{1}{2}-\delta} \widehat{u}(n, \tau)\|_{L_\tau^2} \leq \|\langle \tau - n^3 \rangle^{-2\delta}\|_{L_\tau^{\frac{2p}{p-2}}} \|\langle \tau - n^3 \rangle^{\frac{1}{2}+\delta} \widehat{u}(n, \tau)\|_{L_\tau^p}.$$

i.e. We have $\|\eta\Phi\|_{X_{p,2}^{s, \frac{1}{2}-\delta}} \lesssim \|\eta\Phi\|_{X_{p,2}^{s, \frac{1}{2}+\delta}}$ as long as $\delta > \frac{p-2}{4p}$. Thus, we will work in $X_{p,2}^{s, \frac{1}{2}+\delta}$ in the following.

Let $g(t) = \eta(t) \int_0^t S(-r) \phi(r) dW(r)$. i.e. $\eta(t) \Phi(\cdot, t) = S(t)g(\cdot, t)$. Assume that each β_n is extended to a Brownian motion on \mathbb{R} in such a way that the family $\{\beta_n\}_{n \geq 0}$ is still

independent. Note that for $t \in [0, T]$, we have

$$(21) \quad \widehat{g}(n, t) = \eta(t) \int_0^t \eta(r) e^{-irn^3} \phi_n(r) \chi_{[0, T]}(r) \frac{1}{\sqrt{2}} d\beta_n(r).$$

We have inserted $\eta(r)$ and $\chi_{[0, T]}(r)$ in the integrand since $\eta(r)\chi_{[0, T]}(r) \equiv 1$ for $r \in [0, t] \subset [0, T]$. For notational simplicity, we use $\phi_n(r)$ to denote $\phi_n(r)\chi_{[0, T]}(r)$ in the following. i.e. we assume that ϕ_n is supported on $[0, T]$. By (3), we have $|\phi_n(r)| \leq 1$ for $r \in \mathbb{R}$.

Now, we write the left hand side of (19) as

$$(22) \quad \begin{aligned} \mathbb{E} \left(\|\eta\Phi\|_{X_{p, \frac{1}{2}+\delta, T}} \right) &\lesssim \mathbb{E} \left[\sup_j 2^{js} \left(\sum_{|n| \sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau| \sim 2^k} |\widehat{g}(n, \tau)|^p d\tau \right)^{\frac{1}{p}} \right] \\ &+ \mathbb{E} \left[\sup_j 2^{js} \left(\sum_{|n| \sim 2^j} \int_{|\tau| \leq 2} |\widehat{g}(n, \tau)|^p d\tau \right)^{\frac{1}{p}} \right]. \end{aligned}$$

• **Part 1:** First, we estimate the second term in (22). Let

$$(23) \quad G_n(r, \tau) = \eta(r) e^{-irn^3} \phi_n(r) \int_r^{\infty} \eta(t) e^{-it\tau} dt.$$

Also write $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$ where $\beta_n^{(r)} = \text{Re } \beta_n$ and $\beta_n^{(i)} = \text{Im } \beta_n$. Then, by the stochastic Fubini Theorem, we have, for $m \in \mathbb{N}$,

$$(24) \quad \begin{aligned} \mathbb{E} [|\widehat{g}(n, \tau)|^{2m}] &= \mathbb{E} \left(\left| \int_{\mathbb{R}} \eta(t) e^{-it\tau} \int_{-\infty}^t \eta(r) e^{-irn^3} \phi_n(r) \frac{1}{\sqrt{2}} d\beta_n(r) dt \right|^{2m} \right) \\ &= 2^{-m} \mathbb{E} \left(\left| \int_{-1}^2 G_n(r, \tau) d\beta_n(r) \right|^{2m} \right) \\ &\lesssim \mathbb{E} \left(\left| \int_{-1}^2 \text{Re} G_n(r, \tau) d\beta_n^{(r)}(r) - \int_{-1}^2 \text{Im} G_n(r, \tau) d\beta_n^{(i)}(r) \right|^{2m} \right) \\ &\quad + \mathbb{E} \left(\left| \int_{-1}^2 \text{Im} G_n(r, \tau) d\beta_n^{(r)}(r) + \int_{-1}^2 \text{Re} G_n(r, \tau) d\beta_n^{(i)}(r) \right|^{2m} \right). \end{aligned}$$

Note that $|\text{Re} G_n(r, \tau)|, |\text{Im} G_n(r, \tau)| \leq |G_n(r, \tau)| \leq \|\eta\|_{L^1} |\phi_n(r)| \lesssim \|\eta\|_{L^1} \chi_{[0, T]}(r)$. Thus, we have $\|\text{Re} G_n(r, \tau)\|_{L^2}^{2k} \|\text{Im} G_n(r, \tau)\|_{L^2}^{2(m-k)} \lesssim \|\eta\|_{L^1}^{2m}$ for $k = 0, \dots, m$. Then, by (20) along with the independence of $\phi_n, \beta_n^{(r)}$ and $\beta_n^{(i)}$, we have

$$\|\widehat{g}(n, \tau)\|_{L^{2m}(\Omega)} \leq C = C(\eta, m)$$

independent of n and τ . Hence, for $p \in (2, 4)$, we have

$$(25) \quad \left(\mathbb{E} [|\widehat{g}(n, \tau)|^p] \right)^{\frac{1}{p}} \leq \|\widehat{g}(n, \tau)\|_{L^2(\Omega)}^{\theta} \|\widehat{g}(n, \tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 1,$$

by interpolation, where $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{4}$. Then, the second term in (22) is estimated by

$$(26) \quad \begin{aligned} (22) &\leq \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n| \sim 2^j} \int_{|\tau| \leq 2} \mathbb{E} [|\widehat{g}(n, \tau)|^p] d\tau \right)^{\frac{1}{p}} \lesssim \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n| \sim 2^j} 1 \right)^{\frac{1}{p}} \\ &\sim \left(\sum_{j=0}^{\infty} 2^{(sp+1)j} \right)^{\frac{1}{p}} \leq C < \infty, \end{aligned}$$

since $sp < -1$.

• **Part 2:** Next, we estimate the first term in (22). Let

$$(27) \quad \begin{cases} G_n^{(1)}(r, \tau) = \eta(r) e^{-irn^3} \phi_n(r) \int_r^\infty \eta'(t) \frac{e^{-it\tau}}{i\tau} dt, \\ G_n^{(2)}(r, \tau) = \eta^2(r) e^{-irn^3} \phi_n(r) \frac{e^{-ir\tau}}{i\tau}. \end{cases}$$

Then, by the stochastic Fubini theorem and integration by parts, we have

$$(28) \quad \begin{aligned} \sqrt{2}\widehat{g}(n, \tau) &= \int_{-1}^2 G_n(r, \tau) d\beta_n(r) = \int_{-1}^2 G_n^{(1)}(r, \tau) d\beta_n(r) + \int_{-1}^2 G_n^{(2)}(r, \tau) d\beta_n(r) \\ &=: I_n^{(1)}(\tau) + I_n^{(2)}(\tau). \end{aligned}$$

Thus, we have $|\widehat{g}(n, \tau)|^p \lesssim |I_n^{(1)}(\tau)|^p + |I_n^{(2)}(\tau)|^p$.

First, we estimate the contribution from $G_n^{(1)}$. For $|\tau| \sim 2^k$, we have

$$(29) \quad \left| \int_r^\infty \eta'(t) \frac{e^{-it\tau}}{i\tau} dt \right| \leq |\tau^{-2} \eta'(r)| + \left| \int_r^\infty \eta''(t) \frac{e^{-it\tau}}{\tau^2} dt \right| \leq C_\eta 2^{-2k},$$

by partial integration. Thus, we have $|G_n^{(1)}(r, \tau)| \lesssim 2^{-2k}$. Then, repeating a similar computation as in Part 1, we obtain

$$(30) \quad \left(\mathbb{E}[|I_n^{(1)}(\tau)|^p] \right)^{\frac{1}{p}} \leq \|I_n^{(1)}(\tau)\|_{L^2(\Omega)}^\theta \|I_n^{(1)}(\tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 2^{-2k},$$

by (20) and interpolation. Hence, the contribution to (22) is estimated by

$$(31) \quad \begin{aligned} (22) &\leq \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n| \sim 2^j} \sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau| \sim 2^k} \mathbb{E}[|I_n^{(1)}(\tau)|^p] d\tau \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{j(sp+1)} \sum_{k=1}^\infty 2^{k(-\frac{3p}{2}+\delta p+1)} \right)^{\frac{1}{p}} \leq C < \infty, \end{aligned}$$

since $sp < -1$ and $-\frac{3p}{2} + \delta p + 1 < 0$.

Now, we consider the contribution from $I_n^{(2)}(\tau)$. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have $|I_n^{(2)}(\tau)|^2 \lesssim \left| \int_{-1}^2 G_n^{(2)}(r, \tau) d\beta_n^{(r)}(r) \right|^2 + \left| \int_{-1}^2 G_n^{(2)}(r, \tau) d\beta_n^{(i)}(r) \right|^2$. We only estimate the first term since the second term is estimated in the same way. By Ito formula (c.f. [8]), we have

$$\begin{aligned} \left| \int_{-1}^2 G_n^{(2)}(r, \tau) d\beta_n^{(r)}(r) \right|^2 &= \int_{-1}^2 \eta^4(t) \frac{|\phi_n(t)|^2}{\tau^2} dt \\ &\quad + 2\text{Re} \int_{-1}^2 \int_{-\infty}^t G_n^{(2)}(r, \tau) d\beta_n^{(r)}(r) \overline{G_n^{(2)}(t, \tau)} d\beta_n^{(r)}(t) =: I_n'(\tau) + I_n''(\tau). \end{aligned}$$

The contribution from $I_n'(\tau)$ is at most

$$(32) \quad \begin{aligned} (22) &\lesssim \left(\sum_{j=0}^\infty 2^{jsp} \sum_{|n| \sim 2^j} \sum_{k=1}^\infty 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau| \sim 2^k} |\tau|^{-p} d\tau \left(\int_{-1}^2 \eta^4(t) dt \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\lesssim \|\eta\|_{L^4}^2 \left(\sum_{j=0}^\infty 2^{j(sp+1)} \sum_{k=1}^\infty 2^{k(-\frac{p}{2}+\delta p+1)} \right)^{\frac{1}{p}} \leq C < \infty, \end{aligned}$$

since $sp < -1$ and $\delta < \frac{p-2}{2p}$.

We finally estimate the contribution from $I_n''(\tau)$. Write $I_n''(\tau) = \int_{-1}^2 H_n(t) d\beta_n^{(r)}(t)$, where $H_n(t) = \int_{-\infty}^t \tilde{H}_n(r, t) d\beta_n^{(r)}(r)$ with

$$(33) \quad \tilde{H}_n(r, t) = 2\tau^{-2} \operatorname{Re}(\eta^2(r)\eta^2(t)e^{i(t-r)n^3} \phi_n(r) \overline{\phi_n(t)} e^{i(t-r)\tau}).$$

Then, by Ito isometry and $|\phi_n(\omega, t)| \leq 1$ for all $(\omega, t) \in \Omega \times \mathbb{R}$, we have

$$(34) \quad \begin{aligned} \mathbb{E}[|I_n''(\tau)|^2] &= \mathbb{E}\left[\left(\int_{-1}^2 H_n(t) d\beta_n^{(r)}(t)\right)^2\right] \sim \int_{-1}^2 \mathbb{E}[H_n^2(t)] dt \\ &= \int_{-1}^2 \mathbb{E}\left[\left(\int_{-\infty}^t \tilde{H}_n(r, t) d\beta_n^{(r)}(r)\right)^2\right] dt = \int_{-1}^2 \int_{-1}^t \mathbb{E}[|\tilde{H}_n(r, t)|^2] dr dt \\ &\lesssim \tau^{-4} \int_{-1}^2 \int_{-1}^t \eta^4(r)\eta^4(t) dr dt \lesssim \tau^{-4}. \end{aligned}$$

Hence, the contribution from $I_n''(\tau)$ is at most

$$(35) \quad \begin{aligned} (22) &\lesssim \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n| \sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau| \sim 2^k} \mathbb{E}[|I_n''(\tau)|^{\frac{p}{2}}] d\tau \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n| \sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau| \sim 2^k} (\mathbb{E}[|I_n''(\tau)|^2])^{\frac{p}{4}} d\tau \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{j(sp+1)} \sum_{k=1}^{\infty} 2^{k(-\frac{p}{2}+\delta p+1)} \right)^{\frac{1}{p}} \leq C < \infty, \end{aligned}$$

for $p \leq 4$, $sp < -1$, and $\delta < \frac{p-2}{2p}$. \square

We state a corollary to the proof of Proposition 4.1 for a general diagonal covariance operator $\phi(t, \omega) = \operatorname{diag}(\phi_n(t, \omega); n \in \mathbb{Z})$, which is independent of $\{\beta_n\}_{n \geq 1}$.

Corollary 4.2. *Let $0 < T \leq 1$, $p = 2+$, and $s, s' \in \mathbb{R}$ with $s < s'$. Moreover, let $b = \frac{1}{2} - \delta$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. i.e. $(b-1) \cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in (10) with $\phi \in L^p([0, T] \times \Omega; \widehat{b}_{p, \infty}^{s'})$, independent of $\{\beta_n\}_{n \geq 1}$, we have*

$$(36) \quad \mathbb{E}[\|\eta\Phi\|_{X_{p,2}^{s,b,T}}] \leq C(\eta, s, s', p) \|\phi\|_{L^p([0,T] \times \Omega; \widehat{b}_{p, \infty}^{s'})}.$$

In particular, $\Phi \in X_{p,2}^{s, \frac{1}{2}-\delta, T}$ almost surely.

Proof. In the proof of Proposition 4.1, we used $|\phi_n(t)| \leq 1$ whenever $\phi_n(t)$ appeared. Now, we briefly go through the proof of Proposition 4.1, keeping track of $\phi_n(t)$. Since ϕ is independent of $\{\beta_n\}_{n \geq 1}$, we regard β_n and ϕ_n as $\beta_n(t, \omega) = \beta_n(t, \omega_1)$ and $\phi_n(t, \omega) = \phi_n(t, \omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$.

In (25), we have $\mathbb{E}[|\widehat{g}(n, \tau)|^p] \lesssim \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p$. Then, in (26), we have

$$\begin{aligned} (22) &\leq \left(\sum_{j=0}^{\infty} 2^{j s p} \sum_{|n| \sim 2^j} \int_{|\tau| \leq 2} \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p d\tau \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=0}^{\infty} 2^{j(s-s')p} 2^{j s' p} \sum_{|n| \sim 2^j} \|\phi_n(\cdot, \omega_2)\|_{L^p([0, T] \times \Omega_2)}^p \right)^{\frac{1}{p}} \\ &\lesssim \|\phi\|_{L^p([0, T] \times \Omega; \widehat{b}_{p, \infty}^{s'})} \end{aligned}$$

since $s - s' < 0$. A similar modification in (30) and (31) (and (32)) takes care of the contribution from $I_n^{(1)}(\tau)$ (and $I_n'(\tau)$, respectively.) Now, as for $I_n''(\tau)$, we first integrate only over Ω_1 in (34) and obtain

$$\mathbb{E}_{\Omega_1} [|I_n''(\tau)|^2] \lesssim \tau^{-4} \int_{-1}^2 \int_{-1}^t \eta^4(r) \eta^4(t) |\phi_n(r)|^2 |\phi_n(t)|^2 dr dt \lesssim \tau^{-4} \|\phi_n\|_{L^2[0, T]}^4.$$

Then, in (35), we have

$$\mathbb{E}[|I_n''(\tau)|^{\frac{p}{2}}] = \mathbb{E}_{\Omega_2} [\|I_n''(\tau)\|_{L^{\frac{p}{2}}(\Omega_1)}^{\frac{p}{2}}] \leq \mathbb{E}_{\Omega_2} [\|I_n''(\tau)\|_{L^2(\Omega_1)}^{\frac{p}{2}}] \lesssim \tau^{-p} \mathbb{E}_{\Omega_2} \|\phi_n(\cdot, \omega_2)\|_{L^2[0, T]}^p$$

for $p \in [2, 4]$. The rest follows as before. \square

Now, we discuss the continuity of the stochastic convolution. In the remaining of this section, we show that the stochastic convolution $\Phi(t)$ defined in (10) belongs to $C([0, T]; \widehat{b}_{p, \infty}^s(\mathbb{T}))$ almost surely. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have

$$(37) \quad \Phi(t) = \frac{1}{\sqrt{2}} \sum_{n \neq 0} \int_0^t S(t-r) \phi_n(r) e_n d\beta_n^{(r)}(r) + i \frac{1}{\sqrt{2}} \sum_{n \neq 0} \int_0^t S(t-r) \phi_n(r) e_n d\beta_n^{(i)}(r),$$

since $\phi e_0 = 0$ and $\phi e_n = \phi_n e_n$, $n \neq 0$. In the following, we only show the continuity of the first stochastic convolution in (37), which we shall denote by $\Phi^{(r)}(t)$. Also, let $W^{(r)}(t) = \frac{1}{\sqrt{2}} \sum_n \beta_n^{(r)}(t) e_n$. As in Da Prato [7], we use the factorization method based on the elementary identity

$$(38) \quad \int_r^t (t-t')^{\alpha-1} (t'-r)^{-\alpha} dt' = \frac{\pi}{\sin \pi \alpha},$$

with $\alpha \in (0, 1)$ for $0 \leq r \leq t' \leq t$. Using (38), we can write the first term in (37) as

$$(39) \quad \Phi^{(r)}(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t S(t-t') (t-t')^{\alpha-1} Y(t') dt',$$

where

$$(40) \quad Y(t') = \int_0^{t'} S(t'-r) (t'-r)^{-\alpha} \phi(r) dW^{(r)}(r).$$

First, we present the following lemma which provides a criterion for the continuity of (39) in terms of the L^{2m} -integrability of $Y(t')$.

Lemma 4.3 (Lemma 2.7 in [7]). *Let $T > 0$, $\alpha \in (0, 1)$, and $m > \frac{1}{2\alpha}$. For $f \in L^{2m}([0, T]; \widehat{b}_{p, \infty}^s(\mathbb{T}))$, let*

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t') dt', \quad 0 \leq t \leq T.$$

Then, $F \in C([0, T]; \widehat{b}_{p, \infty}^s(\mathbb{T}))$. Moreover, there exists $C = C(m, T)$ such that

$$\|F(t)\|_{\widehat{b}_{p, \infty}^s} \leq C \|f\|_{L^{2m}([0, T]; \widehat{b}_{p, \infty}^s)}, \quad 0 \leq t \leq T.$$

Remark 4.4. Although Lemma 2.7 in [7] is stated for a Hilbert space H , its proof makes no use of the Hilbert space structure of H . Thus the same result holds for $\widehat{b}_{p, \infty}^s(\mathbb{T})$ as well.

In view of Lemma 4.3, it suffices to show that $Y(t') \in L^{2m}([0, T]; \widehat{b}_{p, \infty}^s(\mathbb{T}))$ a.s.

Proposition 4.5. *Let $T > 0$, $m \geq 2$, $s = -\frac{1}{2}+$, and $p = 2+$ such that $sp < -1$. Let ϕ be as in (3). Then, the stochastic convolution $\Phi^{(r)}(t)$ is continuous from $[0, T]$ into $\widehat{b}_{p, \infty}^s$ almost surely. Moreover, there exists*

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|\Phi^{(r)}(t)\|_{\widehat{b}_{p, \infty}^s}^{2m} \right) \leq C(m, T, s, p) < \infty.$$

Proof. Let $\alpha \in (\frac{1}{2m}, \frac{1}{2})$ and Y be as in (40). First, note that Y is real-valued since $\phi_{-n}(s)e_{-n} = \overline{\phi_n(s)e_n}$ and $\beta_{-n}^{(r)} = \beta_n^{(r)}$. Note that $\{\beta_n^{(r)}\}_{n \neq 0}$ and ϕ are independent since ϕ depends only on β_0 . Thus, we can regard $\beta_n^{(r)}$ and ϕ as $\beta_n^{(r)}(\omega) = \beta_n^{(r)}(\omega_1)$ and $\phi(\omega) = \phi(\omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Then, for each fixed ω_2 and $t' \in [0, t]$, $\widehat{Y}(t')(n)$ is a Gaussian random variable on Ω_1 with $\text{Var}_{\Omega_1}(\widehat{Y}(t')(n)) = \mathbb{E}_{\Omega_1}[|\widehat{Y}(t')(n)|^2]$.

Let $G_n(r, \omega_2) = (t' - r)^{-\alpha} e^{i(t'-r)n^3} \phi_n(r, \omega_2)$. Note that $|G_n(r, \omega_2)| = (t' - r)^{-\alpha}$ for $0 < r < t'$ and $n \neq 0$. By Ito isometry, we have

$$\begin{aligned} \mathbb{E}_{\Omega_1}[|\widehat{Y}(t')(n)|^2] &= \frac{1}{2} \mathbb{E}_{\Omega_1} \left[\left| \int_0^{t'} G_n(r, \omega_2) d\beta(r, \omega_1) \right|^2 \right] \\ &= \frac{1}{2} \int_0^{t'} |G_n(r, \omega_2)|^2 dr \sim \int_0^{t'} (t' - r)^{-2\alpha} dr. \end{aligned}$$

Then, by Minkowski integral inequality (with $p = 2+ < 2m$) after replacing \sup_j by \sum_j , we have

$$\begin{aligned} \mathbb{E}_{\Omega_1} \left(\|Y(t', \cdot, \omega_2)\|_{\widehat{b}_{p, \infty}^s}^{2m} \right) &= \mathbb{E}_{\Omega_1} \left[\left(\sup_j \sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\widehat{Y}(t')(n)|^p \right)^{\frac{2m}{p}} \right] \\ &\lesssim \left(\sum_{j=0}^{\infty} \sum_{|n| \sim 2^j} 2^{jsp} \left(\mathbb{E}_{\Omega_1}[|\widehat{Y}(t')(n)|^{2m}] \right)^{\frac{p}{2m}} \right)^{\frac{2m}{p}} \\ &\sim \left(\sum_{j=0}^{\infty} 2^{j(sp+1)} \right)^{\frac{2m}{p}} \left(\int_0^{t'} (t' - r)^{-2\alpha} dr \right)^m \lesssim \left(\frac{(t')^{1-2\alpha}}{1-2\alpha} \right)^m, \end{aligned}$$

since $sp < -1$. Therefore, we have

$$\begin{aligned} \int_0^T \mathbb{E}(\|Y(t')\|_{\widehat{b}_{p,\infty}^s}^{2m}) dt' &= \int_0^T \mathbb{E}_{\Omega_2} \mathbb{E}_{\Omega_1}(\|Y(t')\|_{\widehat{b}_{p,\infty}^s}^{2m}) dt' \\ &\lesssim \int_0^T \left(\frac{(t')^{1-2\alpha}}{1-2\alpha} \right)^m dt' \lesssim T^{(1-2\alpha)m+1} < C(m, T, s, p) < \infty. \end{aligned}$$

In particular, it follows that $Y(\cdot, \omega) \in L^{2m}([0, T]; \widehat{b}_{p,\infty}^s)$ almost surely. Then, the desired result follows from Lemma 4.3. \square

5. NONLINEAR ESTIMATE ON THE SECOND ITERATION

Now, we present the crucial nonlinear analysis. First, we briefly go over Bourgain's argument in [2]. By writing the integral equation, the deterministic KdV (5) is equivalent to

$$(41) \quad u(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}(u, u)(t),$$

where $\mathcal{N}(\cdot, \cdot)$ is given by

$$(42) \quad \mathcal{N}(u_1, u_2)(t) := \int_0^t S(t-t') \partial_x(u_1 u_2)(t') dt'.$$

In the following, we assume that the initial condition u_0 has the mean 0, which implies that $u(t)$ has the spatial mean 0 for each $t \in \mathbb{R}$. We use (n, τ) , (n_1, τ_1) , and (n_2, τ_2) to denote the Fourier variables for uu , the first factor, and the second factor u of uu in $\mathcal{N}(u, u)$, respectively. i.e. we have $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$. By the mean 0 assumption on u and by the fact that we have $\partial_x(uu)$ in the definition of $\mathcal{N}(u, u)$, we assume $n, n_1, n_2 \neq 0$. We also use the following notation:

$$\sigma_0 := \langle \tau - n^3 \rangle \text{ and } \sigma_j := \langle \tau_j - n_j^3 \rangle.$$

One of the main ingredients is the observation due to Bourgain [1]:

$$(43) \quad n^3 - n_1^3 - n_2^3 = 3nn_1n_2, \text{ for } n = n_1 + n_2,$$

which in turn implies that

$$(44) \quad \text{MAX} := \max(\sigma_0, \sigma_1, \sigma_2) \gtrsim \langle nn_1n_2 \rangle.$$

Now, define

$$(45) \quad A_j = \{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \sigma_j = \text{MAX}\},$$

and let $\mathcal{N}_j(u, u)$ denote the contribution of $\mathcal{N}(u, u)$ on A_j . By the standard bilinear estimate as in [1], [11], we have

$$(46) \quad \|\mathcal{N}_0(u, u)\|_{-\frac{1}{2}+\delta, \frac{1}{2}-\delta} \leq o(1) \|u\|_{-\frac{1}{2}-\delta, \frac{1}{2}-\delta}^2,$$

where $o(1) = T^\theta$ with some $\theta > 0$ by considering the estimate on a short time interval $[-T, T]$ (e.g. Lemma 3.4). See (2.17), (2.26), and (2.68) in [2]. Here, we abuse the notation and use $\|\cdot\|_{s,b} = \|\cdot\|_{X^{s,b}}$ to denote the local-in-time version as well. Note that the temporal regularity $b = \frac{1}{2} - \delta < \frac{1}{2}$. This allowed us to gain the spatial regularity by 2δ . Clearly, we can not expect to do the same for $\mathcal{N}_1(u, u)$. (By symmetry, we do not consider $\mathcal{N}_2(u, u)$ in the following.) The bilinear estimate (7) is known to fail for any $s \in \mathbb{R}$ if $b < \frac{1}{2}$ due to the contribution from $\mathcal{N}_1(u, u)$. See [11]. Following the notation in [2], let

$$(47) \quad I_{s,b} = \|\mathcal{N}_1(u, u)\|_{X^{s,b}} \text{ and } \alpha := \frac{1}{2} - \delta < \frac{1}{2}.$$

Then, by Lemma 3.2 and duality with $\|d(n, \tau)\|_{L_{n, \tau}^2} \leq 1$, we have

$$(48) \quad \begin{aligned} I_{-\alpha, 1-\alpha} &= \|\mathcal{N}_1(u, u)\|_{-\alpha, 1-\alpha} \\ &\lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \widehat{u}(n_1, \tau_1) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha}, \end{aligned}$$

where

$$(49) \quad c(n_2, \tau_2) = \langle n_2 \rangle^{-(1-\alpha)} \sigma_2^\alpha \widehat{u}(n_2, \tau_2) \text{ so that } \|c\|_{L_{n, \tau}^2} = \|u\|_{-(1-\alpha), \alpha} = \|u\|_{-\frac{1}{2}-\delta, \frac{1}{2}-\delta}.$$

The main idea here is to consider the second iteration, i.e. substitute (41) for $\widehat{u}(n_1, \tau_1)$ in (48), thus leading to a trilinear expression. Since $\sigma_1 = \text{MAX} \gtrsim \langle nn_1 n_2 \rangle \gg 1$ on A_1 , we can assume that

$$(50) \quad \widehat{u}(n_1, \tau_1) = (\mathcal{N}(u, u))^\wedge(n_1, \tau_1) \sim \frac{|n_1|}{\sigma_1} \sum_{n_1=n_3+n_4} \int_{\tau_1=\tau_3+\tau_4} \widehat{u}(n_3, \tau_3) \widehat{u}(n_4, \tau_4) d\tau_4.$$

Note that $\widehat{u}(n_1, \tau_1)$ can not come from $S(t)u_0$ of (41) since we have $\sigma_1 \sim 1$ for the linear part. Moreover, by the standard computation [1], we have

$$(51) \quad \begin{aligned} \mathcal{N}(u, u)(x, t) &= -i \sum_{k=1}^{\infty} \frac{i^k t^k}{k!} \sum_{n \neq 0} e^{i(nx+n^3 t)} \int \eta(\lambda - n^3) \widehat{\partial_x u^2}(n, \lambda) d\lambda \\ &\quad + i \sum_{n \neq 0} e^{inx} \int \frac{(1-\eta)(\tau - n^3)}{\tau - n^3} \widehat{\partial_x u^2}(n, \tau) e^{i\tau t} d\tau \\ &\quad + i \sum_{n \neq 0} e^{i(nx+n^3 t)} \int \frac{(1-\eta)(\lambda - n^3)}{\lambda - n^3} \widehat{\partial_x u^2}(n, \lambda) d\lambda \\ &=: \mathcal{M}_1(u, u)(x, t) + \mathcal{M}_2(u, u)(x, t) + \mathcal{M}_3(u, u)(x, t). \end{aligned}$$

Note that $(\mathcal{M}_1(u, u))^\wedge(n_1, \tau_1)$ and $(\mathcal{M}_3(u, u))^\wedge(n_1, \tau_1)$ are distributions supported on $\{\tau_1 - n_1^3 = 0\}$. i.e. $\sigma_1 \sim 1$. Hence, the only contribution for the second iteration on A_1 comes from $\mathcal{M}_2(u, u)$ whose Fourier transform is given in (50). This shows the validity of the assumption (50).

Note that the σ_1 appearing in the denominator allows us to cancel $\langle n \rangle^{1-\alpha}$ and $\langle n_2 \rangle^{1-\alpha}$ in the numerator in (48). Then, $I_{-\alpha, 1-\alpha}$ can be estimated by

$$(52) \quad \lesssim \sum_{\substack{n=n_1+n_2 \\ n_1=n_3+n_4}} \int_{\substack{\tau=\tau_1+\tau_2 \\ \tau_1=\tau_3+\tau_4}} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{|n_1|}{\sigma_1} \widehat{u}(n_3, \tau_3) \widehat{u}(n_4, \tau_4) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha}.$$

Then, Bourgain divided the argument into several cases, depending on the sizes of $\sigma_0, \dots, \sigma_4$. Here, the key algebraic relation is

$$(53) \quad n^3 - n_2^3 - n_3^3 - n_4^3 = 3(n_2 + n_3)(n_3 + n_4)(n_4 + n_2), \quad \text{with } n = n_2 + n_3 + n_4.$$

Then, Bourgain proved -see (2.69) in [2]-

$$(54) \quad I_{-\alpha, 1-\alpha} \leq o(1) \|u\|_{-(1-\alpha), \alpha} I_{-\alpha, 1-\alpha} + o(1) \|u\|_{-(1-\alpha), \alpha}^3 + o(1) \|u\|_{-(1-\alpha), \alpha},$$

assuming the a priori estimate (8): $|\widehat{u}(n, t)| < C$ for all $n \in \mathbb{Z}, t \in \mathbb{R}$. Indeed, the estimates involving the first two terms on the right hand side of (54) were obtained without (8), and

only the last term in (54) required (8), -see ‘‘Estimation of (2.62)’’ in [2]-, which was then used to deduce

$$(55) \quad \|\widehat{u}(n, \cdot)\|_{L^2_\tau} < C.$$

The a priori estimate (8) is derived via the isospectral property of the KdV flow and is false for a general function in $X^{-(1-\alpha), \alpha}$. (It is here that the smallness of the total variation $\|\mu\|$ is used.)

Our goal is to carry out a similar analysis for SKdV (2) on the second iteration *without* the a priori estimates (8) and (55) coming from the complete integrability of KdV. We achieve this goal by considering the estimate in $X_{p,2}^{-\alpha, \alpha} = X_{p,2}^{-\frac{1}{2}+\delta, \frac{1}{2}-\delta}$, where $p = 2+$ and $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. By (15) and (17) (recall $-\alpha = -\frac{1}{2} + \delta$ and $-(1-\alpha) = -\frac{1}{2} - \delta$), we have

$$(56) \quad \|u\|_{X_{p,2}^{-\alpha, \alpha}} \leq \|u\|_{X^{-\alpha, \alpha}}, \text{ and } \|u\|_{X^{-(1-\alpha), \alpha}} \lesssim \|u\|_{X_{p,2}^{-\alpha, \alpha}}.$$

Then, it follows from (46) and (56) that

$$(57) \quad \|\mathcal{N}_0(u, u)\|_{X_{p,2}^{-\alpha, \alpha}} \leq o(1)\|u\|_{X_{p,2}^{-\alpha, \alpha}}^2.$$

Now, we consider the estimate on $\|\mathcal{N}_1(u, u)\|_{X_{p,2}^{-\alpha, \alpha}}$. From (56) and $\alpha < 1 - \alpha$, it suffices to control $I_{-\alpha, 1-\alpha}$. As in the deterministic case, we consider the second iteration, and substitute (4) for $\widehat{u}(n_1, \tau_1)$ in (48). As before, there is no contribution from $S(t)u_0$, or $\mathcal{M}_1(u, u)$, $\mathcal{M}_3(u, u)$ defined in (51). Now, there are two contributions:

- (i) $\mathcal{N}_1(\mathcal{M}_2(u, u), u)$ from the deterministic nonlinear part: In this case, we can use the estimates from [2] *except* when the a priori bound (8) was assumed. i.e. we need to estimate the contribution from (2.62) in [2]:

$$(58) \quad R_\alpha := \sum_n \int_{\tau=\tau_2+\tau_3+\tau_4} \chi_B \frac{d(n, \tau)}{\langle n \rangle^{1+\alpha} \sigma_0^\alpha} \widehat{u}(-n, \tau_2) \widehat{u}(n, \tau_3) \widehat{u}(n, \tau_4) d\tau_2 d\tau_3 d\tau_4,$$

where $\|d(n, \tau)\|_{L^2_{n, \tau}} \leq 1$ and $B = \{\sigma_0, \sigma_2, \sigma_3, \sigma_4 < |n|^\gamma\}$ with some small parameter $\gamma > 0$. Note that this corresponds to the case $n_2 = -n$ and $n_3 = n_4 = n$ in (52) after some reduction. In our analysis, we directly estimate R_α in terms of $\|u\|_{X_{p,2}^{-\alpha, \alpha}}$.

The key observation is that we can take the spatial regularity $s = -\alpha$ to be greater than $-\frac{1}{2}$ by choosing $p > 2$.

- (ii) $\mathcal{N}_1(\Phi, u)$ from the stochastic convolution Φ in (10): In view of (56), we estimate

$$(59) \quad \mathbb{E}[\|\mathcal{N}_1(\eta\Phi, u)\|_{X^{-\alpha, 1-\alpha}}]$$

via the stochastic analysis from Section 4.

Remark 5.1. In fact, we do not need to take an expectation in (59) since we establish local well-posedness pathwise in ω , i.e. for almost every *fixed* ω . Nonetheless, we estimate (59) with the expectation since it shows how F_1^N and F_2^N defined in (71) arise along with their estimates.

• **Estimate on (i):** In [2], the parameter $\gamma = \gamma(\alpha)$, subject to the conditions (2.43) and (2.60) in [2], played a certain role in estimating R_α along with the a priori bound (8). However, it plays no role in our analysis. By Cauchy-Schwarz and Young’s inequalities, we have

$$(58) \leq \sum_n \|d(n, \cdot)\|_{L^2_\tau} \langle n \rangle^{-1-\alpha} \|\widehat{u}(-n, \tau_2)\|_{L^{\frac{6}{\tau_2}}} \|\widehat{u}(n, \tau_3)\|_{L^{\frac{6}{\tau_3}}} \|\widehat{u}(n, \tau_4)\|_{L^{\frac{6}{\tau_4}}}$$

By Hölder inequality (with appropriate \pm signs) and the fact that $-1 - \alpha < -3\alpha$,

$$\begin{aligned} &\leq \sum_n \|d(n, \cdot)\|_{L^2_\tau} \prod_{j=2}^4 \langle n \rangle^{-\alpha} \|\sigma_j^{-\alpha}\|_{L^3_{\tau_j}} \|\sigma_j^\alpha \widehat{u}(\pm n, \tau_j)\|_{L^2_{\tau_j}} \\ &\leq \|d(\cdot, \cdot)\|_{L^2_{n,\tau}} \|u\|_{X_{6,2}^{-\alpha,\alpha}}^3 \leq \|u\|_{X_{p,2}^{-\alpha,\alpha}}^3, \end{aligned}$$

where the last two inequalities follow by choosing $\alpha > \frac{1}{3}$ and $p = 2+ < 6$.

• **Estimate on (ii):** We use the notation from the proof of Proposition 4.1. It follows from (28) and $\eta(t)\Phi(\cdot, t) = S(t)g(\cdot, t)$ that

$$(\eta\Phi)^\wedge(n_1, \tau_1) = \widehat{g}(n_1, \tau_1 - n_1^3) = \frac{1}{\sqrt{2}} \mathbf{I}_{n_1}^{(1)}(\tau_1 - n_1^3) + \frac{1}{\sqrt{2}} \mathbf{I}_{n_1}^{(2)}(\tau_1 - n_1^3).$$

Recall that $\sigma_1 = \langle \tau_1 - n_1^3 \rangle \gtrsim \langle nn_1n_2 \rangle$. Also, recall from the proof of Proposition 4.1 that $|\phi_{n_1}(r)| = \chi_{[0,T]}(r)$ is independent of ω .

◦ Contribution from $\mathbf{I}_{n_1}^{(1)}(\tau_1 - n_1^3)$: From (48) with (27), (28), and (29), we estimate (59) by

$$(60) \lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1^2} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right]$$

By Cauchy-Schwarz inequality in ω and Ito isometry,

$$(61) \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^\alpha} \frac{\|\phi_{n_1}\|_{L^2[0,T]} \|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_1^{\frac{3}{2}-\delta} \langle n_1 \rangle^{\frac{1}{2}+\delta} \sigma_2^\alpha}$$

By $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$ -Hölder inequality along with Lemma 3.3, (16), (18), (49), and (56)

$$\begin{aligned} &\lesssim T^\theta \|d\|_{L^2_{n,\tau}} \|\phi\|_{L^2([0,T]; H^{-\frac{1}{2}-\delta})} \|c\|_{L^2(\Omega; L^2_{n,\tau})} \leq T^\theta \|\phi\|_{L^p([0,T]; \widehat{b}_{p,\infty}^-)} \|u\|_{L^2(\Omega; X^{-(1-\alpha),\alpha})} \\ &\lesssim T^\theta \|\phi\|_{L^p([0,T]; \widehat{b}_{p,\infty}^-)} \|u\|_{L^2(\Omega; X_{p,2}^{-\alpha,\alpha})}. \end{aligned}$$

Remark 5.2. Strictly speaking, we need to take the supremum over $\{\|d\|_{L^2_{n,\tau}} = 1\}$ inside the expectation in (60). However, we do not worry about this issue for simplicity of the presentation, since we have

$$\begin{aligned} (59) &\leq \|\mathcal{N}_1(\eta\Phi, u)\|_{L^2(\Omega; X^{-\alpha, 1-\alpha})} \\ &\leq \left(\sum_n \int \frac{\langle n \rangle^{2-2\alpha}}{\sigma_0^{2\alpha}} \mathbb{E} \left| \int_0^T |\phi_{n_1}(r)| \sum_{\substack{n=n_1+n_2 \\ \tau=\tau_1+\tau_2}} \int \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_1^2 \sigma_2^\alpha} d\tau_1 d\beta_{n_1}(r) \right|^2 d\tau \right)^{\frac{1}{2}} \\ &= \sup_{\|d\|_{L^2_{n,\tau}}=1} (61) \end{aligned}$$

by Ito isometry. Also, recall that we have $\mathbf{I}_{n_1}^{(1)}(\tau_1 - n_1^3) = \int_0^T G_{n_1}^{(1)}(r, \tau_1 - n_1^3) d\beta_{n_1}(r)$ where $G_n^{(1)}(r, \tau)$ is defined in (27). Hence, strictly speaking, we should replace $G_{n_1}^{(1)}(r, \tau_1 - n_1^3)$ by $\sigma_1^{-2} |\phi_{n_1}(r)|$ in (60) only after the application of Ito isometry. Once again, we do not worry about this issue for simplicity of the presentation. The same remark applies in the following as well.

◦ Contribution from $I_{n_1}^{(2)}(\tau_1 - n_1^3)$:

First, suppose that $\max(\sigma_0, \sigma_2) \gtrsim \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Say $\sigma_0 \geq \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Then, (59) is estimated by

$$(62) \quad \begin{aligned} & \lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right] \\ & \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^{\alpha-200\delta}} \frac{\|\phi_{n_1}\|_{L^2[0, T]}}{\sigma_1^{\frac{1}{2}+\delta} \langle n_1 \rangle^{\frac{1}{2}+\delta}} \frac{\|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha} \end{aligned}$$

Then, we can conclude this case as before by $L_{x,t}^4, L_{x,t}^2, L_{x,t}^4$ -Hölder inequality as long as $\alpha - 200\delta > \frac{1}{3}$, which can be guaranteed by taking $\delta > 0$ sufficiently small, or equivalently, taking $p > 2$ sufficiently close to 2.

Hence, assume $\max(\sigma_0, \sigma_2) \ll \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Recall the following lemma from [5, (7.50) and Lemma 7.4].

Lemma 5.3. *Let*

$$(63) \quad \Omega(n) = \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2 \rangle^{\frac{1}{100}}) \text{ for some } n_1 \in \mathbb{Z} \text{ with } n = n_1 + n_2\}.$$

Then, we have

$$(64) \quad \int \langle \tau - n^3 \rangle^{-\frac{3}{4}} \chi_{\Omega(n)}(\tau - n^3) d\tau \lesssim 1.$$

Note that (64) is stated with $\langle \tau - n^3 \rangle^{-1}$ in [5]. However, by examining the proof of Lemma 7.4 in [5], one immediately sees that (64) is valid with $\langle \tau - n^3 \rangle^{-\beta}$ for any $\beta > \frac{2}{3} + \frac{1}{100}$.

Then, (59) is estimated by

$$\lesssim \mathbb{E} \left[\sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^\alpha} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2 \rangle^{1-\alpha} c(n_2, \tau_2)}{\sigma_2^\alpha} \right]$$

By Cauchy-Schwarz inequality and Ito isometry,

$$(65) \quad \lesssim \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n, \tau)}{\sigma_0^\alpha} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3) \|\phi_{n_1}\|_{L^2[0, T]}}{\sigma_1^{\frac{1}{2}-\delta} \langle n_1 \rangle^{\frac{1}{2}+\delta}} \frac{\|c(n_2, \tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}$$

By $L_{x,t}^4, L_{x,t}^2, L_{x,t}^4$ -Hölder inequality along with Lemmata 3.3, 5.3, (16), (18), (49), and (56),

$$\begin{aligned} & \lesssim T^\theta \|d\|_{L_{n,\tau}^2} \|\langle n_1 \rangle^{-\frac{1}{2}-\delta} \|\phi_{n_1}\|_{L^2[0, T]} \|\chi_{\Omega(n_1)}(\tau_1 - n_1^3) \sigma_1^{-\frac{1}{2}+\delta}\|_{L_\tau^2} \|c\|_{L^2(\Omega; L_{n,\tau}^2)} \\ & \leq T^\theta \|\phi\|_{L^2([0, T]; H^{-\frac{1}{2}-\delta})} \|u\|_{L^2(\Omega; X^{-(1-\alpha), \alpha})} \lesssim T^\theta \|\phi\|_{L^p([0, T]; \widehat{b}_{p,\infty}^{-\alpha})} \|u\|_{L^2(\Omega; X_{p,2}^{-\alpha, \alpha})}. \end{aligned}$$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Fix mean zero $u_0 \in \widehat{b}_{p,\infty}^{-\alpha'}(\mathbb{T})$ and ϕ as in (3), where $\alpha' = \frac{1}{2} - \delta -$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$ such that $(-\alpha')p < -1$. Consider sequences of initial data $u_0^N \in L^2(\mathbb{T})$ and diagonal covariance operator $\phi^N \in HS(L^2; L^2)$, given by

$$(66) \quad u_0^N = \mathbb{P}_{\leq N} u_0 = \sum_{|n| \leq N} \widehat{u}_0(n) e^{inx} \text{ and } \phi^N(t, \omega) := \text{diag}(\phi_n(t, \omega); 0 < |n| \leq N)$$

where ϕ_n is given in (3). Now, fix $\alpha = \frac{1}{2} - \delta > \alpha'$ as in (47). Note that such u_0^N converges to u_0 in $\mathcal{F}L^{-\alpha,p}(\mathbb{T})$, and thus in $\widehat{b}_{p,\infty}^{-\alpha}(\mathbb{T})$. Also, ϕ^N converges to ϕ in $\mathcal{F}L^{-\frac{1}{2}-p}(\mathbb{T})$ for each t and ω , and thus in $\widehat{b}_{p,\infty}^{-\frac{1}{2}-}(\mathbb{T})$. Then, by Monotone Convergence Theorem, ϕ^N converges to ϕ in $L^p([0,1] \times \Omega; \widehat{b}_{p,\infty}^{-\frac{1}{2}-})$. (Indeed, the convergence is in $L^\infty([0,1] \times \Omega; \widehat{b}_{p,\infty}^{-\frac{1}{2}-})$, since we have $|\phi_n(t, \omega)| = 1$ for all n , independent of $t \in \mathbb{R}$ and $\omega \in \Omega$.) Note that a slight loss of the regularity $-\alpha < -\alpha'$ was necessary since u_0^N defined in (66) does not necessarily converge to u_0 in $\widehat{b}_{p,\infty}^{-\alpha'}(\mathbb{T})$ due to the L^∞ nature of the norm over the dyadic blocks. We can avoid such a loss of the regularity if we start with $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$.

Now, let $\Gamma^N = \Gamma_{u_0^N}^N$ be the map defined by

$$(67) \quad \Gamma^N v = \Gamma_{u_0^N}^N v := S(t)u_0^N - \frac{1}{2}\mathcal{N}(v, v) + \eta\Phi^N,$$

where Φ^N is the stochastic convolution defined in (10) with the covariance operator ϕ^N . By the well-posedness result in [8], there exists a unique global solution $u^N \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T})) \cap C(\mathbb{R}^+; B_{2,1}^0(\mathbb{T}))$ a.s. to (67) for each N since $\phi^N \in HS(L^2; L^2)$.

Now, we put all the estimates together. Note that all the implicit constants are independent of N . Also, when there is no superscript N , it means that $N = \infty$. From Lemma 3.1, we have

$$(68) \quad \|S(t)u_0^N\|_{X_{p,2}^{s,b,T}} \leq C_1 \|u_0^N\|_{\widehat{b}_{p,\infty}^s}$$

for any $s, b \in \mathbb{R}$ with $C_1 = C_1(b)$. In particular, by taking $b > \frac{1}{2}$, we see that $S(t)u_0$ is continuous on $[0, T]$ with values in $\widehat{b}_{p,\infty}^s$. Also, by taking $b < \frac{1}{2}$, we gain a power of T . From the definition of $\mathcal{N}_j(\cdot, \cdot)$ and (57), we have

$$(69) \quad \|\mathcal{N}(u^N, u^N)\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq C_2 T^{\theta_1} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^2 + 2\|\mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-\alpha,\alpha,T}}.$$

Also, from (47) and (56), we have

$$(70) \quad \|\mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-\alpha,1-\alpha,T}} \leq I_{-\alpha,1-\alpha}^N.$$

Recall that $\eta\Phi \in X_{p,2}^{-\alpha,\alpha}$ a.s. from Proposition 4.1. Moreover, by defining F_1^N and F_2^N on $\mathbb{T} \times \mathbb{R} \times \Omega$ via their Fourier transforms:

$$(71) \quad \begin{aligned} \widehat{F_1^N}(n, \tau) &= \langle n \rangle^{-\frac{1}{2}-\delta} (\sigma_0^{-\frac{3}{2}+\delta} + \sigma_0^{-\frac{1}{2}-\delta}) \int_0^T |\phi_n(r)| d\beta_n(r), \quad \text{and} \\ \widehat{F_2^N}(n, \tau) &= \langle n \rangle^{-\frac{1}{2}-\delta} \chi_{\Omega(n)}(\tau - n^3) \sigma_0^{-\frac{1}{2}+\delta} \int_0^T |\phi_n(r)| d\beta_n(r) \end{aligned}$$

for $|n| \leq N$, we have $F_1^N, F_2^N \in L^2(\Omega; L_{x,t}^2)$ by Ito isometry and Lemma 5.3, which is basically shown in the estimate on (ii). See (61) and (65). Then, from (54) and the estimates on (i) and (ii), we have

$$(72) \quad I_{-\alpha,1-\alpha}^N \leq C_3 (T^{\theta_2} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} I_{-\alpha,1-\alpha}^N + T^{\theta_3} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}),$$

where $L_\omega^N = L^N(F_1^N, F_2^N)(\omega) := \|F_1^N(\omega)\|_{L_{x,t}^2} + \|F_2^N(\omega)\|_{L_{x,t}^2} < \infty$ a.s. Moreover, L_ω^N is non-decreasing in N .

For fixed $R > 0$, choose $T > 0$ small such that $C_3 T^{\theta_2} R \leq \frac{1}{2}$. Then, from (72), we have

$$(73) \quad I_{-\alpha,1-\alpha}^N \leq 2C_3 (T^{\theta_3} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}),$$

for $\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq R$. From (67)~(73), we have

$$(74) \quad \begin{aligned} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} &= \|\Gamma^N u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq C_1 \|u_0^N\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \frac{1}{2} C_2 T^{\theta_1} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^2 \\ &\quad + 2C_3 (T^{\theta_3} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^3 + T^{\theta_4} L_\omega^N \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}) + C_4 \|\eta \Phi^N(\omega)\|_{X_{p,2}^{-\alpha,\alpha}}, \end{aligned}$$

and

$$(75) \quad \begin{aligned} \|u^N - u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} &= \|\Gamma^N u^N - \Gamma^M u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &\leq C_1 \|u_0^N - u_0^M\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \frac{1}{2} C_2 T^{\theta_1} (\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} + \|u^M\|_{X_{p,2}^{-\alpha,\alpha,T}}) \|u^N - u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &\quad + C_5 T^{\theta_3} (\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}^2 + \|u^M\|_{X_{p,2}^{-\alpha,\alpha,T}}^2) \|u^N - u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &\quad + 2C_3 T^{\theta_4} L_\omega^N \|u^N - u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} + 2C_3 T^{\theta_4} \widetilde{L}_\omega^{N,M} \|u^M\|_{X_{p,2}^{-\alpha,\alpha,T}} \\ &\quad + C_4 \|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-\alpha,\alpha}}, \end{aligned}$$

where

$$(76) \quad \widetilde{L}_\omega^{N,M} := \|F_1^N - F_1^M\|_{L_{x,t}^2} + \|F_2^N - F_2^M\|_{L_{x,t}^2}.$$

Note that in estimating the difference $\Gamma^N u^N - \Gamma^M u^M$ on A_1 , one needs to consider

$$(77) \quad \widetilde{I}_{-\alpha,1-\alpha} := \|\mathcal{N}_1(u^N, u^N) - \mathcal{N}_1(u^M, u^M)\|_{-\alpha,1-\alpha}$$

as in [2]. We can follow the argument on pp.135-136 in [2], except for R_α defined in (58), yielding the third term on the right hand side of (75). As for R_α , we can write

$$(78) \quad \mathcal{N}(\mathcal{N}(u, u), u) - \mathcal{N}(\mathcal{N}(v, v), v) = \mathcal{N}(\mathcal{N}(u + v, u - v), u) + \mathcal{N}(\mathcal{N}(v, v), u - v)$$

as in (3.4) in [2], and then we can repeat the computation done for R_α in Estimate on (i), also yielding the third term on the right hand side of (75).

By definition of u_0^N , we have $2C_1 \|u_0^N\|_{\widehat{b}_{p,\infty}^{-\alpha}} \leq 2C_1 \|u_0\|_{\widehat{b}_{p,\infty}^{-\alpha}} + \frac{1}{2}$ for N sufficiently large. Also, since ϕ^N converges to ϕ in $L^p([0, 1] \times \Omega; \widehat{b}_{p,\infty}^{-\alpha,+})$, it follows from Corollary 4.2 and the estimate on (ii) -see (61), (62), and (65)- that $\mathbb{E}[\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-\alpha,\alpha}}]$ and $\mathbb{E}[\widetilde{L}_\omega^{N,\infty}]$ defined in (76) converge to 0. Hence, $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-\alpha,\alpha}} + \widetilde{L}_\omega^{N,\infty} \rightarrow 0$ a.s. after selecting a subsequence (which we still denote with the index N .) Then, by Egoroff's theorem, given $\varepsilon > 0$, there exists a set Ω_ε with $\mathbb{P}(\Omega_\varepsilon^c) < 2^{-1}\varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-\alpha,\alpha}} + \widetilde{L}_\omega^{N,\infty} \rightarrow 0$ uniformly in Ω_ε . In particular, $2C_4 \|\eta \Phi^N\|_{X_{p,2}^{-\alpha,\alpha}} \leq 2C_4 \|\eta \Phi\|_{X_{p,2}^{-\alpha,\alpha}} + \frac{1}{2}$ for large N uniformly on Ω_ε . In the following, we will work on Ω_ε .

Now, let $R_\omega = 2(C_1 \|u_0\|_{\widehat{b}_{p,\infty}^{-\alpha}} + C_4 \|\eta \Phi(\omega)\|_{X_{p,2}^{-\alpha,\alpha}}) + 1$, and define the stopping time T_ω by

$$(79) \quad T_\omega = \inf\{T > 0 : \max(C_3 T^{\theta_2} R_\omega, P_1(T, R_\omega, \omega), P_2(T, R_\omega, \omega)) \geq \frac{1}{2}\},$$

where

$$\begin{cases} P_1(T, R_\omega, \omega) = \frac{1}{2} C_2 T^{\theta_1} R_\omega + 2C_3 T^{\theta_3} (R_\omega)^2 + 2C_3 T^{\theta_4} L_\omega, & \text{from (74)} \\ P_2(T, R_\omega, \omega) = C_2 T^{\theta_1} R_\omega + 2C_5 T^{\theta_3} (R_\omega)^2 + 2C_3 T^{\theta_4} L_\omega, & \text{from (75)}. \end{cases}$$

The first condition in the definition of T_ω guarantees (73), and hence (74) and (75), for $\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq R_\omega$. The second condition along with (74) indeed guarantees that

$$(80) \quad \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq R_\omega$$

for $T \leq T_\omega$ from the following observation. Since we have the temporal regularity $b = \alpha < \frac{1}{2}$, we have $\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} = \|\chi_{[0,T]} u^N\|_{X_{p,2}^{-\alpha,\alpha}}$, where $\chi_{[0,T]}$ denotes the characteristic function of the time interval $[0, T]$. See Bourgain [3]. Hence, $\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}}$ is continuous in T since

$$(81) \quad \left| \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T+\delta}} - \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T}} \right| \leq \|u^N\|_{X_{p,2}^{-\alpha,\alpha}[T,T+\delta]} \lesssim \delta^\theta \|u^N\|_{X^{0,-\frac{1}{2}}[T,T+\delta]}$$

for sufficiently small $\delta > 0$. Note that the last term in (81) is finite for small δ since the local-in-time solutions constructed in [8] are controlled in this norm (indeed in a stronger norm adapted to the Besov space $B_{2,1}^{0,-}$.) Then, (80) follows from (74), the second condition in (79), and the continuity of the norm in T since (80) clearly holds at $T = 0$.

From (75) along with the third condition in (79), we have

$$(82) \quad \begin{aligned} \|u^N - u^M\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} &\leq 2C_1 \|u_0^N - u_0^M\|_{\widehat{b}_{p,\infty}^{-\alpha}} + 4C_3 T^{\theta_4} R_\omega \widetilde{L}_\omega^{N,M} \\ &\quad + 2C_4 \|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-\alpha,\alpha}}. \end{aligned}$$

The right hand side of (82) goes to 0 as $N, M \rightarrow \infty$ since u_0^N is Cauchy in $\widehat{b}_{p,\infty}^{-\alpha}$ and $\|\eta(\Phi^N - \Phi^M)\|_{X_{p,2}^{-\alpha,\alpha}} + \widetilde{L}_\omega^{N,M} \rightarrow 0$ on Ω_ε uniformly in N, M . Let u denote the limit in $X_{p,2}^{-\alpha,\alpha,T_\omega}$.

In the following, we give a brief discussion to show that the limit u is a solution to (4). Clearly, $S(t)u_0^N$ and $\eta\Phi^N$ converge to $S(t)u_0$ and $\eta\Phi$ in $X_{p,2}^{-\alpha,\alpha,T_\omega}$. It follows from (57) that $\mathcal{N}_0(u^N, u^N)$ converges $\mathcal{N}_0(u, u)$ in $X_{p,2}^{-\alpha,\alpha,T_\omega}$. In view of (73), (75), and (77), we see that $\mathcal{N}_j(u^N, u^N)$ is Cauchy in a slightly stronger space $X_{p,2}^{-\alpha,1-\alpha,T_\omega}$, $j = 1, 2$. Let v_j denote the corresponding limit. Thus, from (67), we have

$$(83) \quad u = S(t)u_0 - \frac{1}{2}\mathcal{N}_0(u, u) - \frac{1}{2}(v_1 + v_2) + \eta\Phi.$$

Now, we need to show that $\mathcal{N}_j(u^N, u^N)$ indeed converges to $\mathcal{N}_j(u, u)$, $j = 1, 2$. By symmetry, we only consider $\mathcal{N}_1(u, u) - \mathcal{N}_1(u^N, u^N)$. As before, we substitute (83) (and (67)) in the first factor u (and u^N) of $\mathcal{N}_1(\cdot, \cdot)$, respectively. There are three contributions to consider.

- **(A)** Contribution from the stochastic terms: We have

$$(84) \quad \mathcal{N}_1(\eta\Phi, u) - \mathcal{N}_1(\eta\Phi^N, u^N) = \mathcal{N}_1(\eta(\Phi - \Phi^N), u) + \mathcal{N}_1(\eta\Phi^N, u - u^N).$$

From Estimate on (ii), we have

$$\|(84)\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \lesssim \widetilde{L}_\omega^{N,\infty} \|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} + L_\omega^N \|u^N - u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \rightarrow 0$$

as $N \rightarrow \infty$, since $\|u\|_{X_{p,2}^{-\alpha,\alpha,T}} \leq R_\omega$ and $\widetilde{L}_\omega^{N,\infty} \rightarrow 0$ uniformly on Ω_ε .

- **(B)** Contribution from $\mathcal{N}_0(\cdot, \cdot)$: In this case, we consider

$$(85) \quad \mathcal{N}_1(\mathcal{N}_0(u, u), u) - \mathcal{N}_1(\mathcal{N}_0(u^N, u^N), u^N).$$

Note that we have $\sigma_1 \geq \sigma_0, \sigma_2, \sigma_3, \sigma_4$ from the definition of $\mathcal{N}_1(\cdot, \cdot)$ and $\mathcal{N}_0(\cdot, \cdot)$. See (50) and (52). Indeed, we have $\sigma_1 \geq \sigma_0, \sigma_2$ since we are on A_1 defined in (45), and also $\sigma_1 \geq \sigma_3, \sigma_4$

since we are on the support of $\mathcal{N}_0(\cdot, \cdot)$ in the first factor of $\mathcal{N}_1(\cdot, \cdot)$. Once again, one can easily follow the argument on p.136 in [2] and show

$$\|(85)\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \lesssim (\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}}^2 + \|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}}^2) \|u^N - u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \rightarrow 0.$$

In treating $R_\alpha - R_\alpha^N$ defined in (58), one needs to proceed as before, using (78) and Estimate on (i).

• **(C)** Contribution from v_j and $\mathcal{N}_j(u^N, u^N)$, $j = 1$ or 2 : By symmetry, assume $j = 1$. In this case, we have $\sigma_1 \geq \sigma_0, \sigma_2$ but $\sigma_3 \geq \sigma_1, \sigma_4$. i.e. we control (54) by the first term on the right hand side. See (II.1) on p.126 in [2]. Now, we need to estimate

$$(86) \quad \begin{aligned} & \mathcal{N}_1(v_1, u) - \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u^N) \\ &= \mathcal{N}_1(v_1 - \mathcal{N}_1(u^N, u^N), u) + \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u - u^N) =: \text{I} + \text{II}. \end{aligned}$$

Then, by proceeding as in [2] with (56) and (73), we have

$$\|\text{II}\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} \lesssim I_{-\alpha,1-\alpha}^N \|u - u^N\|_{X^{-(1-\alpha),\alpha,T_\omega}} \lesssim \|u - u^N\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \rightarrow 0.$$

By proceeding as in (II.1) in [2] with $|n_1|^\alpha$ replaced by $|n_1|^{1-\alpha}$, followed by (56), we have

$$\begin{aligned} \|\text{I}\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{-(1-\alpha),1-\alpha} \|u\|_{-(1-\alpha),\alpha} \\ &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} \|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \rightarrow 0 \end{aligned}$$

since $v_1 = \lim_{N \rightarrow \infty} \mathcal{N}_1(u^N, u^N)$ in $X_{p,2}^{-\alpha,1-\alpha,T_\omega}$ by definition.

Hence, we have $u = \Gamma_{u_0} u$ for each $\omega \in \Omega_\varepsilon$. i.e. u is a mild solution to (2) on $[0, T_\omega]$. Let $\Omega^{(1)} = \Omega_\varepsilon$. Now, we can recursively construct $\Omega^{(j+1)} \subset \Omega \setminus \bigcup_{k=1}^j \Omega^{(k)}$ for $j = 1, 2, \dots$ with $\mathbb{P}(\Omega \setminus \bigcup_{k=1}^j \Omega^{(k)}) < 2^{-j}\varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-\alpha,\alpha}}$ and $\tilde{L}_\omega^{N,\infty}$ converge to 0 uniformly in each $\Omega^{(j)}$. Then, by repeating the argument, we can construct a solution u on $\bigcup_{j=1}^\infty \Omega^{(j)}$. Note that $\mathbb{P}(\Omega \setminus \bigcup_{j=1}^\infty \Omega^{(j)}) = 0$.

We have constructed a solution u to (2) in $X_{p,2}^{-\alpha,\alpha,T_\omega}$ with $u_0 \in \widehat{b}_{p,\infty}^{-\alpha'}$. Since u is a solution, the a priori estimate (74) holds with the regularity $(s, b) = (-\alpha', \alpha')$ in place of $(-\alpha, \alpha)$. Then, we easily see that $u \in X_{p,2}^{-\alpha',\alpha',T_\omega}$, by redefining R_ω and T_ω with this regularity. In the remaining of the paper, we work only with the spatial regularity $s = -\alpha'$, i.e. there is no approximating sequences any more. Hence, for notational simplicity, we will use $-\alpha$ in place of $-\alpha'$ to denote the spatial regularity of the solution in the following.

We still need to take care of several issues. Note that the temporal regularity $b = \alpha = \frac{1}{2} - \delta$ of the solution u is less than $\frac{1}{2}$. In particular, we need to show that the solution u is continuous from $[0, T_\omega]$ into $\widehat{b}_{p,\infty}^{-\alpha}$. We also need to show its uniqueness and continuous dependence on the initial data.

From Proposition 4.5, $\eta\Phi \in C([0, T_\omega]; \widehat{b}_{p,\infty}^{-\alpha})$ a.s. Also, it follows from (68) with $b = \frac{1}{2} + \delta$, (70), (73), and symmetry on σ_1 and σ_2 , that

$$S(t)u_0 + \mathcal{N}_1(u, u) + \mathcal{N}_2(u, u) \in X_{p,2}^{-\alpha, \frac{1}{2} + \delta, T_\omega} \subset C([0, T_\omega]; \widehat{b}_{p,\infty}^{-\alpha})$$

a.s. Now, we consider $\mathcal{N}_0(u, u)$, i.e. when $\sigma_0 = \text{MAX}$. Note that the contribution comes only from $\mathcal{M}_2(u, u)$ defined in (51). Let $\mathcal{N}_3(u, u)$ denotes the contribution of $\mathcal{N}_0(u, u)$ on $\{\max(\sigma_1, \sigma_2) \gtrsim \langle nn_1 n_2 \rangle^{\frac{1}{100}}\}$, and $\mathcal{N}_4(u, u) = \mathcal{N}_0(u, u) - \mathcal{N}_3(u, u)$.

• **Case (a):** First, we consider $\mathcal{N}_3(u, u)$. i.e. $\max(\sigma_1, \sigma_2) \gtrsim \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Say $\sigma_1 \gtrsim \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Then, by Lemma 3.2 and (15), we have

$$\|\mathcal{N}_3(u, u)\|_{X_{p,2}^{-\alpha, \frac{1}{2}+\delta, T_\omega}} \lesssim \|\partial_x(u^2)\|_{X_{p,2}^{-\alpha, -\frac{1}{2}+\delta, T_\omega}} \lesssim \|\partial_x(u^2)\|_{X^{-\alpha, -\frac{1}{2}+\delta, T_\omega}}$$

Then, by duality and (44), we have

$$\begin{aligned} &= \sup_{\|d\|_{L_{\tilde{n},\tau}^2}=1} \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^{\frac{1}{2}-\delta}} \prod_{j=1}^2 \frac{\langle n_j \rangle^{1-\alpha} c(n_j, \tau_j)}{\sigma_j^\alpha} d\tau d\tau_1 \\ &\lesssim \sup_{\|d\|_{L_{\tilde{n},\tau}^2}=1} \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n, \tau) \frac{c(n_1, \tau_1)}{\sigma_1^{\alpha-200\delta}} \frac{c(n_2, \tau_2)}{\sigma_2^\alpha} d\tau d\tau_1 \end{aligned}$$

where $c(n, \tau)$ is defined in (49). Then, by $L_{x,t}^2, L_{x,t}^4, L_{x,t}^4$ -Hölder inequality along with Lemma 3.3, (49), and (56),

$$\leq \|c\|_{L_{\tilde{n},\tau}^2}^2 \leq \|u\|_{X^{-(1-\alpha), \alpha}}^2 \lesssim \|u\|_{X_{p,2}^{-\alpha, \alpha}}^2 < \infty.$$

• **Case (b):** Now, consider $\mathcal{N}_4(u, u)$. i.e. $\max(\sigma_1, \sigma_2) \ll \langle nn_1n_2 \rangle^{\frac{1}{100}}$. Note that it suffices to show that $\mathcal{N}_0(u, u) \in X_{p,1}^{-\alpha, 0, T_\omega}$, since $X_{p,1}^{-\alpha, 0, T_\omega} \subset C([0, T_\omega]; \widehat{b}_{p,\infty}^{-\alpha})$. Then, by Cauchy-Schwarz inequality, Lemma 5.3 and duality, we have

$$\begin{aligned} \|\mathcal{N}_4(u, u)\|_{X_{p,1}^{-\alpha, 0, T_\omega}} &\leq \|\partial_x(u^2)\|_{X_{2,1}^{-\alpha, -1, T_\omega}} \leq \left\| \|\langle n \rangle^{-\alpha} \langle \tau - n^3 \rangle^{-1} \chi_{\Omega(n)}(\tau - n^3) \widehat{\partial_x(u^2)}(n, \tau) \|_{L_\tau^1} \right\|_{L_n^2} \\ &\leq \|\langle \tau - n^3 \rangle^{-\frac{1}{2}+\delta} \chi_{\Omega(n)}(\tau - n^3)\|_{L_\tau^2} \|\partial_x(u^2)\|_{-\alpha, -\frac{1}{2}-\delta} \\ &\lesssim \sup_{\|d\|_{L_{\tilde{n},\tau}^2}=1} \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{1-\alpha} d(n, \tau)}{\sigma_0^{\frac{1}{2}+\delta}} \prod_{j=1}^2 \frac{\langle n_j \rangle^{1-\alpha} c(n_j, \tau_j)}{\sigma_j^\alpha} d\tau d\tau_1 \\ &\lesssim \sup_{\|d\|_{L_{\tilde{n},\tau}^2}=1} \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n, \tau) \frac{c(n_1, \tau_1)}{\sigma_1^\alpha} \frac{c(n_2, \tau_2)}{\sigma_2^\alpha} d\tau d\tau_1. \end{aligned}$$

The rest follows as before. Hence, the solution u is continuous from $[0, T_\omega]$ to $\widehat{b}_{p,\infty}^{-\alpha}$.

Lastly, we show the uniqueness and the continuous dependence of the solutions on the initial data. Let u and v be the mild solutions of (2) on $[0, T_\omega]$ with initial data u_0 and v_0 respectively. i.e.

$$(87) \quad u - v = \Gamma_{u_0} u - \Gamma_{v_0} v = S(t)(u_0 - v_0) - \frac{1}{2}(\mathcal{N}(u, u) - \mathcal{N}(v, v)),$$

where Γ is defined in (67). Moreover, assume that

$$(88) \quad \|u_0\|_{\widehat{b}_{p,\infty}^{-\alpha}}, \|v_0\|_{\widehat{b}_{p,\infty}^{-\alpha}}, \|u\|_{X_{p,2}^{-\alpha, \alpha, T_\omega}}, \|v\|_{X_{p,2}^{-\alpha, \alpha, T_\omega}} \leq R.$$

Let $\widetilde{\mathcal{N}}_j(u, v) := -\frac{1}{2}(\mathcal{N}_j(u, u) - \mathcal{N}_j(v, v))$ for $j = 1, \dots, 4$. First, note that $\|\widetilde{\mathcal{N}}_4(u, v)\|_{X_{p,1}^{-\alpha, \varepsilon, T_\omega}} \lesssim R^2 < \infty$ from (a slight variation of) Case (b), and we have

$$\|(u - v) - \widetilde{\mathcal{N}}_4(u, v)\|_{X_{p,1}^{-\alpha, \varepsilon, T_\omega}} \leq \left\| S(t)(u_0 - v_0) + \sum_{j=1}^3 \widetilde{\mathcal{N}}_j(u, v) \right\|_{X_{p,2}^{-\alpha, \frac{1}{2}+\delta, T_\omega}} \lesssim C_1(R) < \infty$$

by Cauchy-Schwarz inequality with $\varepsilon < \delta$, followed by (68), (70), (73), Case (a), and (88). Then, by interpolation and Cauchy-Schwarz inequality, we have

$$(89) \quad \begin{aligned} \|u - v\|_{C([0, T_\omega]; \widehat{b}_{p, \infty}^-)} &\lesssim \|u - v\|_{X_{p, 1}^{-\alpha, 0, T_\omega}} \lesssim \|u - v\|_{X_{p, 1}^{-\alpha, -\delta, T_\omega}}^\beta \|u - v\|_{X_{p, 1}^{-\alpha, \varepsilon, T_\omega}}^{1-\beta} \\ &\lesssim C_2(R) \|u - v\|_{X_{p, 2}^{-\alpha, \frac{1}{2} - \delta, T_\omega}}^\beta \end{aligned}$$

with $\beta = \frac{\varepsilon}{\varepsilon + \delta} \in (0, 1)$. From (68) and the nonlinear estimates (see (69), (73), (75), (77)), we have

$$\|u - v\|_{X_{p, 2}^{-\alpha, \frac{1}{2} - \delta, T_\omega}} \lesssim \|u_0 - v_0\|_{\widehat{b}_{p, \infty}^-} + C_3(R) T_\omega^\theta \|u - v\|_{X_{p, 2}^{-\alpha, \frac{1}{2} - \delta, T_\omega}}.$$

Hence, for sufficiently small $T > 0$, we have

$$(90) \quad \|u - v\|_{X_{p, 2}^{-\alpha, \frac{1}{2} - \delta, T_\omega}} \lesssim \|u_0 - v_0\|_{\widehat{b}_{p, \infty}^-}.$$

Therefore, it follows from (89) and (90) that the solution map is Hölder continuous with the bound

$$\|u - v\|_{C([0, T_\omega]; \widehat{b}_{p, \infty}^-)} \leq C_4(R) \|u_0 - v_0\|_{\widehat{b}_{p, \infty}^-}^\beta.$$

In particular, the solution is unique. This completes the proof of Theorem 1. □

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TADAIRO OH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE ST, RM 6290, TORONTO, ON M5S 2E4, CANADA
E-mail address: oh@math.toronto.edu