

# INVARIANCE OF THE GIBBS MEASURE FOR THE SCHRÖDINGER-BENJAMIN-ONO SYSTEM

TADAIRO OH\*

**Key words.** Schrödinger, Benjamin-Ono, Gibbs measure, a.s global well-posedness, ill-posedness

**AMS subject classifications.** 35Q53, 35G25

**Abstract.** We prove the invariance of the Gibbs measure for the periodic Schrödinger-Benjamin-Ono system (when the coupling parameter  $|\gamma| \neq 0, 1$ ) by establishing a new local well-posedness in a modified Sobolev space and constructing the Gibbs measure (which is in the sub- $L^2$  setting for the Benjamin-Ono part.) We also show the ill-posedness result in  $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  for  $s < \frac{1}{2}$  when  $|\gamma| \neq 0, 1$  and for any  $s \in \mathbb{R}$  when  $|\gamma| = 1$ .

**1. Introduction.** In this paper, we consider the Schrödinger-Benjamin-Ono (SBO) system:

$$\begin{cases} iu_t + u_{xx} = \alpha v u \\ v_t + \gamma \mathcal{H} v_{xx} = \beta (|u|^2)_x, & (x, t) \in \mathbb{T} \times \mathbb{R} \\ (u, v)|_{t=0} = (u_0, v_0) \end{cases} \quad (1.1)$$

where  $\mathbb{T} = [0, 2\pi)$ ,  $u$  is a complex valued function,  $v$  is a real-valued function, and  $\alpha, \beta, \gamma$  are nonzero real constants. In (1.1),  $\mathcal{H}$  denotes the Hilbert transform whose Fourier multiplier is given by  $-i \operatorname{sgn}(n)$ .  $D = |\partial_x| = \mathcal{H} \partial_x$  is defined via  $\widehat{Df}(n) = |n| \widehat{f}(n)$ .

The system (1.1) appears in Funakoshi-Oikawa [18], describing the motion of two fluids with different densities under capillary-gravity waves in a deep water flow. The Schrödinger part describes the short surface wave, and the Benjamin-Ono part describes the long internal wave. The system also appears in the sonic-Langmuir wave interaction in plasma physic (Karpman [22]), in the capillary gravity interaction waves (Djordjevic-Redekopp [16], Grimshaw [20]), and in the general theory of water wave interaction in a nonlinear medium (Benney [3, 4].)

The several conservation laws are known for the SBO system:

$$E_1(v) = \int v \, dx, \quad E_2(u) = \int |u|^2 \, dx, \quad E_3(u, v) = \operatorname{Im} \int u \bar{u}_x \, dx + \frac{\alpha}{2\beta} \int v^2 \, dx \quad (1.2)$$

$$\text{and} \quad H(u, v) = \frac{1}{2} \int |u_x|^2 \, dx - \frac{\alpha\gamma}{4\beta} \int (D^{\frac{1}{2}} v)^2 \, dx + \frac{\alpha}{2} \int v |u|^2 \, dx. \quad (1.3)$$

Note that  $H(u, v)$  is the Hamiltonian for (1.1) and indeed, in terms of the Hamiltonian formulation, (1.1) can be written as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & \frac{2\beta}{\alpha} \partial_x \end{pmatrix} \frac{dH(u, v)}{d(\bar{u}, v)}, \quad (1.4)$$

where  $\frac{dH}{d(\bar{u}, v)}$  is the Gateaux derivative with respect to the  $L^2$  inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \operatorname{Re} \int u_1 \bar{u}_2 \, dx + \int v_1 v_2 \, dx.$$

---

\*Department of Mathematics, University of Toronto, 40 St. George St, Rm 6290, Toronto, ON M5S 2E4, Canada

In this formulation, one of the natural questions is; Is the Gibbs measure of the form  $d\mu = Z^{-1}e^{-\nu H(u,v)} \prod_{x \in \mathbb{T}} du(x) \otimes dv(x)$  invariant under the flow of (1.1)? At this point,  $d\mu$  is merely a formal expression. It is known (c.f. Zhidkov [36]) that when  $\frac{\alpha\gamma}{\beta} < 0$ , the Gaussian part of the Gibbs measure, i.e.

$$d\rho = \tilde{Z}^{-1} \exp\left(-\nu\left(\frac{1}{2} \int |u_x|^2 - \frac{\alpha\gamma}{4\beta} \int (D^{\frac{1}{2}}v)^2\right)\right) \prod_{x \in \mathbb{T}} du(x) \otimes dv(x), \quad (1.5)$$

is supported in  $\cap_{s < \frac{1}{2}} H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$ . Following Bourgain [8], we show that the Gibbs measure  $d\mu$  is a well-defined probability measure with a suitable cutoff in terms of the  $L^2$  norm of  $u$  and the mean of  $v$  (Lemma 4.5, Corollary 4.6.)

Now, let's turn to the well-posedness theory of (1.1). In the non-periodic setting, several results are known. When  $|\gamma| \neq 1$ , Bekiranov-Ogawa-Ponce [2] showed that (1.1) is locally well-posed in  $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$  for  $s \geq 0$ . When  $|\gamma| = 1$ , Pecher [32] showed the local well-posedness for  $s > 0$ . In view of the conservation laws, when  $\frac{\alpha\gamma}{\beta} < 0$ , these local results extend to the global ones for  $s \geq 1$ . Using the  $I$ -method developed by Colliander-Keel-Staffilani-Takaoka-Tao [14, 15], Pecher [32] also proved the global well-posedness for  $s > \frac{1}{3}$  when  $\frac{\alpha\gamma}{\beta} < 0$ . Note that the  $I$ -method automatically provides a polynomial upper bound on the time growth of the norm of the solutions. Recently, when  $|\gamma| \neq 1$ , Angulo-Matheus-Pilod proved the global well-posedness for  $s = 0$  (without assuming  $\frac{\alpha\gamma}{\beta} < 0$ ), following the method developed by Colliander-Holmer-Tzirakis [13]. The method is based on estimating the doubling time of  $\|v(t)\|_{H_x^{-\frac{1}{2}}}$  in terms of the conserved quantity  $\|u(t)\|_{L_x^2}$ . Note that this method provides an exponential upper bound on the time growth of  $\|v(t)\|_{H_x^{-\frac{1}{2}}}$ .

In the periodic setting, there seems to be only few results known at this point. Assuming  $|\gamma| \neq 0, 1$ , Angulo-Matheus-Pilod [1] showed that (1.1) is locally well-posed in  $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  for  $s \geq \frac{1}{2}$ . They also showed the existence and stability of the periodic traveling wave solutions. In [1], the local well-posedness is established via contraction argument by establishing the following bilinear estimates:

$$\|uv\|_{X^{s, -\frac{1}{2}+}} \lesssim \|u\|_{X^{s, \frac{1}{2}}} \|v\|_{X^{s-\frac{1}{2}, \frac{1}{2}}} \quad (1.6)$$

$$\|\partial_x(u_1 \bar{u}_2)\|_{X^{s-\frac{1}{2}, -\frac{1}{2}+}} \lesssim \|u_1\|_{X^{s, \frac{1}{2}}} \|u_2\|_{X^{s, \frac{1}{2}}} \quad (1.7)$$

for  $s \geq \frac{1}{2}$  where  $X^{s,b}$  and  $X_\gamma^{s,b}$  are the Bourgain spaces corresponding to the linear parts of Schrödinger and Benjamin-Ono equations whose norms are given by

$$\|u\|_{X^{s,b}} = \|\langle n \rangle^s \langle \tau + n^2 \rangle^b \widehat{u}(n, \tau)\|_{L_{n,\tau}^2} \quad (1.8)$$

$$\|v\|_{X_\gamma^{s,b}} = \|\langle n \rangle^s \langle \tau + \gamma |n| n \rangle^b \widehat{v}(n, \tau)\|_{L_{n,\tau}^2}, \quad (1.9)$$

where  $\langle \cdot \rangle = 1 + |\cdot|$ . It is also shown in [1] that both estimates (1.6) and (1.7) fail for  $s < \frac{1}{2}$  when  $|\gamma| \neq 0, 1$  and that they fail for any  $s \in \mathbb{R}$  when  $|\gamma| = 1$ .

In Appendix, we show that the solution map of (1.1) is not smooth for  $s < \frac{1}{2}$  when  $|\gamma| \neq 0, 1$  and for all  $s \in \mathbb{R}$  when  $|\gamma| = 1$ . More precisely, let  $\Phi^t : (u_0, v_0) \mapsto (u(t), v(t)) \in H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  be the solution map of (1.1) for  $|t| \ll 1$ . Then,

**THEOREM 1.1.** (a) *Let  $|\gamma| \neq 0, 1$ . If the solution map  $\Phi^t$  is  $C^2$  on  $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$ , then  $s \geq \frac{1}{2}$ .* (b) *If  $|\gamma| = 1$ , then the solution map can never be  $C^2$  for any  $s \in \mathbb{R}$ .*

In particular, Theorem 1.1 (a) states that the flow of (1.1) via the usual contraction argument with the Bourgain norm is *not* defined in  $\cap_{s < \frac{1}{2}} H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  containing the support of the Gibbs measure  $\mu$ . Thus, we need to seek for a new local well-posedness result in a space containing the support of  $\mu$ .

From the standard argument (c.f. Kenig-Ponce-Vega [23]), the proof of the bilinear estimate  $\|uv\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X_\gamma^{s-\frac{1}{2},b}}$  with  $b = \frac{1}{2}$  or  $\frac{1}{2}+$  comes down to establishing an effective upper bound (which needs to be no more than  $\sim 1$  in view of Bourgain's periodic  $L^4$  Strichartz estimate [5]). See Lemma 3.3) on

$$\frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^{s-\frac{1}{2}}} \frac{1}{\max(\langle \tau + n^2 \rangle^{1-b}, \langle \tau_1 + n_1^2 \rangle^b, \langle \tau_2 + \gamma |n_2| n_2 \rangle^b)}, \quad (1.10)$$

where  $n = n_1 + n_2$  and  $\tau = \tau_1 + \tau_2$  with  $n, n_1, n_2 \in \mathbb{Z}$ . Note that

$$\begin{aligned} \max(\langle \tau + n^2 \rangle, \langle \tau_1 + n_1^2 \rangle, \langle \tau_2 + \gamma |n_2| n_2 \rangle) &\gtrsim |-(\tau + n^2) + (\tau_1 + n_1^2) + (\tau_2 + \gamma |n_2| n_2)| \\ &= |n_2| |R_n(n_2)|, \end{aligned} \quad (1.11)$$

where  $R_n(n_2) = (\gamma \operatorname{sgn}(n_2) + 1)n_2 - 2n$ . For simplicity, assume  $n, n_2 > 0$ . Then, when  $n_2 \sim \frac{2n}{1+\gamma}$ , we have  $R_n(n_2) \sim 0$ . In particular, if  $|\gamma| \neq 1$ , then we have  $|n| \sim |n_1| \sim |n_2|$  in such a situation. Then, we have (1.10)  $\lesssim \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^{s-\frac{1}{2}}} \sim \langle n \rangle^{-s+\frac{1}{2}}$ , which forces us to restrict ourselves to the case  $s \geq \frac{1}{2}$ . However, note that for each  $n \in \mathbb{Z}$ , there are at most two values of  $n_2$ , i.e.  $n_2 = \lfloor \frac{2n}{1+\gamma} \rfloor, \lfloor \frac{2n}{1+\gamma} \rfloor + 1$ , which make  $R_n(n_2) \sim 0$ . For all other values of  $n_2$ , we have  $|R_n(n_2)| \gtrsim 1$ . Thus, with  $b = \frac{1}{2}$ , we have (1.10)  $\lesssim \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^{s-\frac{1}{2}} |n_2|^{\frac{1}{2}}} \sim \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \lesssim 1$ , on  $A = \{(n, n_1, n_2) : n = n_1 + n_2, |n_2 - \frac{2n}{1+\gamma}| > 1\}$  as long as  $s \geq 0$ .

This motivates us to consider the initial value problem (1.1) with the initial condition  $(u_0, v_0) \in H^{s_1, s_2}(\mathbb{T}) := H^{s_1, s_2}(\mathbb{T}) \times H^{s_1-\frac{1}{2}, s_2-\frac{1}{2}}(\mathbb{T})$ , where  $H^{s_1, s_2}$  is defined via the norm given by

$$\|\phi\|_{H^{s_1, s_2}} = \|\phi\|_{H^{s_1}} + \sup_n \langle n \rangle^{s_2} |\widehat{\phi}(n)| < \infty \quad (1.12)$$

for some  $s_1, s_2$  with  $0 < s_1 = \frac{1}{2}- < \frac{1}{2} < s_2 = 1- < 1$  (with some additional conditions to be determined later.)

From the above heuristic argument, we see that  $s \geq 0$  is enough to establish the crucial bilinear estimates on  $A$ . In particular,  $s_1 = \frac{1}{2}-$  is a sufficient regularity on  $A$ . On the other hand, the resonances at  $n_2 = \lfloor \frac{2n}{1+\gamma} \rfloor, \lfloor \frac{2n}{1+\gamma} \rfloor + 1$  forces  $s \geq \frac{1}{2}$ . However, for each fixed  $n \in \mathbb{Z}$ , there are only two values of  $n_2$  causing the resonances, which can be controlled by  $\sup_n \langle n \rangle^{s_2} |\widehat{\phi}(n)|$  in (1.12) with the higher regularity  $s_2 = 1- > \frac{1}{2}$ . Then, via a contraction in the modified Bourgain space, we prove

**THEOREM 1.2.** *Let  $s_1 = \frac{1}{2}-, s_2 = 1-$  with  $s_2 < 2s_1$ . Assume  $|\gamma| \neq 0, 1$ . Then, the SBO system (1.1) is locally well-posed in  $H^{s_1, s_2}(\mathbb{T})$ .*

Note that  $H^{s_1, s_2} \subsetneq H^{s_1} \times H^{s_1-\frac{1}{2}}$ . However, from the theory of abstract Wiener spaces (Gross [21], Kuo [24]), the Gaussian measure  $d\rho$  in (1.5) is a countably additive probability measure supported on  $H^{s_1, s_2}$  for  $0 < s_1 < \frac{1}{2} < s_2 < 1$ . (See Bourgain [8] for mKdV and [9] for the Zakharov system.) Then, a slight modification Bourgain's argument [8] (for super-cubic NLS) with Theorem 1.2 yields the following result.

**THEOREM 1.3.** *Let  $\frac{\alpha\gamma}{\beta} < 0$ . The Gibbs measure  $\mu$  for the SBO system (1.1) is invariant under the flow. Moreover, (1.1) is globally well-posed almost surely on the statistical ensemble.*

**2. Notations.** On  $\mathbb{T}$ , the spatial Fourier domain is  $\mathbb{Z}$ . Let  $dn$  be the normalized counting measure on  $\mathbb{Z}$ , and we say  $f \in L^p(\mathbb{Z})$ ,  $1 \leq p < \infty$ , if

$$\|f\|_{L^p(\mathbb{Z})} = \left( \int_{\mathbb{Z}} |f(n)|^p dn \right)^{\frac{1}{p}} := \left( \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{\frac{1}{p}} < \infty.$$

If  $p = \infty$ , we have the obvious definition involving the supremum. We often drop  $2\pi$  for simplicity. If the function depends on both  $x$  and  $t$ , we use  $\wedge_x$  (and  $\wedge_t$ ) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use  $\wedge$  to denote the spatial Fourier transform, temporal Fourier transform, and the space-time Fourier transform, depending on the context.

Let  $X^{s,b}$  and  $X_\gamma^{s,b}$  be as in (1.8) and (1.9). Given any time interval  $I = [t_1, t_2] \subset \mathbb{R}$ , we define the local in time  $X^{s,b}(\mathbb{T} \times I)$  (or simply  $X^{s,b}[t_1, t_2]$ ) by

$$\|u\|_{X^{s,b}_I} = \|u\|_{X^{s,b}(\mathbb{T} \times I)} = \inf \{ \|\tilde{u}\|_{X^{s,b}(\mathbb{T} \times \mathbb{R})} : \tilde{u}|_I = u \}.$$

We define the local-in-time versions of other function spaces analogously.

Let  $\eta \in C_c^\infty(\mathbb{R})$  be a smooth cutoff function supported on  $[-2, 2]$  with  $\eta \equiv 1$  on  $[-1, 1]$  and let  $\eta_T(t) = \eta(T^{-1}t)$ . We use  $c, C$  to denote various constants, usually depending only on  $s_1, s_2, b, \alpha, \beta$ , and  $\gamma$ . If a constant depends on other quantities, we will make it explicit. We use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$ . Similarly, we use  $A \sim B$  to denote  $A \lesssim B$  and  $B \lesssim A$  and use  $A \ll B$  when there is no general constant  $C$  such that  $B \leq CA$ . We also use  $a+$  (and  $a-$ ) to denote  $a + \varepsilon$  (and  $a - \varepsilon$ ), respectively, for arbitrarily small  $\varepsilon \ll 1$ .

**3. New local well-posedness on the modified Sobolev space.** In this section, we establish the local well-posedness of (1.1) with the initial data in  $\mathbf{H}^{s_1, s_2} = H^{s_1, s_2} \times H^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}}$  for  $s_1 = \frac{1}{2}-$  and  $s_2 = 1-$  via a contraction on a ball in the modified Bourgain space  $X^{s_1, s_2, b} \times X_\gamma^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, b}$ . First, define the modified Bourgain spaces  $X^{s_1, s_2, b}$  and  $X_\gamma^{s_1, s_2, b}$  whose norms are given by

$$\|u\|_{X^{s_1, s_2, b}} = \|u\|_{X^{s_1, b}} + \|u\|_{X^{s_2, \infty, b}} \quad (3.1)$$

$$\|v\|_{X_\gamma^{s_1, s_2, b}} = \|v\|_{X_\gamma^{s_1, b}} + \|v\|_{X_\gamma^{s_2, \infty, b}}, \quad (3.2)$$

where  $X^{s_1, b}$  and  $X_\gamma^{s_1, b}$  are defined in (1.8) and (1.9), and

$$\|u\|_{X^{s_2, \infty, b}} = \|\langle n \rangle^{s_2} \langle \tau + n^2 \rangle^b \widehat{u}(n, \tau)\|_{L_n^\infty L_\tau^2} \quad (3.3)$$

$$\|v\|_{X_\gamma^{s_2, \infty, b}} = \|\langle n \rangle^{s_2} \langle \tau + \gamma |n| \rangle^b \widehat{v}(n, \tau)\|_{L_n^\infty L_\tau^2}. \quad (3.4)$$

Recall that when  $b > \frac{1}{2}$ , the  $X^{s_1, b} \times X_\gamma^{s_1 - \frac{1}{2}, b}$  norm controls the  $C([-T, T]; H^{s_1} \times H^{s_1 - \frac{1}{2}})$  norm. Also, when  $b > \frac{1}{2}$ , by Sobolev embedding, we have

$$\sup_n \langle n \rangle^{s_2} |\widehat{u}(n, t)| = \sup_n \langle n \rangle^{s_2} |e^{in^2 t} \widehat{u}(n, t)| \lesssim \sup_n \langle n \rangle^{s_2} \|e^{in^2 t} \widehat{u}(n, t)\|_{H_t^b} = \|u\|_{X^{s_2, \infty, b}},$$

for any  $t \in \mathbb{R}$ . A similar result holds if we replace  $\langle \tau + n^2 \rangle^b$  by  $\langle \tau + \gamma |n| \rangle^b$ . Hence, the  $X^{s_1, s_2, b} \times X_\gamma^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, b}$  norm controls the  $C([-T, T]; \mathbf{H}^{s_1, s_2})$  norm.

By writing (1.1) in the integral form, we see that  $(u, v)$  is a solution to (1.1) with the initial condition  $(u_0, v_0)$  for  $|t| \leq T \leq 1$  if and only if

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi_{(u_0, v_0)}^t(u, v) := \begin{pmatrix} \eta(t)U(t)u_0 + i\alpha\eta_T(t) \int_0^t U(t-t')uv(t')dt' \\ \eta(t)V(t)v_0 - \beta\eta_T(t) \int_0^t V(t-t')\partial_x(|u(t')|^2)dt' \end{pmatrix}, \quad (3.5)$$

where  $U(t) = e^{it\partial_x^2}$  and  $V(t) = e^{-\gamma t\mathcal{H}\partial_x^2}$ . First, note that  $(\eta(t)U(t)u_0)^\wedge(n, \tau) = \widehat{\eta}(\tau + n^2)\widehat{u_0}(n)$  and  $(\eta(t)V(t)v_0)^\wedge(n, \tau) = \widehat{\eta}(\tau + \gamma|n|n)\widehat{v_0}(n)$ . Hence, we have

$$\|\eta(t)U(t)u_0\|_{X^{s_1, s_2, b}} \lesssim \|u_0\|_{H^{s_1, s_2}} \quad (3.6)$$

$$\|\eta(t)V(t)v_0\|_{X_\gamma^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, b}} \lesssim \|v_0\|_{H^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}}}. \quad (3.7)$$

Now, let  $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$  and  $T \leq 1$ . Then, from (2.25) in Lemma 2.1 (ii) in Ginibre-Tsutsumi-Velo [19], we have

$$\|\eta_T(U *_R F)\|_{X^{s_1, b}} \lesssim T^{1-b+b'} \|F\|_{X^{s_1, b'}} \quad (3.8)$$

$$\|\eta_T(V *_R G)\|_{X_\gamma^{s_1 - \frac{1}{2}, b}} \lesssim T^{1-b+b'} \|G\|_{X_\gamma^{s_1 - \frac{1}{2}, b'}}, \quad (3.9)$$

where  $*_R$  denotes the retarded convolution, i.e.  $U *_R F(t) = \int_0^t U(t-t')F(t')dt'$ . Also, from (2.24) in Lemma 2.1 (ii) in [19], we have  $\|\eta_T(U *_R F)\|_{H_t^b} \lesssim T^{1-b+b'} \|U(-t)F(t)\|_{H_t^b}$ . Noting that  $\|u\|_{X^{s_2, \infty, b}} = \|(U(-t)u)^\wedge(n, t)\|_{L_n^\infty H_t^b}$ , we have

$$\|\eta_T(U *_R F)\|_{X^{s_2, \infty, b}} \lesssim T^{1-b+b'} \|F\|_{X^{s_2, \infty, b'}}. \quad (3.10)$$

The same computation holds if we replace  $U(t)$  by  $V(t)$ , and thus we obtain

$$\|\eta_T(V *_R G)\|_{X_\gamma^{s_2, \infty, b}} \lesssim T^{1-b+b'} \|G\|_{X_\gamma^{s_2, \infty, b'}}. \quad (3.11)$$

Then, the local well-posedness of (1.1) in  $H^{s_1, s_2}$  follows from the standard argument [5, 23] once we prove the following bilinear estimates.

**PROPOSITION 3.1.** *Let  $b = \frac{1}{2} +$  and  $b' = -\frac{1}{2} +$  with  $1 + b' > b$ . Then, for  $0 < s_1 < \frac{1}{2} \leq s_2 < 1$  with  $s_2 < 2s_1$ , we have*

$$\|uv\|_{X^{s_1, s_2, b'}} \lesssim \|u\|_{X^{s_1, s_2, b}} \|v\|_{X_\gamma^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, b}}. \quad (3.12)$$

**PROPOSITION 3.2.** *Let  $b, b', s_1, s_2$  be as in Proposition 3.1. Then, we have*

$$\|\partial_x(u_1 \overline{u_2})\|_{X_\gamma^{s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, b'}} \lesssim \|u_1\|_{X^{s_1, s_2, b}} \|u_2\|_{X^{s_1, s_2, b}}. \quad (3.13)$$

More precisely, for given  $\theta = 0+$ , choose  $b = \frac{1}{2} + \theta$  and  $b' = -\frac{1}{2} + 2\theta$ . We prove (3.12) for  $s_1 \geq 2\theta$ ,  $s_2 \geq \frac{1}{2}$ ,  $s_2 \leq \min(2s_1 - 4\theta, s_1 + \frac{1}{2} - 2\theta)$ , and (3.13) for  $s_1 \geq 2\theta$ ,  $s_2 \geq \frac{1}{2}$ ,  $s_2 \leq \min(2s_1 - 2\theta, s_1 + \frac{1}{2} - 4\theta)$ . Note that for  $s_1 < \frac{1}{2}$ , it is enough to take  $s_2 \leq 2s_1 - 4\theta$  in both cases.

Note that we have  $1 - b + b' = \theta > 0$ . Thus, the linear estimates (3.6)  $\sim$  (3.11) yield a small positive power of  $T$  and establish the contraction property of  $\Phi_{(u_0, v_0)}^t(\cdot, \cdot)$  for  $|t| \leq T \ll 1$ .

Before proving of Propositions 3.1 and 3.2, first recall the  $L^4$  Strichartz estimate due to Bourgain [5]. Also, see Molinet [27].

**LEMMA 3.3.** *Let  $\gamma \neq 0$ . Then, we have*

$$\|u\|_{L_{x,t}^4} \lesssim \|u\|_{X^{0, \frac{3}{8}}}, \quad \text{and} \quad \|v\|_{L_{x,t}^4} \lesssim \|v\|_{X_\gamma^{0, \frac{3}{8}}}.$$

*Proof of Proposition 3.1.* In the proof, we use  $(n, \tau)$ ,  $(n_1, \tau_1)$ ,  $(n_2, \tau_2)$  to denote the Fourier variables for  $uv$ ,  $u$ ,  $v$ , respectively. i.e. we have  $n = n_1 + n_2$  and  $\tau = \tau_1 + \tau_2$  with  $n, n_1, n_2 \in \mathbb{Z}$  and  $\tau, \tau_1, \tau_2 \in \mathbb{R}$ . Also, let  $\sigma = \langle \tau + n^2 \rangle$ ,  $\sigma_1 = \langle \tau_1 + n_1^2 \rangle$ , and  $\sigma_2 = \langle \tau_2 + \gamma |n_2| n_2 \rangle$ . Then, as in (1.11), we have

$$\text{MAX} := \max(\sigma, \sigma_1, \sigma_2) \gtrsim 1 + |n_2| |R_n(n_2)|, \quad (3.14)$$

where  $R_n(n_2) = (\gamma \operatorname{sgn}(n_2) + 1)n_2 - 2n$ . Note that for fixed  $n \in \mathbb{Z}$ , we have  $R_n(n_2) = 0$  when  $n_2 = \frac{2n}{1 + \gamma \operatorname{sgn}(n_2)}$ , (which may not be an integer.) Hence, we have  $|R_n(n_2)| \gtrsim 1$  for  $n_2 \neq \left[ \frac{2n}{1 + \gamma \operatorname{sgn}(n_2)} \right], \left[ \frac{2n}{1 + \gamma \operatorname{sgn}(n_2)} \right] + 1$ .

• **Part 1:** We first prove (3.12) for the  $X^{s_1, b'}$  part of the  $X^{s_1, s_2, b'}$  norm.

Define the bilinear operator  $\mathcal{B}_{s_1, \theta}(\cdot, \cdot)$  by

$$\mathcal{B}_{s_1, \theta}(f, g)(n, \tau) = \frac{1}{2\pi} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}}} \frac{f(n_1, \tau_1) g(n_2, \tau_2)}{\sigma^{\frac{1}{2} - 2\theta} \sigma_1^{\frac{1}{2} + \theta} \sigma_2^{\frac{1}{2} + \theta}} d\tau_1. \quad (3.15)$$

Then, it suffices to prove

$$\|\mathcal{B}_{s_1, \theta}(f, g)\|_{L_{n, \tau}^2} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2}. \quad (3.16)$$

Define the non-resonance set  $A$  by

$$A = \left\{ (n, n_1, n_2) : n = n_1 + n_2, \left| n_2 - \frac{2n}{1 + \gamma \operatorname{sgn}(n_2)} \right| \geq 1 \right\}.$$

Then, on  $A$ , we have  $\text{MAX} \gtrsim \langle n_2 \rangle$ . Hence, if  $|n_1| \gtrsim |n|$ , then we have

$$\frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}}} \frac{1}{\sigma^{\frac{1}{2} - 2\theta} \sigma_1^{\frac{1}{2} + \theta} \sigma_2^{\frac{1}{2} + \theta}} \leq \frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}} \text{MAX}^{\frac{1}{2} - 2\theta}} \lesssim \frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - 2\theta}} \lesssim 1 \quad (3.17)$$

for  $s_1 \geq 2\theta$ . Now, suppose  $|n_1| \ll |n|$ . Then, we have  $|n_2| \sim |n|$ . Moreover, we have

$$|R_n(n_2)| = |(1 + \tilde{\gamma})n_2 - 2n| \geq |2 - (1 + \tilde{\gamma})||n| - |(1 + \tilde{\gamma})n_1| \sim |n| \sim |n_2|. \quad (3.18)$$

where  $\tilde{\gamma} = \gamma \operatorname{sgn}(n_2)$ . Thus,  $\text{MAX} \gtrsim \langle n_2 \rangle^2$  in this case, and we have

$$\frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}}} \frac{1}{\sigma^{\frac{1}{2} - 2\theta} \sigma_1^{\frac{1}{2} + \theta} \sigma_2^{\frac{1}{2} + \theta}} \lesssim \frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 + \frac{1}{2} - 4\theta}} \lesssim 1 \quad (3.19)$$

for  $s_1 \geq 0$  (with  $\theta$  sufficiently small. i.e.  $\theta \leq \frac{1}{8}$ .)

If  $\text{MAX} = \sigma$ , then by Hölder inequality and Lemma 3.3, we have

$$\|\mathcal{B}_{s_1, \theta}(f, g)\|_{L_{n, \tau}^2} \leq \|(\sigma_1^{-\frac{1}{2} - \theta} f)^\vee\|_{L_{x, t}^4} \|(\sigma_2^{-\frac{1}{2} - \theta} g)^\vee\|_{L_{x, t}^4} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2}$$

on  $A$ . If  $\text{MAX} = \sigma_1$ , then by Hölder inequality and Lemma 3.3, we have, for any  $h \in L_{n, \tau}^2$ ,

$$\langle \mathcal{B}_{s_1, \theta}(f, g), h \rangle_{L_{n, \tau}^2} \leq \|f\|_{L_{n, \tau}^2} \|(\sigma_2^{-\frac{1}{2} - \theta} g)^\vee\|_{L_{x, t}^4} \|(\sigma_1^{-\frac{1}{2} + 2\theta} h)^\vee\|_{L_{x, t}^4} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2} \|h\|_{L_{n, \tau}^2}$$

on  $A$  and this is equivalent to (3.16) via duality. A similar computation holds when  $\text{MAX} = \sigma_2$ . Hence, (3.16) holds on  $A$  for  $s_1 \geq 2\theta$ .

Now, we consider the estimate on  $A^c$ , i.e. near the resonances. In this case, we show

$$\|\tilde{\mathcal{B}}_{s_1, s_2, \theta}(f, g)\|_{L_{n, \tau}^2} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_n^\infty L_\tau^2}, \quad (3.20)$$

where  $\tilde{\mathcal{B}}_{s_1, s_2, \theta}(\cdot, \cdot)$  is given by

$$\tilde{\mathcal{B}}_{s_1, s_2, \theta}(f, g)(n, \tau) = \frac{1}{2\pi} \sum_{\substack{n=n_1+n_2 \\ (n, n_1, n_2) \in A^c}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_2 - \frac{1}{2}}} \frac{f(n_1, \tau_1) g(n_2, \tau_2)}{\sigma^{\frac{1}{2}-2\theta} \sigma_1^{\frac{1}{2}+\theta} \sigma_2^{\frac{1}{2}+\theta}} d\tau_1. \quad (3.21)$$

First, note that since  $|\gamma| \neq 1$ , it follows that  $|n| \sim |n_1| \sim |n_2|$  on  $A^c$ . Thus, we have  $\frac{\langle n \rangle^{s_1}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_2 - \frac{1}{2}}} \sim \langle n \rangle^{\frac{1}{2} - s_2} \lesssim 1$  for  $s_2 \geq \frac{1}{2}$ . On  $A^c$ , we can not expect any contribution from  $\sigma$ ,  $\sigma_1$ , and  $\sigma_2$ . However, for fixed  $n$ , there are only finitely many (2 or 4) values of  $n_2$  in  $A^c$ . i.e. there is virtually no sum over  $n_2$  in this case. Let  $F(n_1, \tau_1) = \sigma_1^{-\frac{1}{2}-\theta} f(n_1, \tau_1)$  and  $G(n_2, \tau_2) = \sigma_2^{-\frac{1}{2}-\theta} g(n_2, \tau_2)$ . Then, by Hölder inequality in  $t$  and Sobolev inequality, we have

$$\begin{aligned} \left\| \int_{\tau=\tau_1+\tau_2} F(n_1, \tau_1) G(n_2, \tau_2) d\tau_1 \right\|_{L_\tau^2} &\leq \| (F(n_1, \cdot))^{\vee t} \|_{L_t^4} \| (G(n_2, \cdot))^{\vee t} \|_{L_t^4} \\ &= \| e^{in_1^2 t} (F(n_1, \cdot))^{\vee t}(t) \|_{L_t^4} \| e^{i\gamma|n_2|n_2 t} (G(n_2, \cdot))^{\vee t}(t) \|_{L_t^4} \\ &\lesssim \| U(-t) (F(n_1, \cdot))^{\vee t} \|_{H_t^{\frac{1}{4}}} \| V(-t) (G(n_2, \cdot))^{\vee t} \|_{H_t^{\frac{1}{4}}}. \end{aligned} \quad (3.22)$$

Then, for  $n_2 = \lfloor \frac{2n}{1+\gamma \text{sgn}(n_2)} \rfloor$  or  $\lfloor \frac{2n}{1+\gamma \text{sgn}(n_2)} \rfloor + 1$ , we have,

$$\begin{aligned} \text{LHS of (3.20)} &\lesssim \left\| \int_{\tau=\tau_1+\tau_2} F(n_1, \tau_1) G(n_2, \tau_2) d\tau_1 \right\|_{L_{n, \tau}^2} \\ &\lesssim \langle \tau + n_2^2 \rangle^{\frac{1}{4}} F(n_1, \tau) \| \langle \tau + \gamma|n_2|n_2 \rangle^{\frac{1}{4}} G(n_2, \tau) \|_{L_{n_2}^\infty L_\tau^2} \leq \|f\|_{L_{n, \tau}^2} \|g\|_{L_n^\infty L_\tau^2}. \end{aligned}$$

• **Part 2:** Next, we prove (3.12) for the  $X^{s_2, \infty, b'}$  part of the  $X^{s_1, s_2, b'}$  norm.

For  $|n| \lesssim 1$ , we have  $L_n^\infty$  norm  $\sim L_n^2$  norm and  $\langle n \rangle^{s_2} \sim \langle n \rangle^{s_1}$ , i.e. the proof reduces to Part 1. Hence, we assume  $|n| \gtrsim 1$ .

On  $A$ , we have  $\text{MAX} \gtrsim \langle n_2 \rangle$ . Thus, if  $|n_1|, |n_2| \gtrsim |n|$ , then we have

$$\frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}}} \frac{1}{\sigma^{\frac{1}{2}-2\theta} \sigma_1^{\frac{1}{2}+\theta} \sigma_2^{\frac{1}{2}+\theta}} \lesssim \frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - 2\theta}} \lesssim 1 \quad (3.23)$$

for  $2s_1 \geq s_2 + 2\theta$ . On the other hand, if  $|n_1| \ll |n|$ , we have  $|n_2| \sim |n|$  and  $|R_n(n_2)| \sim |n|$  as in (3.18). i.e.  $\text{MAX} \sim \langle n \rangle^2$ . Then, we have LHS of (3.23)  $\lesssim \frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_1} \langle n \rangle^{2s_1 - 4\theta}} \lesssim 1$  for  $2s_1 \geq s_2 + 4\theta$ . Also, if  $|n_2| \ll |n|$ , then we have  $|n_1| \sim |n|$  and  $|R_n(n_2)| \sim |n|$ . i.e.  $\text{MAX} \sim \langle n \rangle \langle n_2 \rangle$ . Then, we have LHS of (3.23)  $\lesssim \frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - 2\theta} \langle n \rangle^{\frac{1}{2} - 2\theta}} \lesssim 1$  for  $s_1 \geq 2\theta$  and  $s_1 + \frac{1}{2} \geq s_2 + 2\theta$ .

In this case, it suffices to show

$$\|\mathcal{B}'_{s_1, s_2, \theta}(f, g)\|_{L_n^\infty L_\tau^2} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2}, \quad (3.24)$$

where  $\mathcal{B}'_{s_1, s_2, \theta}(\cdot, \cdot)$  is given by

$$\mathcal{B}'_{s_1, s_2, \theta}(f, g)(n, \tau) = \frac{1}{2\pi} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_1 - \frac{1}{2}}} \frac{f(n_1, \tau_1)g(n_2, \tau_2)}{\sigma^{\frac{1}{2}-2\theta} \sigma_1^{\frac{1}{2}+\theta} \sigma_2^{\frac{1}{2}+\theta}} d\tau_1. \quad (3.25)$$

Suppose  $\text{MAX} = \sigma$ . Then, from (3.22) and Young's inequality in  $n$ , we have

$$\begin{aligned} \text{LHS of (3.24)} &\lesssim \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} F(n_1, \tau_1)G(n_2, \tau_2)d\tau_1 \right\|_{L_n^\infty L_\tau^2} \\ &\lesssim \|\langle \tau + n_1^2 \rangle^{\frac{1}{4}} F(n_1, \tau)\|_{L_{n_1, \tau}^2} \|\langle \tau + \gamma|n_2|n_2 \rangle^{\frac{1}{4}} G(n_2, \tau)\|_{L_{n_2, \tau}^2} \leq \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2}, \end{aligned}$$

where  $F$  and  $G$  are as in Part 1. If  $\text{MAX} = \sigma_1$ , then by Hölder inequality in  $t$ , Young's inequality in  $n$ , and Sobolev inequality (as in (3.22)), we have

$$\begin{aligned} \langle \mathcal{B}'_{s_1, s_2, \theta}(f, g), h \rangle_{L_{n, \tau}^2} &\lesssim \langle f * (\sigma_2^{-\frac{1}{2}-\theta} g), \sigma^{-\frac{1}{2}+2\theta} h \rangle_{L_{n, \tau}^2} \\ &\leq \|f\|_{L_{n, \tau}^2} \|(\sigma_2^{-\frac{1}{2}-\theta} g)^{\vee t}\|_{L_n^2 L_t^4} \|(\sigma^{-\frac{1}{2}+2\theta} h)^{\vee t}\|_{L_n^1 L_t^4} \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2} \|h\|_{L_n^1 L_\tau^2} \end{aligned}$$

for any  $h \in L_n^1 L_\tau^2$ , and this is equivalent to (3.24) via duality. A similar computation holds when  $\text{MAX} = \sigma_2$ . Hence, (3.24) holds on  $A$  for  $s_1 \geq 2\theta$  and  $s_2 \leq \min(2s_1 - 4\theta, s_1 + \frac{1}{2} - 2\theta)$ .

On  $A^c$ , we have  $|n| \sim |n_1| \sim |n_2|$ . Thus,  $\frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_2} \langle n_2 \rangle^{s_2 - \frac{1}{2}}} \lesssim 1$  for  $s_2 \geq \frac{1}{2}$ . Also, recall that for fixed  $n$ , there are only finitely many (2 or 4) values of  $n_2$  on  $A^c$ . i.e. there is no sum over  $n_2$  in this case. Then, as in (3.22), we have

$$\begin{aligned} &\left\| \frac{1}{2\pi} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n \rangle^{s_2}}{\langle n_1 \rangle^{s_2} \langle n_2 \rangle^{s_2 - \frac{1}{2}}} \frac{f(n_1, \tau_1)g(n_2, \tau_2)}{\sigma^{\frac{1}{2}-2\theta} \sigma_1^{\frac{1}{2}+\theta} \sigma_2^{\frac{1}{2}+\theta}} d\tau_1 \right\|_{L_n^\infty L_\tau^2} \\ &\lesssim \left\| \int_{\tau=\tau_1+\tau_2} F(n_1, \tau_1)G(n_2, \tau_2)d\tau_1 \right\|_{L_n^\infty L_\tau^2} \\ &\lesssim \|\langle \tau + n_1^2 \rangle^{\frac{1}{4}} F(n_1, \tau)\|_{L_{n_1, \tau}^2} \|\langle \tau + \gamma|n_2|n_2 \rangle^{\frac{1}{4}} G(n_2, \tau)\|_{L_{n_2, \tau}^2} \leq \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2}, \end{aligned}$$

where  $F$  and  $G$  are as in Part 1. This completes the proof of Proposition 3.1.  $\square$

*Proof of Proposition 3.2.* Noting that  $|n| \langle n \rangle^{r_1 - \frac{1}{2}} \leq \langle n \rangle^{r_1 + \frac{1}{2}}$ , our goal is to show the boundedness of the multilinear functional  $\mathcal{I}_{r_1, r_2, r_3, \theta}$  given by

$$\mathcal{I}_{r_1, r_2, r_3, \theta} = \frac{1}{4\pi^2} \sum_{\substack{n, n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} M_{r_1, r_2, r_3, \theta} f(n_1, \tau_1)g(n_2, \tau_2)h(n, \tau)d\tau_1 d\tau,$$

for  $h \in L_{n, \tau}^2$  or  $L_n^1 L_\tau^2$  and  $r_j = s_1$  or  $s_2$  with  $j = 1, 2, 3$ , where

$$M_{r_1, r_2, r_3, \theta} = \frac{\langle n \rangle^{r_1 + \frac{1}{2}}}{\langle n_1 \rangle^{r_2} \langle n_2 \rangle^{r_3}} \frac{1}{\tilde{\sigma}^{\frac{1}{2}-2\theta} \tilde{\sigma}_1^{\frac{1}{2}+\theta} \tilde{\sigma}_2^{\frac{1}{2}+\theta}}$$

with  $\tilde{\sigma} = \langle \tau + \gamma|n|n \rangle$ ,  $\tilde{\sigma}_1 = \langle \tau_1 + n_1^2 \rangle$ , and  $\tilde{\sigma}_2 = \langle \tau_2 - n_2^2 \rangle$ . From a computation analogous to (1.11), we have

$$\text{MAX} := \max(\tilde{\sigma}, \tilde{\sigma}_1, \tilde{\sigma}_2) \gtrsim 1 + |n| |\tilde{R}_{n_1}(n)|,$$

where  $\tilde{R}_{n_1}(n) = (\gamma \operatorname{sgn}(n) + 1)n - 2n_1$ . Note that for fixed  $n_1 \in \mathbb{Z}$ , we have  $\tilde{R}_{n_1}(n) = 0$  when  $n = \frac{2n_1}{1 + \gamma \operatorname{sgn}(n)}$ . Thus, we have  $|\tilde{R}_{n_1}(n)| \gtrsim 1$  for  $n \neq \left[\frac{2n_1}{1 + \gamma \operatorname{sgn}(n)}\right], \left[\frac{2n_1}{1 + \gamma \operatorname{sgn}(n)}\right] + 1$ . We indicate how this proposition follows as a corollary to (the proof of) Proposition 3.1, basically by replacing  $(n, \tau), (n_1, \tau_1), (n_2, \tau_2)$  here with  $(n_2, \tau_2), (n, \tau), (-n_1, -\tau_1)$ .

First, consider the  $X_\gamma^{s_1 - \frac{1}{2}, b'}$  part of (3.13). Let  $B = \{(n, n_1, n_2) : n = n_1 + n_2, |n - \frac{2n_1}{1 + \gamma \operatorname{sgn}(n)}| \geq 1\}$ . On  $B$ , we have  $\operatorname{MAX} \gtrsim \langle n \rangle$ . Thus, if  $|n_1|, |n_2| \gtrsim |n|$ , then, we have  $M_{s_1, s_1, s_1, \theta} \lesssim \langle n \rangle^{-s_1 + 2\theta} \lesssim 1$  for  $s_1 \geq 2\theta$ . If  $|n_1| \ll |n|$ , then we have  $|n_2| \sim |n|$  and  $\operatorname{MAX} \gtrsim \langle n \rangle^2$ . Also, if  $|n_2| \ll |n|$ , then we have  $|n_1| \sim |n|$  and  $\operatorname{MAX} \gtrsim \langle n \rangle^2$ . (See (3.18).) In both cases, we have  $M_{s_1, s_1, s_1, \theta} \lesssim \langle n \rangle^{-\frac{1}{2} + 4\theta} \lesssim 1$ . Hence, by repeating the first half of Part 1 in the proof of Proposition 3.1, we have  $|\mathcal{I}_{s_1, s_1, s_1, \theta}| \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2} \|h\|_{L_{n, \tau}^2}$ , for all  $h \in L_{n, \tau}^2$ .

On  $B^c$ , we do not expect any contribution from  $\tilde{\sigma}, \tilde{\sigma}_1, \tilde{\sigma}_2$ . However, we have  $|n| \sim |n_1| \sim |n_2|$ . Thus,  $\frac{\langle n \rangle^{s_1 + \frac{1}{2}}}{\langle n_1 \rangle^{s_1} \langle n_2 \rangle^{s_2}} \lesssim \langle n \rangle^{\frac{1}{2} - s_2} \lesssim 1$  for  $s_2 \geq \frac{1}{2}$ . Also, note that for fixed  $n_1$ , there are only finitely many values of  $n$  on  $B^c$ . Hence, by repeating the second half of Part 1 in the proof of Proposition 3.1, we have  $|\mathcal{I}_{s_1, s_1, s_2, \theta}| \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_n^\infty L_\tau^2} \|h\|_{L_{n, \tau}^2}$ , for all  $h \in L_{n, \tau}^2$ . This proves the  $X_\gamma^{s_1 - \frac{1}{2}, b'}$  part of (3.13).

Next, consider the  $X_\gamma^{s_2 - \frac{1}{2}, \infty, b'}$  part of (3.13). On  $B$ , we have  $\operatorname{MAX} \gtrsim \langle n \rangle$ . If  $|n_1|, |n_2| \gtrsim |n|$ , then we have  $M_{s_2, s_1, s_1, \theta} \lesssim \langle n \rangle^{s_2 - 2s_1 + 2\theta} \lesssim 1$  for  $2s_1 \geq s_2 + 2\theta$ . If  $|n_1| \ll |n|$  or  $|n_2| \ll |n|$ , then we have  $\operatorname{MAX} \sim \langle n \rangle^2$  and  $M_{s_2, s_1, s_1, \theta} \lesssim \langle n \rangle^{s_2 - s_1 - \frac{1}{2} + 4\theta} \lesssim 1$  for  $s_1 + \frac{1}{2} \geq s_2 + 4\theta$ . Hence, by repeating the first half of Part 2 in the proof of Proposition 3.1, we have  $|\mathcal{I}_{s_2, s_1, s_1, \theta}| \lesssim \|f\|_{L_{n, \tau}^2} \|g\|_{L_{n, \tau}^2} \|h\|_{L_n^1 L_\tau^2}$ , for all  $h \in L_n^1 L_\tau^2$ .

On  $B^c$ , we have  $|n| \sim |n_1| \sim |n_2|$ . Thus,  $\frac{\langle n \rangle^{s_2 + \frac{1}{2}}}{\langle n_1 \rangle^{s_2} \langle n_2 \rangle^{s_2}} \lesssim \langle n \rangle^{\frac{1}{2} - s_2} \lesssim 1$  for  $s_2 \geq \frac{1}{2}$ . Hence, by repeating the second half of Part 2 in the proof of Proposition 3.1, we have  $|\mathcal{I}_{s_2, s_2, s_2, \theta}| \lesssim \|f\|_{L_n^\infty L_\tau^2} \|g\|_{L_n^\infty L_\tau^2} \|h\|_{L_n^1 L_\tau^2}$ , for all  $h \in L_n^1 L_\tau^2$ . This proves the  $X_\gamma^{s_2 - \frac{1}{2}, \infty, b'}$  part of (3.13).  $\square$

**4. Construction of the Gibbs measure.** In this section, we discuss the construction of the Gibbs measure  $\mu$  for (1.1) following Bourgain [8]. Once we construct the Gibbs measure, we can easily adapt the argument in Bourgain [8, 11] to extend the local well-posedness result (Theorem 1.2) to the global well-posedness almost surely on the statistical ensemble and to establish the invariance of the Gibbs measure (Theorem 1.3.) The argument is standard and we omit the details. For the details of the argument, see Burq-Tzvetkov [12, §6-7], Oh [30, §6], and Tzvetkov [33, §8-10], [34, §7-9] in the context of NLW, KdV systems, and NLS respectively.

Given a Hamiltonian flow

$$\begin{cases} \dot{p}_i = \frac{\partial H}{\partial q_i} \\ \dot{q}_i = -\frac{\partial H}{\partial p_i} \end{cases}$$

on  $\mathbb{R}^{2n}$  with Hamiltonian  $H = H(p_1, \dots, p_n, q_1, \dots, q_n)$ , Liouville's theorem states that the Lebesgue measure on  $\mathbb{R}^{2n}$  is invariant under the flow. From the conservation of the Hamiltonian  $H$ , the Gibbs measures  $e^{-\nu H} \prod_{i=1}^n dp_i dq_i$  are also invariant, where  $\nu > 0$  is the reciprocal temperature.

In the context of NLS, Lebowitz-Rose-Speer [26] considered the Gibbs measure of the form  $d\mu = \exp(-\nu H(u)) \prod_{x \in \mathbb{T}} du(x)$  where  $H(u)$  is the Hamiltonian given by  $H(u) = \frac{1}{2} \int |u_x|^2 \pm \frac{1}{p} \int |u|^p dx$ . In the focusing case (with  $-$ ),  $H(u)$  is not bounded from below and this causes a problem. Using the conservation of the  $L^2$  norm, they instead

considered the Gibbs measure of the form  $d\mu = Z^{-1} \exp(-\nu H(u)) \chi_{\{\|u\|_{L^2} \leq R\}} \prod_{x \in \mathbb{T}} du(x)$ , i.e. with an  $L^2$ -cutoff. This turned out to be a well-defined probability measure on  $H^{\frac{1}{2}-}(\mathbb{T}) = \bigcap_{s < \frac{1}{2}} H^s(\mathbb{T})$  (for  $p < 6$  with any  $R > 0$ , and  $p = 6$  with sufficiently small  $R$ .) Bourgain [8] continued this study and proved the invariance of  $\mu$  under the flow of NLS and the global well-posedness almost surely on the statistical ensemble. Note that [8] appeared before the so-called Bourgain's method [6] or the  $I$ -method [15], i.e. there was virtually no method available to establish any GWP result from a LWP result whose regularity was between two conservation laws. This was the case for NLS for  $4 < p \leq 6$ . We use this idea to obtain a.s. GWP of the SBO system (1.1). The same idea was applied by the author to show the invariance of the Gibbs measures and a.s GWP for coupled KdV systems under some Diophantine conditions [30]. Recently, Burq-Tzvetkov [12] used similar ideas to prove a.s. GWP for the nonlinear wave equation on the unit ball in  $\mathbb{R}^3$  under the radial symmetry. Also, see Tzvetkov [33], [34] for NLS on the unit disc in  $\mathbb{R}^2$  and Oh [31] for the invariance of the white noise for KdV.

Recall that the mean of  $v$ , the  $L^2$  norm of  $u$ , and the Hamiltonian  $H(u, v) = \frac{1}{2} \int |u_x|^2 dx + \frac{\kappa}{2} \int (D^{\frac{1}{2}} v)^2 dx + \frac{\alpha}{2} \int v |u|^2 dx$  are conserved under the flow of the SBO system (1.1). In the following, we assume  $\kappa := -\frac{\alpha\gamma}{2\beta} > 0$  so that the quadratic part of the Hamiltonian (1.3) is nonnegative. Note that  $\frac{\alpha}{2} \int v |u|^2 dx$  is not sign-definite. In particular,  $\exp(-\nu \frac{\alpha}{2} \int v |u|^2 dx)$  is not bounded from above as in the case of focusing NLS and KdV [26], [8]. This motivates us to define the Gibbs measure of the form  $d\mu = \chi_{\{\|\phi\|_{L^2} \leq R\}} \chi_{\{|\hat{\psi}(0)| \leq R\}} e^{-\nu \frac{\alpha}{2} \int \psi |\phi|^2 dx} d\rho$ , where  $d\rho$  is the Gaussian introduced in (1.5). Here, we associate  $(\phi, \psi)$  with  $(u(t), v(t))$  for fixed  $t \in \mathbb{R}$ . In particular,  $\psi$  is real-valued. For simplicity, we set  $\nu = 1$  for the rest of the paper.

Let  $(a, b) = (a_n, b_n)_{n \in \mathbb{Z}}$  denote the Fourier coefficients of  $(\phi, \psi)$  on  $\mathbb{T}$ . Since  $\psi$  is real-valued, we have  $b_{-n} = \overline{b_n}$ . Let  $R > 0$  be a cutoff as above and consider the cylinder in  $\mathbb{C}^{2N+1} \times \mathbb{R}^{2N+1}$  given by

$$\Omega_{N,R} = \{(a_n, b_n)_{|n| \leq N} : \|a_n\|_{L_n^2} \leq R \text{ and } |b_0| \leq R\}.$$

Here, we abuse the notation and actually identify  $\{b_n\}_{|n| \leq N}$  with  $(b_0, \operatorname{Re} b_1, \dots, \operatorname{Re} b_N, \operatorname{Im} b_1, \dots, \operatorname{Im} b_N) \cong \mathbb{R}^{2N+1}$ . Also, define

$$\Omega_R = \{(a_n, b_n)_{n \in \mathbb{Z}} : \|a_n\|_{L_n^2} \leq R \text{ and } |b_0| \leq R\}.$$

Then, define the measure  $\rho_N$  on  $\mathbb{C}^{2N} \times \mathbb{R}^{2N} = \{(a_n, b_n)_{|n| \leq N, n \neq 0}\}$  with the normalized density

$$d\rho_N = \tilde{Z}_N^{-1} e^{-\frac{1}{2} \sum_{|n| \leq N, n \neq 0} (n^2 |a_n|^2 + \kappa |n| |b_n|^2)} \prod_{|n| \leq N, n \neq 0} d(a_n \otimes b_n),$$

where  $\tilde{Z}_N = \int_{\mathbb{C}^{2N} \times \mathbb{R}^{2N}} e^{-\frac{1}{2} \sum_{|n| \leq N, n \neq 0} (n^2 |a_n|^2 + \kappa |n| |b_n|^2)} \prod_{|n| \leq N, n \neq 0} d(a_n \otimes b_n)$ . Note that this measure is the induced probability measure on  $\mathbb{C}^{2N} \times \mathbb{R}^{2N}$  under the map  $\omega \mapsto \{(n^{-1} f_n(\omega), \kappa^{-\frac{1}{2}} |n|^{-\frac{1}{2}} g_n(\omega)); |n| \leq N, n \neq 0\}$ , where  $\{f_n(\omega)\}$  and  $\{g_n(\omega)\}$  are i.i.d. standard complex Gaussian random variables (with  $g_{-n} = \overline{g_n}$ .) Now, define the Gaussian measure  $\rho$  on  $\{(a_n, b_n)_{n \neq 0}\}$  whose density is given by

$$d\rho = \tilde{Z}^{-1} e^{-\frac{1}{2} \sum_{n \neq 0} (n^2 |a_n|^2 + \kappa |n| |b_n|^2)} \prod_{n \neq 0} d(a_n \otimes b_n), \quad (4.1)$$

where  $\tilde{Z} = \int e^{-\frac{1}{2} \sum_{n \neq 0} (n^2 |a_n|^2 + \kappa |n| |b_n|^2)} \prod_{n \neq 0} d(a_n \otimes b_n)$ .

Recall the following definitions [24]: Given a real separable Hilbert space  $H$  with norm  $\|\cdot\|$ , let  $\mathcal{F}$  denote the set of finite dimensional orthogonal projections  $\mathbb{P}$  of  $H$ . Then, define a cylinder set  $E$  by  $E = \{u \in H : \mathbb{P}u \in F\}$ , where  $\mathbb{P} \in \mathcal{F}$  and  $F$  is a Borel subset of  $\mathbb{P}H$ , and let  $\mathcal{R}$  denote the collection of such cylinder sets. Note that  $\mathcal{R}$  is a field but not a  $\sigma$ -field. Then, the Gauss measure  $\rho$  on  $H$  is defined by

$$\rho(E) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{\|u\|^2}{2}} du$$

for  $E \in \mathcal{R}$ , where  $n = \dim \mathbb{P}H$  and  $du$  is the Lebesgue measure on  $\mathbb{P}H$ . It is known that  $\rho$  is finitely additive but not countably additive in  $\mathcal{R}$ .

A seminorm  $\|\cdot\|$  in  $H$  is called measurable if for every  $\varepsilon > 0$ , there exists  $\mathbb{P}_0 \in \mathcal{F}$  such that  $\rho(\{\|\mathbb{P}u\| > \varepsilon\}) < \varepsilon$  for  $\mathbb{P} \in \mathcal{F}$  orthogonal to  $\mathbb{P}_0$ . Any measurable seminorm is weaker than the norm of  $H$ , and  $H$  is not complete with respect to  $\|\cdot\|$  unless  $H$  is finite dimensional. Let  $B$  be the completion of  $H$  with respect to  $\|\cdot\|$  and denote by  $i$  the inclusion map of  $H$  into  $B$ . The triple  $(i, H, B)$  is called an abstract Wiener space.

Now, regarding  $v \in B^*$  as an element of  $H^* \equiv H$  by restriction, we embed  $B^*$  in  $H$ . Define, for a Borel set  $F \subset \mathbb{R}^n$ ,

$$\tilde{\rho}(\{u \in B : ((u, v_1), \dots, (u, v_n)) \in F\}) = \rho(\{u \in H : (\langle u, v_1 \rangle_H, \dots, \langle u, v_n \rangle_H) \in F\}),$$

where  $v_j$ 's are in  $B^*$  and  $(\cdot, \cdot)$  denote the natural pairing between  $B$  and  $B^*$ . Let  $\mathcal{R}_B$  denote the collection of cylinder sets  $\{u \in B : ((u, v_1), \dots, (u, v_n)) \in F\}$  in  $B$ .

**THEOREM 4.1** (Gross [21]).  *$\tilde{\rho}$  is countably additive in the  $\sigma$ -field generated by  $\mathcal{R}_B$ .*

In the present context, let  $H = \dot{H}^1 \times \dot{H}^{\frac{1}{2}}$  and  $B = H^{s_1, s_2}$  with  $0 < s_1 < \frac{1}{2} < s_2 < 1$ . Then, we have

**PROPOSITION 4.2.** *The seminorm  $\|\cdot\|_B$  is measurable.*

Hence,  $(i, \dot{H}^1 \times \dot{H}^{\frac{1}{2}}, H^{s_1, s_2})$  is an abstract Wiener space, and  $\rho$  in (1.5) and (4.1) is countably additive in  $H^{s_1, s_2}$  for  $0 < s_1 < \frac{1}{2} < s_2 < 1$ . Moreover, we have  $\rho_N \rightarrow \rho$ . (See [36].) For the proof of this proposition, see Oh [30]. Note that Bourgain used this norm (i.e.  $H = \dot{H}^1$ ,  $B = H^{s_1, s_2}$  with  $0 < s_1 < \frac{1}{2} < s_2 < 1$ ) in studying the invariance of the Gibbs measure for mKdV [8].

Given an abstract Wiener space  $(i, H, B)$ , we have the following result due to Fernique [17]. Also, see [24, Theorem 3.1].

**LEMMA 4.3.** *There exists  $c > 0$  such that  $\int_B e^{c\|u\|_B^2} \rho(du) < \infty$ . Hence, there exists  $c' > 0$  such that  $\rho(\|u\|_B > K) \leq e^{-c'K^2}$ .*

Now, define  $\Omega_{N,R}(s_1, s_2, K)$  and  $\Omega_R(s_1, s_2, K)$  by

$$\begin{aligned} \Omega_{N,R}(s_1, s_2, K) &= \left\{ (a_n, b_n)_{|n| \leq N} \in \Omega_{N,R} : \left\| \sum_{|n| \leq N} (a_n, b_n) e^{inx} \right\|_{H^{s_1, s_2}} \leq K \right\} \\ \Omega_R(s_1, s_2, K) &= \left\{ (a_n, b_n)_{n \in \mathbb{Z}} \in \Omega_R : \left\| \sum_{n \in \mathbb{Z}} (a_n, b_n) e^{inx} \right\|_{H^{s_1, s_2}} \leq K \right\}. \end{aligned}$$

Also, let  $\tilde{\Omega}_{N,R}$ ,  $\tilde{\Omega}_R$ ,  $\tilde{\Omega}_{N,R}(s_1, s_2, K)$ ,  $\tilde{\Omega}_R(s_1, s_2, K)$  be the restrictions of  $\Omega_{N,R}$ ,  $\Omega_R$ ,  $\Omega_{N,R}(s_1, s_2, K)$ ,  $\Omega_R(s_1, s_2, K)$  onto their mean 0 parts. Then, basically from Lemma 4.3, we have

LEMMA 4.4. *Let  $0 < s_1 < \frac{1}{2} < s_2 < 1$  For sufficiently large  $K > 0$ , we have*

$$\rho_N(\widetilde{\Omega}_{N,R} \setminus \widetilde{\Omega}_{N,R}(s_1, s_2, K)) \leq e^{-cK^2}, \text{ and } \rho(\widetilde{\Omega}_R \setminus \widetilde{\Omega}_R(s_1, s_2, K)) \leq e^{-cK^2},$$

where  $c$  and the implicit constant are independent of  $N$ .

The proof of this lemma is standard and is omitted. See [8, 30].

For the rest of the paper, let  $0 < s_1 = \frac{1}{2} - \epsilon < \frac{1}{2} < s_2 = 1 - \epsilon < 1$  with  $s_2 < 2s_1$ . Let  $\mathbb{P}_N$  be the projection onto the frequencies  $|n| \leq N$  given by  $\mathbb{P}_N \phi = \phi^N = \sum_{|n| \leq N} a_n e^{inx}$ . Now, define the weighted Wiener measure  $\mu_N$  on  $\mathbb{R}^{2N+1} \times \mathbb{C}^{2N+1} = \{(a_n, b_n)_{|n| \leq N}\}$  by

$$d\mu_N = Z_N^{-1} \exp\left(-\frac{\alpha}{2} \int \mathbb{P}_N \psi |\mathbb{P}_N \phi|^2 dx\right) \chi_{\Omega_{N,R}} d(a_0, b_0) \otimes d\rho_N,$$

where  $Z_N = \int_{\mathbb{C}^{2N+1} \times \mathbb{R}^{2N+1}} \exp\left(-\frac{\alpha}{2} \int \mathbb{P}_N \psi |\mathbb{P}_N \phi|^2 dx\right) \chi_{\Omega_{N,R}} d(a_0, b_0) \otimes d\rho_N$ . Also, define the weighted Wiener measure  $\mu$  on  $\{(a_n, b_n)_{n \in \mathbb{Z}}\}$  by

$$d\mu = Z^{-1} \exp\left(-\frac{\alpha}{2} \int \psi |\phi|^2 dx\right) \chi_{\Omega_R} d(a_0, b_0) \otimes d\rho,$$

where  $Z = \int \exp\left(-\frac{\alpha}{2} \int \psi |\phi|^2 dx\right) \chi_{\Omega_R} d(a_0, b_0) \otimes d\rho$ . At this point,  $Z$  need not be finite and thus  $d\mu$  need not be a well-defined probability measure. Indeed, we have

LEMMA 4.5. *For any  $r < \infty$ , we have*

$$\exp\left(-\frac{\alpha}{2} \int \mathbb{P}_N \psi |\mathbb{P}_N \phi|^2 dx\right) \chi_{\Omega_{N,R}} \in L^r(d(a_0, b_0) \otimes d\rho_N) \quad (4.2)$$

$$\exp\left(-\frac{\alpha}{2} \int \psi |\phi|^2 dx\right) \chi_{\Omega_R} \in L^r(d(a_0, b_0) \otimes d\rho). \quad (4.3)$$

In particular,  $d\mu$  is a well-defined probability measure. Moreover, we have  $d\mu_N \ll d(a_0, b_0) \otimes d\rho_N$  and  $d\mu \ll d(a_0, b_0) \otimes d\rho$ . Then, from Lemma 4.4, we have

COROLLARY 4.6 (tightness of  $\mu_N$  and  $\mu$ ). *For large  $K > 0$ , we have*

$$\mu_N(\Omega_{N,R} \setminus \Omega_{N,R}(s_1, s_2, K)) \leq e^{-cK^2}, \text{ and } \mu(\Omega_R \setminus \Omega_R(s_1, s_2, K)) \leq e^{-cK^2},$$

where  $c$  and the implicit constant are independent of  $N$ .

The proof of Lemma 4.5 is based on Bourgain's argument in [8]. We have an additional difficulty since  $\psi \in H^{0-}(\mathbb{T}) \setminus L^2(\mathbb{T})$  almost surely. i.e. the argument is in the sub- $L^2$  setting and we need to employ a probabilistic argument.

In [10], Bourgain constructed the invariant Gibbs measure for the (Wick-ordered) defocusing cubic NLS on  $\mathbb{T}^2$ , which is supported below  $L^2(\mathbb{T}^2)$ . The main tool was the hypercontractivity related to the product of the Gaussian random variables. Recently, Tzvetkov [35] used the hypercontractivity to construct the Gibbs measure for the Benjamin-Ono equation whose support is also below  $L^2(\mathbb{T})$ . We point out that the invariance of such Gibbs measure for the Benjamin-Ono equation is not known at this point, due to the lack of the well-posedness below  $L^2(\mathbb{T})$ .

In our problem, the nonlinearity  $\psi|\phi|^2$  in the Hamiltonian is relatively simple as compared to that in [10], [35] ( $|\psi|^p$  with  $p = 4$  or  $3$ ), since  $\phi \in H^{\frac{1}{2}-}$  and  $\psi \in H^{0-}$ . It turns out that we do not need to use the hypercontractivity. Instead, we use the properties of abstract Wiener spaces along with a probabilistic lemma (Lemma 4.7.)

*Proof of Lemma 4.5.* We prove only (4.3). First, note that on  $\Omega_R$ , we have  $|a_0|, |b_0| \leq R$ . Then, from Hölder and Young's inequalities with  $p = 2-$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , we have

$$\begin{aligned} \int \psi |\phi|^2 dx &\leq \|\widehat{|\phi|^2}\|_{L_n^p} \|\widehat{\psi}\|_{L_n^{p'}} \leq \frac{1}{p'} \|\widehat{|\phi|^2}\|_{L_n^p}^{p'} + \frac{1}{p} \|\widehat{\psi}\|_{L_n^{p'}}^p \\ &\leq \frac{1}{p'} \|\widehat{\phi}\|_{L_n^{\frac{2p}{1+p}}}^{2p'} + \frac{1}{p} \|\widehat{\psi}\|_{L_n^{p'}}^p \lesssim |a_0|^{2p'} + \|\{\widehat{\phi}\}_{n \neq 0}\|_{L_n^{\frac{2p}{1+p}}}^{2p'} + |b_0|^p + \|\{\widehat{\psi}\}_{n \neq 0}\|_{L_n^{p'}}^p. \end{aligned}$$

Since  $\frac{2p}{1+p} = \frac{4}{3}-$ , we have

$$\begin{aligned} &\left\| e^{-\frac{\sigma}{2} \int \psi |\phi|^2 dx} \chi_{\Omega_R} \right\|_{L^r(d(a_0, b_0) \otimes d\rho)} \\ &\lesssim \left( \int_{\substack{|a_0| \leq R \\ |b_0| \leq R}} e^{c(R^{2p'} + R^p)} \chi_{\Omega_R} d(a_0, b_0) \right)^{\frac{1}{r}} \left\| e^{c(\|\{\widehat{\phi}\}_{n \neq 0}\|_{L_n^{\frac{4}{3}-}^{4+} + \|\{\widehat{\psi}\}_{n \neq 0}\|_{L_n^{2+}}^{2-})} \chi_{\Omega_R} \right\|_{L^r(d\rho)} \\ &\lesssim C_R \left\| \exp(c(\|\{\widehat{\phi}\}_{n \neq 0}\|_{L_n^{\frac{4}{3}-}^{4+} + \|\{\widehat{\psi}\}_{n \neq 0}\|_{L_n^{2+}}^{2-})) \chi_{\Omega_R} \right\|_{L^r(d\rho)}. \end{aligned}$$

Let  $d\rho_1 = d\rho|_{(a_n)_{n \neq 0}}$  and  $d\rho_2 = d\rho|_{(b_n)_{n \neq 0}}$ , i.e.

$$d\rho_1 = \widetilde{Z}_1^{-1} e^{-\frac{1}{2} \sum_{n \neq 0} n^2 |a_n|^2} \prod_{n \neq 0} da_n, \quad \text{and} \quad d\rho_2 = \widetilde{Z}_2^{-1} e^{-\frac{1}{2} \sum_{n \neq 0} \kappa |n| |b_n|^2} \prod_{n \neq 0} db_n,$$

where  $\widetilde{Z}_1 = \int e^{-\frac{1}{2} \sum_{n \neq 0} n^2 |a_n|^2} \prod_{n \neq 0} da_n$  and  $\widetilde{Z}_2 = \int e^{-\frac{1}{2} \sum_{n \neq 0} \kappa |n| |b_n|^2} \prod_{n \neq 0} db_n$ . Then, since  $d\rho = d\rho_1 \otimes d\rho_2$ , it suffices to prove, for arbitrary  $r < \infty$ ,

$$\exp\left(\|\widehat{\phi}\|_{L_n^{\frac{4}{3}-}^{4+}}\right) \chi_{\{\|\phi\|_{L^2} \leq R\}} \in L^r(d\rho_1), \quad (4.4)$$

$$\exp\left(\|\widehat{\psi}\|_{L_n^{2+}}^{2-}\right) \in L^r(d\rho_2), \quad (4.5)$$

for  $\phi$  and  $\psi$  with mean 0. First, note that, by Hölder inequality, we have

$$\|\{\widehat{\phi}(n)\}_{n \sim M}\|_{L_n^{\frac{4}{3}-}} = \left( \sum_{|n| \sim M} |a_n|^{\frac{4}{3}-} \right)^{\frac{3}{4}+} \leq CM^{\frac{1}{4}+} \left( \sum_{|n| \sim M} |a_n|^2 \right)^{\frac{1}{2}}.$$

for any  $M$  dyadic. Then, we see that the proof of (4.4) is basically the same as that of  $\exp\left(\|\phi\|_{L_x^{4+}}^{4+}\right) \chi_{\{\|\phi\|_{L^2} \leq R\}} \in L^r(d\rho_1)$  in Bourgain [8].

Now, we turn to the proof of (4.5). First, assume that  $(i, \dot{H}^{\frac{1}{2}}, \widehat{L}_n^{2+})$  is an abstract Wiener space (with respect to  $\rho_2$ ), where  $\widehat{L}_n^{2+}$  is the space defined via the norm  $\|\psi\|_{\widehat{L}_n^{2+}} = \|\widehat{\psi}\|_{L_n^{2+}}$ . Then, by Lemma 4.3, we have

$$\begin{aligned} &\left\| e^{\|\psi\|_{\widehat{L}_n^{2+}}^{2-}} \right\|_{L^r(d\rho_2)}^r \leq \int_{\{\|\psi\|_{\widehat{L}_n^{2+}} \leq K\}} e^{r\|\psi\|_{\widehat{L}_n^{2+}}^{2-}} d\rho_2 + \sum_{j=0}^{\infty} \int_{\{2^j K \leq \|\psi\|_{\widehat{L}_n^{2+}} < 2^{j+1} K\}} e^{r\|\psi\|_{\widehat{L}_n^{2+}}^{2-}} d\rho_2 \\ &\leq e^{rK^{2-}} + \sum_{j=0}^{\infty} e^{r(2^{j+1}K)^{2-}} \rho_2[\|\psi\|_{\widehat{L}_n^{2+}} \geq 2^j K] \lesssim e^{rK^{2-}} + \sum_{j=1}^{\infty} e^{-c(2^j K)^2 + 2^{2-j} r(2^j K)^{2-}} < \infty. \end{aligned}$$

Hence, it remains to show that  $(i, \dot{H}^{\frac{1}{2}}, \widehat{L}_n^{2+})$  is an abstract Wiener space. Recall that  $\psi = \sum_{n \neq 0} b_n e^{inx} = \sum_{n \neq 0} \frac{g_n(\omega)}{|\kappa|^{\frac{1}{2}} |n|^{\frac{1}{2}}} e^{inx}$ , where  $\{g_n(\omega)\}$  is a sequence of i.i.d.

complex Gaussian random variables with  $g_{-n} = \overline{g_n}$ . Note that it suffices to show that for given  $\varepsilon > 0$  there exists large  $M_0$  such that  $\rho_2[\|\mathbb{P}_{>M_0}\psi\|_{\widehat{L}_n^{2+}} > \varepsilon] < \varepsilon$ , where  $\mathbb{P}_{>M_0}$  is the Dirichlet projection onto the frequencies  $|n| > M_0$ . First, we present a lemma which provides an a.s. decay at high frequencies in the sub- $L_n^2$  setting.

LEMMA 4.7. *Let  $\{g_n\}$  be a sequence of i.i.d Gaussian random variables. Then, for  $M$  dyadic and  $\delta > 0$ , we have*

$$\lim_{M \rightarrow \infty} M^{1-\delta} \frac{\max_{|n| \sim M} |g_n|^2}{\sum_{|n| \sim M} |g_n|^2} = 0, \text{ a.s.} \quad (4.6)$$

*Proof.* We show  $\lim_{n \rightarrow \infty} n^{1-\delta} \frac{\max_{1 \leq j \leq n} |g_j|^2}{\sum_{j=1}^n |g_j|^2} = 0$ , a.s. Let  $X_n = |g_n|^2$ . Then,  $\{X_n\}_{n \in \mathbb{N}}$  is a sequence of i.i.d random variables with all of its moments finite. By Kolmogorov's SLLN, we have  $\frac{S_n}{n} \rightarrow 1$  a.s., where  $S_n = \sum_{j=1}^n X_j$ . Now, fix  $\varepsilon > 0$  and  $\theta > 0$ . By Chebyshev's inequality, we have

$$n^{2+\theta} \mathbb{P}[X_1 > \varepsilon n^\delta] \leq \varepsilon^{-\frac{2+\theta}{\delta}} \int_{\{\omega: X_1(\omega) > \varepsilon n^\delta\}} X_1^{\frac{2+\theta}{\delta}} \mathbb{P}(d\omega) \leq \varepsilon^{-\frac{2+\theta}{\delta}} \mathbb{E}[X_1^{\frac{2+\theta}{\delta}}] < \infty \quad (4.7)$$

for all  $n \in \mathbb{N}$  as long as  $\delta > 0$ . Let  $M_n = \max_{1 \leq j \leq n} X_j$ . Then, by the independence of  $X_j$  and (4.7), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[M_n > \varepsilon n^\delta] \leq \sum_{n=1}^{\infty} n \mathbb{P}[X_1 > \varepsilon n^\delta] = \sum_{n=1}^{\infty} n^{-1-\theta} n^{2+\theta} \mathbb{P}[X_1 > \varepsilon n^\delta] \lesssim \sum_{n=1}^{\infty} n^{-1-\theta} < \infty.$$

Then, by Borel-Cantelli lemma,  $\mathbb{P}[\frac{M_n}{n^\delta} > \varepsilon, \text{ i.o.}] = 0$ . This implies that  $\limsup_{n \rightarrow \infty} \frac{M_n}{n^\delta} \leq \varepsilon$ , a.s. Since  $\limsup_{n \rightarrow \infty} \frac{M_n}{n^\delta}$  is a tail function, it is a.s. constant. Noting that this non-negative constant is bounded above by any  $\varepsilon > 0$ , we conclude that  $\limsup_{n \rightarrow \infty} \frac{M_n}{n^\delta} = 0$ , a.s.  $\square$

Now, fix  $\varepsilon > 0$  and  $\delta \in (0, \frac{1}{2})$ . Then, by Lemma 4.7 and Egoroff's theorem, there exists a set  $E$  with  $\rho_2(E^c) < \frac{1}{2}\varepsilon$  such that the convergence in Lemma 4.7 is uniform on  $E$ . i.e. we can choose dyadic  $M_0$  large such that

$$\frac{\|\{g_n(\omega)\}_{|n| \sim M}\|_{L_n^\infty}}{\|\{g_n(\omega)\}_{|n| \sim M}\|_{L_n^2}} \leq M^{-\delta}, \quad (4.8)$$

for all  $\omega \in E$  and dyadic  $M > M_0$ . In the following, we will work only on  $E$  and drop ' $\cap E$ ' for notational simplicity. However, it should be understood that all the events are under the intersection with  $E$  and thus (4.8) holds.

Let  $\{\sigma_j\}_{j \geq 1}$  be a sequence of positive numbers such that  $\sum \sigma_j = 1$ , and let  $M_j = M_0 2^j$  dyadic. Note that  $\sigma_j = C 2^{-\lambda j} = C M_0^\lambda M_j^{-\lambda}$  for some small  $\lambda > 0$  (to be determined later.) Then, we have

$$\rho_2[\|\mathbb{P}_{>M_0}\psi\|_{\widehat{L}_n^{2+}} > \varepsilon] \leq \sum_{j=0}^{\infty} \rho_2[\|\{b_n\}_{|n| \sim M_j}\|_{L_n^{2+}} > \sigma_j \varepsilon]. \quad (4.9)$$

By Hölder inequality and (4.8), we have

$$\begin{aligned} \|\{b_n\}_{|n| \sim M_j}\|_{L_n^{2+\theta}} &\sim M_j^{-\frac{1}{2}} \|\{g_n\}_{|n| \sim M_j}\|_{L_n^{2+\theta}} \leq M_j^{-\frac{1}{2}} \|\{g_n\}_{|n| \sim M_j}\|_{L_n^2}^{\frac{2}{2+\theta}} \|\{g_n\}_{|n| \sim M_j}\|_{L_n^\infty}^{\frac{\theta}{2+\theta}} \\ &\leq M_j^{-\frac{1}{2}} \|\{g_n\}_{|n| \sim M}\|_{L_n^2} \left( \frac{\|\{g_n\}_{|n| \sim M_j}\|_{L_n^\infty}}{\|\{g_n\}_{|n| \sim M_j}\|_{L_n^2}} \right)^{\frac{\theta}{2+\theta}} \leq M_j^{-\frac{1}{2}-\delta \frac{\theta}{2+\theta}} \|\{g_n\}_{|n| \sim M_j}\|_{L_n^2} \end{aligned}$$

a. s. where  $\theta = 0+$ . Thus, if we have  $\|\{b_n\}_{|n|\sim M_j}\|_{L_n^{2+}} > \sigma_j \varepsilon$ , then we have  $\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j$  where  $R_j := \sigma_j \varepsilon M_j^{\frac{1}{2} + \delta \frac{\theta}{2+\theta}}$ . Now, take  $\lambda$  sufficiently small such that  $\delta \frac{\theta}{2+\theta} - \lambda > 0$ . By a direct computation in the polar coordinates with  $R_j = \sigma_j \varepsilon M_j^{\frac{1}{2} + \delta \frac{\theta}{2+\theta}} = C \varepsilon M_0^\lambda M_j^{\frac{1}{2} + \delta \frac{\theta}{2+\theta} - \lambda} = C \varepsilon M_0^\lambda M_j^{\frac{1}{2} +}$ , we have

$$\rho_2[\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j] \sim \int_{B^c(0, R_j)} e^{-\frac{1}{2}|g|^2} \prod_{|n|\sim M_j} dg_n \lesssim \int_{R_j}^{\infty} e^{-\frac{1}{2}r^2} r^{2 \cdot \#\{|n|\sim M_j\}-1} dr, \quad (4.10)$$

Note that the implicit constant in the inequality is  $\sigma(S^{2 \cdot \#\{|n|\sim M_j\}-1})$ , a surface measure of the  $2 \cdot \#\{|n|\sim M_j\} - 1$  dimensional unit sphere. We drop it since  $\sigma(S^n) = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}) \lesssim 1$ . By change of variable  $t = M_j^{-\frac{1}{2}}r$ , we have  $r^{2 \cdot \#\{|n|\sim M_j\}-2} \lesssim r^{4M_j} \sim M_j^{2M_j} t^{4M_j}$ . Since  $t > M_j^{-\frac{1}{2}}R_j = C \varepsilon M_0^\lambda M_j^{0+}$ , we have  $M_j^{2M_j} = e^{2M_j \ln M_j} < e^{\frac{1}{8}M_j t^2}$  and  $t^{4M_j} < e^{\frac{1}{8}M_j t^2}$  for  $M_0$  sufficiently large. Thus, we have  $r^{2 \cdot \#\{|n|\sim M_j\}-2} < e^{\frac{1}{4}M_j t^2} = e^{\frac{1}{4}r^2}$  for  $r > R$ . Hence, we have

$$\begin{aligned} \rho_2[\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} \gtrsim R_j] &\leq C \int_{R_j}^{\infty} e^{-\frac{1}{4}r^2} r dr \leq e^{-cR_j^2} \\ &= e^{-cC^2 M_0^{2\lambda} M_j^{1+\varepsilon^2}} = e^{-cC^2 M_0^{1+2\lambda+2j+\varepsilon^2}}. \end{aligned} \quad (4.11)$$

From (4.9) and (4.11), we have

$$\rho_2[\|\{b_n\}_{|n|> M_0}\|_{L_n^{2+}} > \frac{1}{2}\varepsilon] \leq \sum_{j=1}^{\infty} \rho_2[\|\{g_n\}_{|n|\sim M_j}\|_{L_n^2} > R_j] \leq e^{-c'M_0^{1+2\lambda+2j+\varepsilon^2}} < \frac{1}{2}\varepsilon,$$

by choosing  $M_0$  sufficiently large. This completes the proof of Lemma 4.5.  $\square$

**Appendix. On the ill-posedness results in  $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  for  $s < \frac{1}{2}$ .** There are so called ‘‘ill-posedness’’ results for dispersive equations such as NLS and KdV. However, this term often refers to the necessary conditions for uniform continuity or smoothness of the solution map  $\Phi^t : u_0 \in H^s \mapsto u(t) \in H^s$ . In such cases, the Cauchy problem is not necessarily ill-posed in the sense of the usual definition, even when these results hold. However, since the contraction argument provides analytic dependence on the initial data, it is often natural to consider a strengthened notion of well-posedness requiring the solution map to be uniformly continuous/smooth. In this latter sense, the following results may be regarded as ‘‘ill-posedness’’ results.

Here, we follow Bourgain’s argument in [7]. Consider the following Cauchy problem:

$$\begin{cases} iu_t + u_{xx} = \alpha v u \\ v_t + \gamma \mathcal{H}v_{xx} = \beta(|u|^2)_x \\ (u(x, 0), v(x, 0)) = (\delta\phi(x), \delta\psi(x)) \in H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T}) \end{cases} \quad (A.1)$$

where  $\delta \geq 0$ . Let  $(u(x, t; \delta), v(x, t; \delta))$  or  $(u(t; \delta), v(t; \delta))$  denote the solution to (A.1). First, note that with  $\delta = 0$ ,  $(u(x, t; 0), v(x, t; 0)) \equiv 0$  is the unique solution. Also, by writing as integral equations, the solution  $(u(t; \delta), v(t; \delta))$  to (A.1) can be written as

$$\begin{cases} u(t; \delta) = \delta U(t)\phi + i\alpha \int_0^t U(t-t')uv(t')dt' \\ v(t; \delta) = \delta V(t)\psi - \beta \int_0^t V(t-t')\partial_x(|u(t')|^2)dt'. \end{cases}$$

where  $U(t) = e^{it\partial_x^2}$  and  $V(t) = e^{-\gamma\mathcal{H}\partial_x^2}$ . By taking derivatives in  $\delta$  at  $\delta = 0$ , we have  $\partial_\delta u(t; 0) = U(t)\phi =: \phi_1$  and  $\partial_\delta v(t; 0) = V(t)\psi =: \psi_1$ . By taking the 2nd derivative in  $\delta$  at  $\delta = 0$ , we have

$$\begin{cases} \partial_\delta^2 u(t; 0) = 2i\alpha \int_0^t U(t-t')(\phi_1\psi_1)(t')dt' =: \phi_2 \\ \partial_\delta^2 v(t; 0) = -2\beta \int_0^t V(t-t')\partial_x(|\phi_1|^2)(t')dt' =: \psi_2. \end{cases}$$

Then, it follows that if the solution map  $\Phi^t : (u_0, v_0) \in H^s \times H^{s-\frac{1}{2}} \mapsto (u(t), v(t)) \in H^s \times H^{s-\frac{1}{2}}$  is  $C^2$  for fixed  $t \ll 1$ , then we must have

$$\|\partial_\delta^2(u, v)(\cdot, t; 0)\|_{H_x^s \times H_x^{s-\frac{1}{2}}} = \|(\phi_2, \psi_2)(\cdot, t)\|_{H_x^s \times H_x^{s-\frac{1}{2}}} \lesssim \|(\phi, \psi)\|_{H^s \times H^{s-\frac{1}{2}}}^2 \quad (\text{A.2})$$

from the smoothness of  $\Phi^t$  at the zero solution. In the following, we present the proof of Theorem 1.1, assuming that the SBO system (1.1) is locally well-posed in  $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$  over a small time interval and fix  $t \ll 1$  such that the solution map  $\Phi^t$  is well-defined.

We'd like to point out the following before presenting the proof of Theorem 1.1. On the one hand, when  $|\gamma| \neq 0, 1$ , we exploit the high-high interaction as in [7] for KdV. On the other hand, when  $|\gamma| = 1$ , we exploit the high-low interaction as in Molinet-Saut-Tzvetkov [28], [29] for the Benjamin-Ono and KP-I equations.

*Proof of Theorem 1.1.* Recall that with  $n = n_1 + n_2$ , we have

$$Q(n, n_1) := \gamma|n|n - n_1^2 + n_2^2 = n((1 + \gamma \operatorname{sgn}(n))n - 2n_1).$$

Let  $c_\gamma = c_\gamma(n) = \frac{1+\gamma \operatorname{sgn}(n)}{2}$  and  $d_\gamma = 1 - c_\gamma$ . Thus,  $Q(n, n_1) = 0$  when  $n_1 = c_\gamma n$  and  $n_2 = d_\gamma n$ . Now, let  $\|\rho\|$  denote the closest integer to  $\rho$ . (If  $\rho - [\rho] = \frac{1}{2}$ , let  $\|\rho\| = [\rho]$ , where  $[\cdot]$  is the integer part function.)

Given  $N \in \mathbb{N}$ , let  $\psi \equiv 0$  and  $\phi(x) = N^{-s}(e^{i\|c_\gamma N\|x} + e^{-i\|d_\gamma N\|x})$  with  $c_\gamma = c_\gamma(N)$  and  $d_\gamma = d_\gamma(N)$ . Then, we have  $\|(\phi, \psi)\|_{H^s \times H^{s-\frac{1}{2}}} = \|\phi\|_{H^s} \sim 1$ . A direct computation shows that  $\psi_1 = \phi_2 \equiv 0$  and

$$\phi_1(x, t) = N^{-s}(e^{i\|c_\gamma N\|x - i\|c_\gamma N\|^2 t} + e^{-i\|d_\gamma N\|x - i\|d_\gamma N\|^2 t}).$$

Using  $\|c_\gamma N\| + \|d_\gamma N\| = N$ , we have

$$V(t-t')\partial_x|\phi_1(x, t')|^2 = -2N^{-2s+1} \sin(Nx - \gamma|N|Nt + Q(N, \|c_\gamma N\|)t').$$

Suppose  $\gamma \in \mathbb{Q}$  with  $|\gamma| \neq 1$ . Then, for  $n = N \in \mathbb{N}$ , we have  $c_\gamma(n) = \frac{1+\gamma}{2} \in \mathbb{Q}$ . Thus, there exist infinitely many  $N \in \mathbb{N}$  such that  $c_\gamma N, d_\gamma N \in \mathbb{Z}$ . Hence, we have  $Q(N, \|c_\gamma N\|) = Q(N, c_\gamma N) = 0$  for all such  $N$ . Then,

$$\psi_2 = 4\beta N^{-2s+1} t \sin(Nx - \gamma|N|Nt)$$

and thus  $\|(\phi_2, \psi_2)\|_{H^s \times H^{s-\frac{1}{2}}} \sim \|\psi_2\|_{H^{s-\frac{1}{2}}} \sim N^{-s+\frac{1}{2}}$ . In view of (A.2), by letting  $N \rightarrow \infty$ , this implies  $s \geq \frac{1}{2}$  if the solution map  $\Phi^t$  is  $C^2$ .

Now, suppose  $\gamma \notin \mathbb{Q}$ . Then, we have  $Q(N, \|c_\gamma N\|) \neq 0$  for any  $n \in \mathbb{N}$ . From (A), we have

$$\begin{aligned} \psi_2(x, t) &= 4\beta N^{-2s+1} \frac{\cos(Nx - \gamma|N|Nt) - \cos(Nx - (\|c_\gamma N\|^2 - \|d_\gamma N\|^2)t)}{Q(N, \|c_\gamma N\|)} \\ &= 8\beta N^{-2s+1} \frac{\sin(Nx - 2\gamma|N|Nt + Q(N, \|c_\gamma N\|)t) \sin(Q(N, \|c_\gamma N\|)t)}{Q(N, \|c_\gamma N\|)}. \end{aligned}$$

Recall that Dirichlet Theorem [25] says that for given  $\rho \in \mathbb{R} \setminus \mathbb{Q}$ , there exist infinitely many  $(p, q) \in \mathbb{Z}^2$  such that  $|\rho - \frac{p}{q}| < \frac{1}{q^2}$ . In our context, it says that there exist infinitely many  $N \in \mathbb{N}$  such that

$$|Q(N, \|c_\gamma N\|)| = \min_{n_1 \in \mathbb{Z}} |Q(N, n_1)| = N^2 \left| c_\gamma - \frac{2n_1}{N} \right| \leq 1.$$

Then, for such  $N$ , we have  $\|(\phi_2, \psi_2)\|_{H^s \times H^{s-\frac{1}{2}}} \sim N^{-s+\frac{1}{2}}$ . In view of (A.2), by letting  $N \rightarrow \infty$ , this implies  $s \geq \frac{1}{2}$  if the solution map  $\Phi^t$  is  $C^2$ .

Finally, we consider the case when  $|\gamma| = 1$ . Without loss of generality, assume  $\gamma = 1$ . Then,  $Q(N, N) = 0$  for all  $N \in \mathbb{N}$ , i.e.  $c_\gamma = 1$  and  $d_\gamma = 0$ . (When  $\gamma = -1$ , we have  $Q(N, N) = 0$  for all  $N \in \mathbb{Z}_{<0}$  and the following argument can be easily modified.) Given  $N \in \mathbb{N}$ , let  $\psi \equiv 0$  and  $\phi(x) = N^{-s} e^{iNx} + 1$ . Then, we have  $\|(\phi, \psi)\|_{H^s \times H^{s-\frac{1}{2}}} = \|\phi\|_{H^s} \sim 1$ . A direct computation shows that  $\psi_1 = \phi_2 \equiv 0$  and  $\phi_1(x, t) = N^{-s} e^{iNx - iN^2 t} + 1$ . Thus, we have  $|\phi_1(x, t)|^2 = 2N^{-s} \cos(Nx - N^2 t) + N^{-2} + 1$  in this case and

$$V(t - t') \partial_x |\phi_1(x, t')|^2 = -2N^{-s+1} \sin(Nx - N^2 t).$$

since  $Q(N, N) = 0$ . Therefore, we have

$$\psi_2 = 4\beta N^{-s+1} t \sin(Nx - N^2 t)$$

and thus  $\|(\phi_2, \psi_2)\|_{H^s \times H^{s-\frac{1}{2}}} \sim \|\psi_2\|_{H^{s-\frac{1}{2}}} \sim N^{\frac{1}{2}}$ . In view of (A.2), by letting  $N \rightarrow \infty$ , this implies that the solution map  $\Phi^t$  can *never* be  $C^2$  for any  $s \in \mathbb{R}$ .  $\square$

**Acknowledgments.** The author would like to thank Prof. Luc Rey-Bellet for mentioning the work of Gross [21] and Kuo [24]. He is also grateful to the anonymous referees for their comments.

#### REFERENCES

- [1] J. Angulo, C. Matheus, D. Pilod, *Global well-posedness and non-linear stability of periodic traveling waves for a Schrödinger-Benjamin-Ono System*, Commun. Pure Appl. Anal. 8 (2009), no. 3, 815–844.
- [2] D. Bekiranov, T. Ogawa, G. Ponce, *Interaction equations for short and long dispersive waves*, J. Funct. Anal. 158 (1998), 357–388.
- [3] D.J. Benney, *Significant interactions between small and large scale surface waves*, Stud. Appl. Math. 55 (1976), 93–106.
- [4] D.J. Benney, *A general theory of interactions between short and long waves*, Stud. Appl. Math. 56 (1977), 81–94.
- [5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II*, Geom. Funct. Anal., 3 (1993), 107–156, 209–262.
- [6] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notice, 5 (1998), 253–283.
- [7] J. Bourgain, *Periodic Korteweg-de Vries equation with measures as initial data*, Sel. Math., New Ser. 3 (1997) 115–159.
- [8] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. 166 (1994), 1–26.
- [9] J. Bourgain, *On the Cauchy and invariant measure problem for the periodic Zakharov system*, Duke Math. J. 76 (1994), no. , 175–202.
- [10] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996), 421–445.
- [11] J. Bourgain, *Nonlinear Schrödinger Equations*, Hyperbolic equations and frequency interactions (Park City, UT, 1995), 3–157, IAS/Park City Math. Ser., 5, Amer. Math. Soc., Providence, RI, 1999.

- [12] N. Burq, N. Tzvetkov, *Invariant measure for a three dimensional nonlinear wave equation*, Int. Math. Res. Not. (2007), no. 22, Art. ID rnm108, 26pp.
- [13] J. Colliander, J. Holmer, N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Trans. Amer. Math. Soc. 360 (2008), no. 9, 4619–4638.
- [14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness for Schrödinger equations with derivative*, SIAM J. Math. Anal. 33 (2001) no.3, 649–669.
- [15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global Well-Posedness for KdV and Modified KdV on  $\mathbb{R}$  and  $\mathbb{T}$* , J. Amer. Math. Soc. 16 (2003), no. 3, 705–749.
- [16] V. Djordjevic, L. Redekopp, *On two-dimensional packet of capillary-gravity waves*, J. Fluid Mech. 79 (1977), 703–714.
- [17] M.X. Fernique, *Intégrabilité des Vecteurs Gaussiens*, Academie des Sciences, Paris, Comptes Rendus, 270, Séries A (1970), 1698–1699.
- [18] M. Funakoshi, M. Oikawa, *The resonant interaction between a long internal gravity wave and a surface gravity wave packet*, J. Phys. Soc. Japan, 52 (1983), 1982–1995.
- [19] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy Problem for the Zakharov System*, J. Funct. Anal., 151 (1997), 384–436.
- [20] R. Grimshaw, *The modulation of an internal gravity-wave packet and the resonance with the mean motion*, Stud. Appl. Math. 56 (1977), 241–266.
- [21] L. Gross, *Abstract Wiener spaces*, Proc. 5th Berkeley Sym. Math. Stat. Prob. 2 (1965), 31–42.
- [22] V. Karpman, *On the dynamics of sonic-Langmuir soliton*, Physica Scripta. 11 (1975), 263–265.
- [23] C. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. 9 (1996), no. 2 573–603.
- [24] H. Kuo, *Gaussian Measures in Banach Spaces*, Lec. Notes in Math. 463, Springer-Verlag, New York, 1975.
- [25] S. Lang, *Introduction to Diophantine approximations*, 2nd ed., Springer-Verlag, New York, 1995.
- [26] J. Lebowitz, H. Rose, E. Speer, *Statistical Mechanics of the Nonlinear Schrödinger Equation*, J. Stat. Phys. 50 (1988), no.3, 657–687.
- [27] L. Molinet, *Global well-posedness in the energy space for the Benjamin-Ono equation on the circle*, Math. Ann. 337 (2007), no. 2, 353–383.
- [28] L. Molinet, J.C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. 33 (2001), no. 4, 982–988.
- [29] L. Molinet, J.C. Saut, N. Tzvetkov, *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation*, Duke Math. J. 115 (2002), no. 2, 353–384.
- [30] T. Oh, *Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems*, Diff. Integ. Equations, Vol. 22, Numbers 7-8 (2009), 637-668.
- [31] T. Oh, *Invariance of the White Noise for KdV*, to appear in Comm. Math. Phys.
- [32] H. Pecher, *Rough Solutions of a Schrödinger-Benjamin-Ono System*, Diff. Integ. Eq. 19 (2006), 517–535.
- [33] N. Tzvetkov, *Invariant measures for the nonlinear Schrödinger equation on the disc*, Dyn. Partial Differ. Equ. 3 (2006), no. 2, 111–160.
- [34] N. Tzvetkov, *Invariant measures for the defocusing nonlinear Schrödinger equation*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 7, 2543–2604.
- [35] N. Tzvetkov, *Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation*, to appear in Probability Theory and Related Fields.
- [36] P. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*, Lec. Notes in Math. 1756, Springer-Verlag, 2001.