

APM384H1F HW Solution:

• HW11:

9.5.11: (a) Let $\mathbf{x}_0^* = (x_0, -y_0)$ where $\mathbf{x}_0 = (x_0, y_0)$. Now, let $G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0| + \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0^*|$. Then, we have

$$\frac{\partial G}{\partial y} \Big|_{y=0} = \frac{1}{4\pi} \frac{2(y-y_0)}{|\mathbf{x} - \mathbf{x}_0|^2} + \frac{1}{4\pi} \frac{2(y+y_0)}{|\mathbf{x} - \mathbf{x}_0^*|^2} \Big|_{y=0} = \frac{1}{4\pi} \frac{-2y_0}{(x-x_0)^2 + y_0^2} + \frac{1}{4\pi} \frac{2y_0}{(x-x_0)^2 + y_0^2} = 0.$$

Moreover, for $\mathbf{x}_0 \in \mathbb{R}_+^2 = \{(x, y) : y > 0\}$, we have $\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ since $\Delta_{\mathbf{x}} \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0^*| = 0$ on \mathbb{R}_+^2 .

(b) On the boundary of \mathbb{R}_+^2 , we have $\frac{\partial G}{\partial \bar{n}} = 0$ and $\frac{\partial u}{\partial \bar{n}} = -\frac{\partial u}{\partial y} \Big|_{y=0} = -h(x)$. By Green's second identity (with symmetry of $G(\mathbf{x}, \mathbf{x}_0)$ in \mathbf{x} and \mathbf{x}_0), we have

$$u(\mathbf{x}) = \iint_{\mathbb{R}_+^2} u \Delta_{\mathbf{x}_0} G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}_0 = \iint_{\mathbb{R}_+^2} f(\mathbf{x}_0) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}_0 + \int_{-\infty}^{\infty} h(x_0) G(\mathbf{x}, x_0, 0) dx_0.$$

9.5.14: For $\mathbf{x}_0 = (x_0, y_0)$ in the first quadrant (i.e. $x_0, y_0 > 0$), define $\mathbf{x}_0^* = (x_0, -y_0)$, $\mathbf{x}_0^{**} = (-x_0, y_0)$, and $\mathbf{x}_0^{***} = (-x_0, -y_0)$. Now, define the Green's function on the first quadrant with zero boundary condition:

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0| - \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0^*| - \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0^{**}| + \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0^{***}|.$$

Then, we have $G = 0$ for $y = 0$ with $x > 0$ and for $x = 0$ with $y > 0$. (You need to show this.) Moreover, $\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ for \mathbf{x}_0 in the first quadrant (since $\mathbf{x}_0^*, \mathbf{x}_0^{**}, \mathbf{x}_0^{***}$ are not in the first quadrant.)

9.5.18: From Chapter 2, we have

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta,$$

where $A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$,

$$A_n a^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta, \quad \text{and} \quad B_n a^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta.$$

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\sum_{n=1}^{\infty} \frac{r^n}{a^n} 2(\cos n\theta \cos n\theta_0 + \sin n\theta \sin n\theta_0) + 1 \right] d\theta_0 \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\sum_{n=1}^{\infty} \frac{r^n}{a^n} \underbrace{2 \cos n(\theta - \theta_0)}_{=e^{in(\theta-\theta_0)} + e^{-in(\theta-\theta_0)}} + 1 \right] d\theta_0 \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\sum_{n=1}^{\infty} \left(\frac{r e^{i(\theta-\theta_0)}}{a} \right)^n + \sum_{n=1}^{\infty} \left(\frac{r e^{-i(\theta-\theta_0)}}{a} \right)^n + 1 \right] d\theta_0. \end{aligned}$$

Recall that $\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}$ for $|\alpha| < 1$.

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\frac{\frac{re^{i(\theta-\theta_0)}}{a}}{1 - \frac{re^{i(\theta-\theta_0)}}{a}} + \frac{\frac{re^{-i(\theta-\theta_0)}}{a}}{1 - \frac{re^{-i(\theta-\theta_0)}}{a}} + 1 \right] d\theta_0 \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\frac{re^{i(\theta-\theta_0)}}{a - re^{i(\theta-\theta_0)}} + \frac{re^{-i(\theta-\theta_0)}}{a - re^{-i(\theta-\theta_0)}} + 1 \right] d\theta_0 \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \left[\frac{2ar \cos(\theta - \theta_0) - 2r^2}{r^2 + a^2 - 2ar \cos(\theta - \theta_0)} + 1 \right] d\theta_0 \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\theta_0) \frac{a^2 - r^2}{r^2 + a^2 - 2ar \cos(\theta - \theta_0)} d\theta_0,
\end{aligned}$$

which is the Poisson's formula.

9.5.19: In Subsection 9.5.9, we computed the Green's function on the disk with zero boundary condition, which we denote $\tilde{G}(\mathbf{x}, \mathbf{x}_0)$. Take $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{D}_+ = \{0 < r < a, 0 < \theta < \pi\}$, and define \mathbf{x}_0^* by $\mathbf{x}_0^* = (x_0, -y_0)$. (In the polar coordinates, $\mathbf{x}_0^* = (r_0, -\theta_0)$ if $\mathbf{x}_0 = (r_0, \theta_0)$.) Now, define $G(\mathbf{x}, \mathbf{x}_0) = \tilde{G}(\mathbf{x}, \mathbf{x}_0) - \tilde{G}(\mathbf{x}, \mathbf{x}_0^*)$ for $\mathbf{x}_0 \in \mathbb{D}_+$. Then, G has the desired properties. (Once again, you need to check the properties.)

9.5.21: Recall that the Green's function on \mathbb{R}^3 is given by $-\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$. Define the Green's function on the ball $B(0, a)$ with zero boundary condition by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}$$

for $\mathbf{x}_0 \in B(0, a)$ and $\mathbf{x}_0^* = \frac{a^2}{\rho_0^2} \mathbf{x}_0$ (which is obtained by following the computation on p.431, just by replacing r_0 by ρ_0). Now, check that the $G(\mathbf{x}, \mathbf{x}_0)$ thus obtained has the desired properties.

9.5.22: (c) The Green's function on \mathbb{R}^3 is given by $G_0 = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$. We will construct the Green's function on the infinite strip by inductive steps. In order to have $G = 0$ for $x = 0$, we need to have a negative source at $(-x_0, y_0, z_0)$. i.e. define $G_1 = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}$. Now, in order to have $G = 0$ at $x = L$, we need to \mathbf{x}_0 and \mathbf{x}_0^* about $x = L$ (which are given by $(-x_0 + 2L, y_0, z_0)$ and $(x_0 + 2L, y_0, z_0)$), and define

$$\begin{aligned}
G_2 = & -\frac{1}{4\pi|\mathbf{x} - (x_0, y_0, z_0)|} + \frac{1}{4\pi|\mathbf{x} - (-x_0, y_0, z_0)|} \\
& + \frac{1}{4\pi|\mathbf{x} - (-x_0 + 2L, y_0, z_0)|} - \frac{1}{4\pi|\mathbf{x} - (x_0 + 2L, y_0, z_0)|}.
\end{aligned}$$

Now, one repeats this process indefinitely, and obtain

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \left(\frac{1}{|\mathbf{x} - (x_0 + 2nL, y_0, z_0)|} - \frac{1}{|\mathbf{x} - (-x_0 + 2nL, y_0, z_0)|} \right).$$

(d) By repeating the argument in part (c), we obtain the Green's function in the infinite

strip $0 < x < L$, $-\infty < y < \infty$ given by

$$\begin{aligned}\tilde{G}(\mathbf{x}, \mathbf{x}_0) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\ln |\mathbf{x} - (x_0 + 2nL, y_0)| - \ln |\mathbf{x} - (-x_0 + 2nL, y_0)| \right) \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \ln \frac{|\mathbf{x} - (x_0 + 2nL, y_0)|}{|\mathbf{x} - (-x_0 + 2nL, y_0)|}.\end{aligned}$$

Now, we need to make a reflection with respect to $\{y = 0\}$ so that the resulting function satisfies the zero boundary condition on $\{y = 0\}$. Hence, we obtain

$$\begin{aligned}G(\mathbf{x}, \mathbf{x}_0) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\ln \frac{|\mathbf{x} - (x_0 + 2nL, y_0)|}{|\mathbf{x} - (-x_0 + 2nL, y_0)|} - \ln \frac{|\mathbf{x} - (x_0 + 2nL, -y_0)|}{|\mathbf{x} - (-x_0 + 2nL, -y_0)|} \right) \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \ln \frac{|\mathbf{x} - (x_0 + 2nL, y_0)| |\mathbf{x} - (-x_0 + 2nL, -y_0)|}{|\mathbf{x} - (-x_0 + 2nL, y_0)| |\mathbf{x} - (x_0 + 2nL, -y_0)|}.\end{aligned}$$

9.5.23: (a)

$$u(\mathbf{x}) = \iint_{\mathbb{R}^2} f(\mathbf{x}_0) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}_0 = \frac{1}{2\pi} \iint_{\mathbb{R}^2} g(|\mathbf{x}_0|) \ln |\mathbf{x} - \mathbf{x}_0| d\mathbf{x}_0$$

(b) Consider

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = g(r) \text{ for } r > 0 \quad (1)$$

$$\iff \frac{d}{dr} \left(r \frac{du}{dr} \right) = rg(r).$$

First, let's try to solve $\frac{d}{dr} \left(r \frac{du}{dr} \right) = F(r)$. What is a reasonable definition of the Green's function G for this problem? Suppose that G satisfies

$$\frac{d}{dr} \left(r \frac{dG(r, r_0)}{dr} \right) = \delta(r - r_0). \quad (2)$$

Then, by integration by parts (e.g. Green's second identity), we have

$$\begin{aligned}& \int_0^\infty \frac{d}{dr} \left(r \frac{du(r)}{dr} \right) G(r, r_0) dr - \int_0^\infty u(r) \frac{d}{dr} \left(r \frac{dG(r, r_0)}{dr} \right) dr \\ &= \underbrace{- \int_0^\infty r \frac{du}{dr} \frac{dG(r, r_0)}{dr} dr + \int_0^\infty \frac{du}{dr} r \frac{dG(r, r_0)}{dr} dr}_{=0} \\ &+ G(r, r_0) r \frac{du(r)}{dr} \Big|_0^\infty - u(r) r \frac{dG(r, r_0)}{dr} \Big|_0^\infty.\end{aligned}$$

i.e.

$$u(r_0) = \int_0^\infty F(r) G(r, r_0) dr - G(r, r_0) r \frac{du(r)}{dr} \Big|_0^\infty + u(r) r \frac{dG(r, r_0)}{dr} \Big|_0^\infty. \quad (3)$$

First, let's solve (2) by integrating (2) in r . Assume that $r \frac{dG}{dr}(r)|_{r=0} = 0$. Then, we get $r \frac{dG}{dr} = \chi_{[r_0, \infty)}(r)$, where χ_A is the characteristic function of a given set A . i.e. $r \frac{dG}{dr} = \chi_{[r_0, \infty)}(r) = 1$ for $r > r_0$ and $= 0$ for $r < r_0$. Note that $\frac{d}{dr}\chi_{[r_0, \infty)}(r) = \delta(r - r_0)$, since $\chi_{[r_0, \infty)}(r)$ has an (upward) jump of height 1 exactly at $r = r_0$. Now, divide by r , and integrate $\frac{dG}{dr} = \frac{1}{r}\chi_{[r_0, \infty)}(r)$. Then, (suppressing r_0 in $G(r, r_0)$), we have

$$G(r) = (\ln r - \ln r_0)\chi_{[r_0, \infty)}(r) + C(r_0).$$

Choose $C(r_0) = \ln r_0$. Then, we have

$$G(r) = \ln r \chi_{[r_0, \infty)}(r) + \ln r_0 \chi_{(0, r_0)}(r).$$

Note that G is continuous (in particular at $r = r_0$) with this choice of $C(r_0)$.

Now, assume u decays fast to 0 as $r \rightarrow \infty$, and $|u(0)|, |\partial_r u(0)| < \infty$. (We could assume $\partial_r u(0) = 0$, since u is really defined on \mathbb{R}^2 and it is independent of θ .) Then, from (3), we have

$$\begin{aligned} u(r_0) &= \int_0^\infty F(r)G(r, r_0)dr \\ &= \int_{r_0}^\infty \ln r F(r)dr + \ln r_0 \int_0^{r_0} F(r)dr. \end{aligned}$$

Hence, the solution to (1) is given by

$$u(r) = \int_r^\infty r_0 \ln r_0 g(r_0) dr_0 + \ln r \int_0^r r_0 g(r_0) dr_0.$$

Note that assuming g decays sufficiently fast as $r_0 \rightarrow \infty$, the first integral is always convergent. On the other hand, as long as $r_0 g(r_0)$ stays bounded near $r_0 = 0$ (which is a reasonable assumption), the second integral is

$$\ln r \int_0^r r_0 g(r_0) dr_0 \sim \ln r \times r \rightarrow 0 \text{ as } r \rightarrow 0.$$

This is the reason why we chose $C(r_0) = \ln r_0$. Other choices would give either the first or the second integral divergent as $r \rightarrow 0$ or $r \rightarrow \infty$.

(c) In this part, we manipulate the result from part (a) to obtain the result in part (b). Fix \mathbf{x} with $|\mathbf{x}| = r$. With $|\mathbf{x}_0| = r_0$, write

$$u(\mathbf{x}) = \frac{1}{2\pi} \iint_{r_0 < r} g(r_0) \ln |\mathbf{x} - \mathbf{x}_0| d\mathbf{x}_0 + \frac{1}{2\pi} \iint_{r_0 > r} g(r_0) \ln |\mathbf{x} - \mathbf{x}_0| d\mathbf{x}_0 =: \text{I} + \text{II}.$$

◦ On $\{r_0 < r\}$: Recall the Green's second identity

$$\iint_{\Omega} u \Delta v - v \Delta u d\mathbf{x} = \oint_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \hat{n} ds. \quad (4)$$

In particular, if u is harmonic on a domain Ω , then we have $\oint_{\partial\Omega} \partial_{\hat{n}} u ds = 0$, where $\partial_{\hat{n}}$ denotes the normal derivative.

Now, note that $\ln|\mathbf{x} - \mathbf{x}_0|$ is harmonic in \mathbf{x}_0 on $\{r_0 < r\}$. From (4) with $u = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x}_0|$ and $v = \frac{r_0^2}{4}$ (note that $\Delta v = 1$ and $\partial_{\hat{n}_0} v = \frac{r_0}{2}$), we have

$$\frac{1}{2\pi} \iint_{r_0 < R} \ln|\mathbf{x} - \mathbf{x}_0| d\mathbf{x}_0 = \frac{1}{2\pi} \oint_{|\mathbf{x}_0|=R} \ln|\mathbf{x} - \mathbf{x}_0| ds_0 \cdot \frac{R}{2}, \quad \text{for } R < r.$$

Let $f(R) = \frac{1}{2\pi} \oint_{|\mathbf{x}_0|=R} \ln|\mathbf{x} - \mathbf{x}_0| ds_0$. Then, from above, we obtain

$$f(R) = \frac{2}{R} \int_0^R f(r_0) dr_0.$$

Thus, $f(r_0) = C_1 r_0$. Keeping \mathbf{x} fixed, take $r_0 \rightarrow 0$. Then, we have $C_1 = \frac{f(r_0)}{r_0} \sim \frac{1}{r_0} \frac{1}{2\pi} 2\pi r_0 \ln r = \ln r$. Therefore, we obtain $I = \ln r \int_0^r r_0 g(r_0) dr_0$.

One can also use the mean value property of the harmonic function and obtain $f(r_0) = r_0 \frac{1}{2\pi r_0} \oint_{|\mathbf{x}_0|=r_0} \ln|\mathbf{x} - \mathbf{x}_0| ds_0 = r_0 \ln|\mathbf{x}| = r_0 \ln r$.

○ On $\{r_0 > r\}$:

In the previous part, we showed $\frac{1}{2\pi} \oint_{|\mathbf{x}_0|=R} \ln|\mathbf{x} - \mathbf{x}_0| ds_0 = R \ln r$ for any $R < r$. In particular, we have $\frac{1}{2\pi} \oint_{|\mathbf{x}_0|=R} \ln \frac{|\mathbf{x} - \mathbf{x}_0|}{|\mathbf{x}|} ds_0 = 1$ for any $R < r$. From here, we see that $\frac{1}{2\pi} \oint_{|\mathbf{x}_0|=R} \ln \frac{|\mathbf{x} - \mathbf{x}_0|}{|\mathbf{x}_0|} ds_0 = 1$ for any $R > r$. (Maybe not so trivial to see this.)

Hence, we have

$$II = \frac{1}{2\pi} \int_r^\infty g(r_0) \int_{|\mathbf{x}_0|=r_0} \ln|\mathbf{x} - \mathbf{x}_0| d\theta_0 r_0 dr_0 = \int_r^\infty g(r_0) r_0 \ln r_0 dr_0.$$

Therefore, the solutions in part (a) and (b) agree.

• HW6 (Suggested problems):

3.2.2 (a) Since x is odd, $a_n = 0$ for all $n \geq 0$. $b_n = \frac{1}{L} \int_{-L}^L x \sin \frac{n\pi x}{L} dx = \frac{2L}{n\pi} (-1)^{n+1}$ by integration by parts.

(b) $a_0 = \frac{1}{2L} \int_{-L}^L e^{-x} dx = \frac{-e^{-L} + e^L}{2L}$. For $n \geq 1$, $a_n = \frac{1}{L} \int_{-L}^L e^{-x} \cos \frac{n\pi x}{L} dx$. Let $I_n = \int e^{-x} \cos \frac{n\pi x}{L} dx$. Then, integrating by parts twice, we have

$$I_n = e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} + \frac{L}{n\pi} \int_{-L}^L e^{-x} \sin \frac{n\pi x}{L} dx = e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} - e^{-x} \frac{L^2}{n^2 \pi^2} \cos \frac{n\pi x}{L} - \frac{L^2}{n^2 \pi^2} I_n.$$

$$\text{i.e. } I_n = \frac{n^2 \pi^2}{L^2 + n^2 \pi^2} \left[e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} - e^{-x} \frac{L^2}{n^2 \pi^2} \cos \frac{n\pi x}{L} \right] = \frac{Ln\pi e^{-x}}{L^2 + n^2 \pi^2} \left[\sin \frac{n\pi x}{L} - \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right].$$

Hence, $a_n = \frac{1}{L} I_n = \frac{L}{L^2 + n^2 \pi^2} (-1)^n (e^L - e^{-L})$. The computation for $b_n = \frac{1}{L} \int_{-L}^L e^{-x} \sin \frac{n\pi x}{L} dx$ is similar and hence is omitted.

(c) $b_1 = 1$ and $b_n = 0$ for all $n \geq 2$ and $a_n = 0$ for all $n \geq 0$.

(d) $a_0 = \frac{1}{2L} \int_0^L x dx = \frac{L}{4}$. For $n \geq 1$, we have $a_n = \frac{1}{L} \int_0^L x \cos \frac{n\pi x}{L} dx = \frac{L}{n^2 \pi^2} (\cos n\pi - 1)$, and $b_n = \frac{1}{L} \int_0^L x \sin \frac{n\pi x}{L} dx = \frac{L}{n\pi} (-1)^{n+1}$ by integration by parts.

(e) $a_0 = \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx = \frac{1}{2}$. For $n \geq 1$, $a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos \frac{n\pi x}{L} dx = \frac{2}{n\pi} \sin \frac{n\pi}{2} = \frac{2(-1)^k}{(2k+1)\pi}$ if $n = 2k + 1$ is odd and $= 0$ if n is even. $b_n = 0$ for all $n \geq 1$ since f is even.

3.3.2 (c) For $n \geq 1$, integrating by parts, we have $B_n = \frac{2}{L} \int_{L/2}^L x \sin \frac{n\pi x}{L} dx = \frac{2}{n\pi} \left[-x \cos \frac{n\pi x}{L} \Big|_{L/2}^L + \int_{L/2}^L \cos \frac{n\pi x}{L} dx \right] = \frac{2}{n\pi} \left[(-1)^{n+1} L + \frac{L}{2} \cos \frac{n\pi}{2} - \frac{L}{n\pi} \sin \frac{n\pi}{2} \right] = \frac{2L}{n\pi} (-1 + \frac{1}{2}(-1)^k)$ when $n = 2k$ is even and $= \frac{2L}{n\pi} (1 - \frac{1}{n\pi}(-1)^k)$ when $n = 2k + 1$ is odd.

3.3.5 (c) $A_0 = \frac{1}{L} \int_{L/2}^L x dx = \frac{x^2}{2} \Big|_{L/2}^L = \frac{3L^2}{8}$. For $n \geq 1$, integrating by parts, we have $A_n = \frac{2}{L} \int_{L/2}^L x \cos \frac{n\pi x}{L} dx = \frac{2}{n\pi} \left[x \sin \frac{n\pi x}{L} \Big|_{L/2}^L - \int_{L/2}^L \sin \frac{n\pi x}{L} dx \right] = \frac{2}{n\pi} \left[\frac{L}{2} \sin \frac{n\pi}{2} + \frac{L}{n\pi} (\cos n\pi - \cos \frac{n\pi}{2}) \right] = \frac{2L}{n^2\pi^2} (1 - (-1)^k)$ when $n = 2k$ is even and $= \frac{L}{n\pi} (-1)^k - \frac{2L}{n^2\pi^2}$ when $n = 2k + 1$ is odd.

3.3.7 $e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x$.

3.3.10 For $x > 0$, $f_{\text{even}} = \frac{1}{2}(e^{-x} + x^2)$ and $f_{\text{odd}} = \frac{1}{2}(e^{-x} - x^2)$. For $x < 0$, $f_{\text{even}} = \frac{1}{2}(x^2 + e^{-x})$ and $f_{\text{odd}} = \frac{1}{2}(x^2 - e^{-x})$.

3.3.17 (a) $\int_0^1 \frac{1}{1+x^2} dx = \arctan x \Big|_0^1 = \frac{\pi}{4}$.

(b), (c) $\int_0^1 \frac{1}{1+x^2} dx = \int_0^1 \frac{1}{1-(-x^2)} dx \int_0^1 \sum_{n=0}^{\infty} (-x^2)^n dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Hence, $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

3.5.1 (a) (3.5.6) says $x^2 = Lx - \frac{8L^2}{\pi^3} \sum_{n, \text{ odd}} \frac{1}{n^3} \sin \frac{n\pi x}{L}$. Then, using (3.3.11) and (3.3.12), we have

$$x^2 = \sum_{n=1}^{\infty} \left[\frac{2L^2(-1)^{n+1}}{n\pi} - \frac{4L^2(1 - (-1)^n)}{n^3\pi^3} \right] \sin \frac{n\pi x}{L}. \quad (5)$$

Note $1 - (-1)^n = 2$ if n is odd and $= 0$ if n is even.

(b) x^2 is continuous and $x^2 \Big|_0 = 0$. Thus, (3.5.12) is an equality for $0 \leq x < L$ (i.e. except for L since $x^2 \Big|_L \neq 0$.)

(c) Integrating (5) term by term (and multiplying by 3), we have

$$x^3 = C + \sum_{n=1}^{\infty} \left[-\frac{6L^3(-1)^{n+1}}{n^2\pi^2} + \frac{12L^3(1 - (-1)^n)}{n^4\pi^4} \right] \cos \frac{n\pi x}{L}, \quad (6)$$

where $C = \frac{1}{L} \int_0^L x^3 dx = \frac{L^3}{4}$. (6) is an equality for all $x \in [0, L]$ since x^3 is continuous. (No extra condition is needed for F.C.S.)

3.5.2 (a) Integrating (3.3.11) term by term (and multiplying by 2), we obtain

$$x^2 = C + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} (-1)^n \cos \frac{n\pi x}{L}, \quad (7)$$

where $C = A_0 = \frac{1}{L} \int_0^L x^2 dx = \frac{L^2}{3}$.

(b) Integrating (7) term by term (and multiplying by 3), we obtain

$$x^3 = L^2 x + D + \sum_{n=1}^{\infty} \frac{12L^3}{n^3\pi^3} (-1)^n \sin \frac{n\pi x}{L} = D + \sum_{n=1}^{\infty} \left[\frac{12L^3}{n^3\pi^3} - \frac{2L^3}{n\pi} \right] (-1)^n \sin \frac{n\pi x}{L}, \quad (8)$$

where we used (3.3.11) and (3.3.12) in the second equality. By evaluating (8) at $x = 0$, we see that $D = 0$.

3.5.4 (b)

$$\cosh x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (9)$$

Since $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, we have $\int \cosh x dx = \sinh x + C$. Then, integrating (9) term by term, we have $\sinh x = C + \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos \frac{n\pi x}{L}$. Integrating term by term once again, we obtain

$$\cosh x = Cx + D + \sum_{n=1}^{\infty} \frac{-b_n L^2}{(n\pi)^2} \sin \frac{n\pi x}{L} \quad (10)$$

Evaluating (10) at $x = 0$ gives $D = 1$, and evaluating (10) at $x = L$ gives $C = L^{-1}(\cosh L - 1)$

Now, equate (9) and (10). Then, we have

$$\sum_{n=1}^{\infty} \left(1 + \frac{L^2}{n^2\pi^2}\right) b_n \sin \frac{n\pi x}{L} = \frac{\cosh L - 1}{L} x + 1. \quad (11)$$

Recall $x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}$ and $1 \sim \sum_{n, \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - (-1)^n) \sin \frac{n\pi x}{L}$. Hence, from (11), we have

$$b_n = \frac{n^2\pi^2}{L^2 + n^2\pi^2} \left(\frac{2}{n\pi} (-1)^{n+1} (\cosh L - 1) + \frac{2}{n\pi} (1 - (-1)^n) \right) = \frac{2n\pi}{L^2 + n^2\pi^2} (1 - (-1)^n \cosh L).$$

3.5.7 Evaluating (3.5.6) at $x = \frac{L}{2}$, we obtain $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$.