

APM384H1F HW Solution:

• HW6 (Suggested problems):

3.2.2 (a) Since x is odd, $a_n = 0$ for all $n \geq 0$. $b_n = \frac{1}{L} \int_{-L}^L x \sin \frac{n\pi x}{L} dx = \frac{2L}{n\pi} (-1)^{n+1}$ by integration by parts.

(b) $a_0 = \frac{1}{2L} \int_{-L}^L e^{-x} dx = \frac{-e^{-L} + e^L}{2L}$. For $n \geq 1$, $a_n = \frac{1}{L} \int_{-L}^L e^{-x} \cos \frac{n\pi x}{L} dx$. Let $I_n = \int e^{-x} \cos \frac{n\pi x}{L} dx$. Then, integrating by parts twice, we have

$$I_n = e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} + \frac{L}{n\pi} \int_{-L}^L e^{-x} \sin \frac{n\pi x}{L} dx = e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} - e^{-x} \frac{L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} - \frac{L^2}{n^2\pi^2} I_n.$$

$$\text{i.e. } I_n = \frac{n^2\pi^2}{L^2+n^2\pi^2} \left[e^{-x} \frac{L}{n\pi} \sin \frac{n\pi x}{L} - e^{-x} \frac{L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} \right] = \frac{Ln\pi e^{-x}}{L^2+n^2\pi^2} \left[\sin \frac{n\pi x}{L} - \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right].$$

Hence, $a_n = \frac{1}{L} I_n = \frac{L}{L^2+n^2\pi^2} (-1)^n (e^L - e^{-L})$. The computation for $b_n = \frac{1}{L} \int_{-L}^L e^{-x} \sin \frac{n\pi x}{L} dx$ is similar and hence is omitted.

(c) $b_1 = 1$ and $b_n = 0$ for all $n \geq 2$ and $a_n = 0$ for all $n \geq 0$.

(d) $a_0 = \frac{1}{2L} \int_0^L x dx = \frac{L}{4}$. For $n \geq 1$, we have $a_n = \frac{1}{L} \int_0^L x \cos \frac{n\pi x}{L} dx = \frac{L}{n^2\pi^2} (\cos n\pi - 1)$, and $b_n = \frac{1}{L} \int_0^L x \sin \frac{n\pi x}{L} dx = \frac{L}{n\pi} (-1)^{n+1}$ by integration by parts.

(e) $a_0 = \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx = \frac{1}{2}$. For $n \geq 1$, $a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos \frac{n\pi x}{L} dx = \frac{2}{n\pi} \sin \frac{n\pi}{2} = \frac{2(-1)^k}{(2k+1)\pi}$ if $n = 2k + 1$ is odd and $= 0$ if n is even. $b_n = 0$ for all $n \geq 1$ since f is even.

3.3.2 (c) For $n \geq 1$, integrating by parts, we have $B_n = \frac{2}{L} \int_{L/2}^L x \sin \frac{n\pi x}{L} dx = \frac{2}{n\pi} \left[-x \cos \frac{n\pi x}{L} \Big|_{L/2}^L + \int_{L/2}^L \cos \frac{n\pi x}{L} dx \right] = \frac{2}{n\pi} \left[(-1)^{n+1} L + \frac{L}{2} \cos \frac{n\pi}{2} - \frac{L}{n\pi} \sin \frac{n\pi}{2} \right] = \frac{2L}{n\pi} (-1 + \frac{1}{2} (-1)^k)$ when $n = 2k$ is even and $= \frac{2L}{n\pi} (1 - \frac{1}{n\pi} (-1)^k)$ when $n = 2k + 1$ is odd.

3.3.5 (c) $A_0 = \frac{1}{L} \int_{L/2}^L x dx = \frac{x^2}{2} \Big|_{L/2}^L = \frac{3L^2}{8}$. For $n \geq 1$, integrating by parts, we have $A_n = \frac{2}{L} \int_{L/2}^L x \cos \frac{n\pi x}{L} dx = \frac{2}{n\pi} \left[x \sin \frac{n\pi x}{L} \Big|_{L/2}^L - \int_{L/2}^L \sin \frac{n\pi x}{L} dx \right] = \frac{2}{n\pi} \left[\frac{L}{2} \sin \frac{n\pi}{2} + \frac{L}{n\pi} (\cos n\pi - \cos \frac{n\pi}{2}) \right] = \frac{2L}{n^2\pi^2} (1 - (-1)^k)$ when $n = 2k$ is even and $= \frac{L}{n\pi} (-1)^k - \frac{2L}{n^2\pi^2}$ when $n = 2k + 1$ is odd.

3.3.7 $e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x$.

3.3.10 For $x > 0$, $f_{\text{even}} = \frac{1}{2}(e^{-x} + x^2)$ and $f_{\text{odd}} = \frac{1}{2}(e^{-x} - x^2)$. For $x < 0$, $f_{\text{even}} = \frac{1}{2}(x^2 + e^{-x})$ and $f_{\text{odd}} = \frac{1}{2}(x^2 - e^{-x})$.

3.3.17 (a) $\int_0^1 \frac{1}{1+x^2} dx = \arctan x \Big|_0^1 = \frac{\pi}{4}$.

(b), (c) $\int_0^1 \frac{1}{1+x^2} dx = \int_0^1 \frac{1}{1-(-x^2)} dx \int_0^1 \sum_{n=0}^{\infty} (-x^2)^n dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Hence, $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

3.5.1 (a) (3.5.6) says $x^2 = Lx - \frac{8L^2}{\pi^3} \sum_{n, \text{ odd}} \frac{1}{n^3} \sin \frac{n\pi x}{L}$. Then, using (3.3.11) and (3.3.12), we have

$$x^2 = \sum_{n=1}^{\infty} \left[\frac{2L^2(-1)^{n+1}}{n\pi} - \frac{4L^2(1 - (-1)^n)}{n^3\pi^3} \right] \sin \frac{n\pi x}{L}. \quad (1)$$

Note $1 - (-1)^n = 2$ if n is odd and $= 0$ if n is even.

(b) x^2 is continuous and $x^2 \Big|_0 = 0$. Thus, (3.5.12) is an equality for $0 \leq x < L$ (i.e. except for L since $x^2 \Big|_L \neq 0$.)

(c) Integrating (1) term by term (and multiplying by 3), we have

$$x^3 = C + \sum_{n=1}^{\infty} \left[-\frac{6L^3(-1)^{n+1}}{n^2\pi^2} + \frac{12L^3(1-(-1)^n)}{n^4\pi^4} \right] \cos \frac{n\pi x}{L}, \quad (2)$$

where $C = \frac{1}{L} \int_0^L x^3 dx = \frac{L^3}{4}$. (2) is an equality for all $x \in [0, L]$ since x^3 is continuous. (No extra condition is needed for F.C.S.)

3.5.2 (a) Integrating (3.3.11) term by term (and multiplying by 2), we obtain

$$x^2 = C + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} (-1)^n \cos \frac{n\pi x}{L}, \quad (3)$$

where $C = A_0 = \frac{1}{L} \int_0^L x^2 dx = \frac{L^2}{3}$.

(b) Integrating (3) term by term (and multiplying by 3), we obtain

$$x^3 = L^2 x + D + \sum_{n=1}^{\infty} \frac{12L^3}{n^3\pi^3} (-1)^n \sin \frac{n\pi x}{L} = D + \sum_{n=1}^{\infty} \left[\frac{12L^3}{n^3\pi^3} - \frac{2L^3}{n\pi} \right] (-1)^n \sin \frac{n\pi x}{L}, \quad (4)$$

where we used (3.3.11) and (3.3.12) in the second equality. By evaluating (4) at $x = 0$, we see that $D = 0$.

3.5.4 (b)

$$\cosh x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (5)$$

Since $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, we have $\int \cosh x dx = \sinh x + C$. Then, integrating (5) term by term, we have $\sinh x = C + \sum_{n=1}^{\infty} \frac{-b_n L}{n\pi} \cos \frac{n\pi x}{L}$. Integrating term by term once again, we obtain

$$\cosh x = Cx + D + \sum_{n=1}^{\infty} \frac{-b_n L^2}{(n\pi)^2} \sin \frac{n\pi x}{L} \quad (6)$$

Evaluating (6) at $x = 0$ gives $D = 1$, and evaluating (6) at $x = L$ gives $C = L^{-1}(\cosh L - 1)$

Now, equate (5) and (6). Then, we have

$$\sum_{n=1}^{\infty} \left(1 + \frac{L^2}{n^2\pi^2} \right) b_n \sin \frac{n\pi x}{L} = \frac{\cosh L - 1}{L} x + 1. \quad (7)$$

Recall $x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}$ and $1 \sim \sum_{n, \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - (-1)^n) \sin \frac{n\pi x}{L}$. Hence, from (7), we have

$$b_n = \frac{n^2\pi^2}{L^2 + n^2\pi^2} \left(\frac{2}{n\pi} (-1)^{n+1} (\cosh L - 1) + \frac{2}{n\pi} (1 - (-1)^n) \right) = \frac{2n\pi}{L^2 + n^2\pi^2} (1 - (-1)^n \cosh L).$$

3.5.7 Evaluating (3.5.6) at $x = \frac{L}{2}$, we obtain $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$.