

APM384H1F HW:

Please submit the HW with the cover sheet (which you can find on the course website) with the HW # and the group #. Make sure to *print* your **registered** names and ID numbers and staple together all the pages. When you write up a solution, write it with clear handwriting. (0 point will be given if a solution is unreadable.) HW is due at the beginning of the class. *No late HW is accepted.*

If you have 6 people in your group, two students need to sign for one of the problems. (If your name does not appear on the cover sheet, you will receive 0 for the HW.) Even if you have less than 5 people in your group (for the moment), you still need to hand in all 5 parts but choose *only* 1 part (to be graded for 50% of the HW grade.) Write “Everyone” for the remaining part of the HW.

Make sure to write down sufficient details to receive credits.

• HW10: Due on Dec 03, 09 (THU)

A: Consider the 1-D linear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = 0, & x, t \in \mathbb{R} \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \quad (1)$$

Prove “Dispersive estimate”:

$$|u(x, t)| \leq \frac{1}{\sqrt{4\pi|t|}} \int_{\mathbb{R}} |f(x)| dx, \quad t \neq 0.$$

B: Riemann-Lebesgue Lemma says: A Fourier transform \hat{f} of an L^1 function f (i.e. $\int_{\mathbb{R}} |f(x)| dx < \infty$) is continuous and decays to 0 as $|\omega| \rightarrow \infty$. In short, $\mathcal{F}(L^1(\mathbb{R})) \subset C_0(\mathbb{R})$. In this exercise, you will prove this with a stronger assumption.

(a) Suppose $f' \in L^1(\mathbb{R})$ as well. i.e. $\int_{\mathbb{R}} |f'(x)| dx < \infty$. Show that $|\hat{f}(\omega)| \rightarrow 0$ as $|\omega| \rightarrow \infty$. **HINT:** Express $\hat{f}(\omega)$ in the Fourier integral.

(b) Suppose $xf(x) \in L^1(\mathbb{R})$ as well. Show \hat{f} is continuous. **HINT:** Show $|\hat{f}(\omega) - \hat{f}(\eta)| \rightarrow 0$ as $|\omega - \eta| \rightarrow 0$. Write $\hat{f}(\omega) - \hat{f}(\eta)$ on the Fourier side. You may find Mean Value Theorem useful. i.e. Given g differentiable, we have $|g(x) - g(y)| \leq \max_{\theta \in [x, y]} |g'(\theta)| \times |x - y|$.

C: Consider (complex-valued) f on $[-L, L]$. Suppose we have $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-\frac{in\pi x}{L}}$ for all $x \in [-L, L]$, where $\hat{f}(n)$ is the n th Fourier coefficient.

(a) Prove Plancherel identity. i.e. Show

$$\int_{-L}^L |f(x)|^2 dx = 2L \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

HINT: Use the orthogonality of the complex exponentials. You may switch integrals and sums without justifying it.

- (b) Suppose $f \in L^2([-L, L])$, i.e. $\|f\|_{L^2([-L, L])} = \left(\int_{-L}^L |f(x)|^2 dx\right)^{\frac{1}{2}} < \infty$. Show $\hat{f}(n) \rightarrow 0$ as $n \rightarrow \infty$. i.e. we are proving Riemann-Lebesgue lemma under a stronger assumption. (Riemann-Lebesgue Lemma: The Fourier coefficient $\hat{f}(n)$ of any function $f \in L^1([-L, L])$ tends to 0 as $|n| \rightarrow \infty$. The proof for $f \in L^1$ is more difficult.)

HINT: What happens to the n th term if a given series is convergent?

D: Consider the 3-D Schrödinger equation $iu_t + \Delta u = 0$. Assume u and all its derivatives go to 0 as $|\vec{x}| \rightarrow \infty$.

- (a) Show Conservation of mass: $\int_{\mathbb{R}^3} |u(\vec{x}, t)|^2 d\vec{x} \equiv \text{constant}$.
 (b) Show Conservation of momentum: $-i \int_{\mathbb{R}^3} u(\vec{x}, t) \nabla \bar{u}(\vec{x}, t) d\vec{x} \equiv \text{constant vector}$.
 (c) Show Conservation of energy: $\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(\vec{x}, t)|^2 d\vec{x} \equiv \text{constant}$.

HINT: Since it is in \mathbb{R}^3 , you need to use Divergence Theorem (or Green's formula) rather than integration by parts.

E: Consider the 1-D cubic Schrödinger equation $iu_t + u_{xx} = |u|^2 u$. Assume u and all its derivatives go to 0 as $|x| \rightarrow \infty$.

- (a) Show Conservation of mass: $\int_{\mathbb{R}} |u(x, t)|^2 dx \equiv \text{constant}$.
 (b) Show Conservation of momentum: $-i \int_{\mathbb{R}} u(x, t) \bar{u}_x(x, t) dx \equiv \text{constant}$.
 (c) Show Conservation of energy: $\frac{1}{2} \int_{\mathbb{R}} |u_x(x, t)|^2 dx + \frac{1}{4} \int_{\mathbb{R}} |u(x, t)|^4 dx \equiv \text{constant}$.

HINT for D and E: Given a complex-valued $f(x)$ that decays to 0 at infinity, we have $\int_{\mathbb{R}} \partial_x f(x) dx = 0$ by Fundamental Theorem of Calculus. Also, given a complex number $z = x + iy$, we have

$$\begin{cases} \text{Real part of } z = \text{Re } z = x = \frac{z + \bar{z}}{2} \\ \text{Imaginary part of } z = \text{Im } z = y = \frac{z - \bar{z}}{2i} \end{cases}$$

You can assume u is “nice” so that you can interchange a derivative and an integral without justification.

NOTE: By Plancherel identity, we have $-i \int_{\mathbb{R}} u(x, t) \bar{u}_x(x, t) dx = 2\pi \int_{\mathbb{R}} \omega |\hat{f}(\omega)|^2 d\omega = \text{expectation (or mean) of the momentum}$. i.e. Conservation of momentum says that the mean momentum (or mean velocity if you like) is preserved under a given dynamics.

- **HW9: Due on Nov 30, 09 (MON) A:** 10.4.3 **B:** 10.4.4 **C:** 10.4.7 (a), (b) **D:** 10.5.12 (Do not use Fourier Cosine transform.) **HINT:** Use the even extension on f and u . i.e. Redefine f and u on \mathbb{R} by

$$u(x, t) = \begin{cases} u(x, t), & x > 0 \\ u(-x, t), & x < 0 \end{cases}, \text{ and } f(x) = \begin{cases} f(x), & x > 0 \\ f(-x), & x < 0 \end{cases}.$$

Note that $\partial_x u(0, t) = \partial_x f(0) = 0$ thanks to this even extension. You may assume that the heat equation for the even extended u holds at $\{x = 0\} \times \{t > 0\}$. However, you *need* to show that the even extended u satisfies the heat equation $\{x < 0\} \times \{t > 0\}$.

Since this new f is even, the contribution from the $\sin(\omega x)$ (in $e^{i\omega x}$) in taking the Fourier transform is 0. (This is how Fourier Cosine series arises, but this HINT allows you to solve this problem directly with Fourier transform.) The same comment holds for $\hat{u}(\omega, t)$.

E: 10.6.10

- **HW8: Due on Nov 19, 09 (THU) A:** 10.3.3, 10.3.6, 10.3.7 **B:** 10.3.5, 10.3.11 **C:** 10.3.8, 10.3.15, 10.4.1 (Use integration by parts twice.) **D:** 10.3.14, 10.4.2 **E:** 10.3.18, 10.4.11

- **HW7: Due on Nov 12, 09 (THU)**

A: 3.4.1 **B:** 3.4.6, 3.4.12 **C:** 3.4.9 **D:** 3.4.10 **E:** 3.4.11

- **HW6: Suggested problems**

The answers will be posted on Nov 2 (MON) except for graphing.

3.2.2, 3.3.1, 3.3.2 (c), 3.3.5 (c), 3.3.7, 3.3.10, 3.3.17, 3.5.1, 3.5.2, 3.5.4 (b), 3.5.7

- **HW5: Due on Oct 29, 08 (THU)**

A: 12.2.2, 12.3.6 **B:** 12.2.3, 12.2.6 **C:** 12.2.5 (a), 12.2.7 **D:** 12.2.5 (b), 12.2.8 **E:** 12.2.5 (d), 12.3.3

- **HW4: Due on Oct 22, 08 (THU)**

A: 4.4.2 **B:** 4.4.3 (b), 4.4.5 **C:** 4.4.7, 4.4.8 **D:** 4.4.9, 4.4.12 **E:** 4.4.10, 4.4.11

- **HW3: Due on Oct 15, 09 (THU)**

A: 2.4.1, (a), (b), 2.5.1 (b), (e)

B: 2.4.2, 2.5.2 (a), (b)

C: 2.4.3, 2.5.12. In Part (c), you only get uniqueness up to constants. In Part (d), show uniqueness when $h > 0$ (which corresponds to Newton's law of cooling when the exterior temperature is 0.) Do not worry about the second sentence in (d). Assume that the solution u is C^2 (= twice continuously differentiable) in the interior of the domain D , and is C^1 up to the boundary ∂D (i.e. u and ∇u are continuous up to the boundary.)

HINT: Recall that Green's formula

$$\int_D u \Delta v d\vec{x} = \oint_{\partial D} u \nabla v \cdot \hat{n} dS - \int_D \nabla u \cdot \nabla v d\vec{x}$$

is obtained from applying Divergence Theorem on $u\nabla v = (u\partial_x v, u\partial_y v, u\partial_z v)$. i.e. for 2.5.12 (a), apply Divergence Theorem to $u\nabla u$ to obtain the Green's formula when $u = v$. Although 2.5.12 (a) is stated in 3-d, this result holds in any dimension.

D: 2.4.4 (i.e. Show $c_1 = c_2 = 0$ for $\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ by following the computation done in class (for Section 2.3.) Also, 2.4.5, 2.5.5 (c), (d)

E: 2.4.6, 2.5.15 (b), (d)

• **HW2: Due on Oct 08, 09 (THU)**

A: 2.3.1 (a), (b), (c), (f)

B: 2.3.2 (a), (d), (e) (Find the eigenfunctions as well.)

C: 2.3.3 (c), (d)

D: 2.3.8 *Hint: Let $v(x, t) = e^{\alpha t}u(x, t)$. Then, what PDE and BCs does v satisfy?*

E: 2.3.10

• **HW1: Due on Oct 01, 09 (THU)**

A: 1.2.9, **B:** 1.4.1 (a), (c), (g), **C:** 1.4.1 (h), 1.4.7 (b), (c), **D:** 1.5.3, 1.5.4, **E:** 1.5.8, 1.5.9 (b), 1.5.11.