FACULTY OF APPLIED SCIENCE
University of Toronto

MAT234: Differential Equations

Midterm Exam #2, March 28, 2018

Total: 70 points

Mine

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Toronto Email Address: ______________________

Signature: ______________________

There are 4 questions in total. Some questions have multiple parts.

DO NOT REMOVE any pages from the test booklet. If you happen to need to use the last page clearly indicate on whatever question you're working on that it is continued on the extra page.

Do NOT begin until you are instructed to do so.

When you are told the test has ended you MUST stop writing at once. Failure to do so is an academic offence.

Aids of any kind are Forbidden.

Problem 1. (12 points). Find the general solution of given differential equation

$$y'' - y' - 2y = 2e^{-t}$$

Homogeneous: $\lambda^2 - \lambda - 2 = 0 \to (\lambda - 2)(\lambda + 1) = 0$

$\to y_h(t) = c_1 e^{2t} + c_2 e^{-t}$

Particular: Since e^{-t} is homog. we try $y = Ate^{-t}$

Then $y' = Ae^{-t} - Ate^{-t} = Ae^{-t} - y$

Then $y'' = -Ae^{-t} - y' = -Ae^{-t} - (Ae^{-t} - y)$

$= -2Ae^{-t} + y$

$y'' - y' - 2y = -2Ae^{-t} + y - (Ae^{-t} - y) - 2y$

$= -3Ae^{-t} = 2e^{-t}$

$\to A = -2/3$

So $\boxed{y = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3} te^{-t}}$

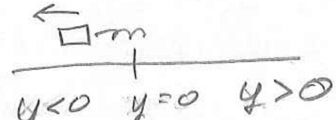
Problem 2. (23 points). A 2 kilogram mass is attached to a spring with stiffness $k = 50N/m$. The mass is displaced 0.25 meters to the left of the equilibrium point and given a velocity of 1 m/s in a direction to the left of equilibrium. Neglecting damping, find the equation of motion of the mass along with the amplitude, period and frequency. How long after release does the mass pass through the equilibrium position? *Hint: It will help to write your solution using the phase-amplitude form!*

$m = 2\,[kg] \quad k = 50\,N/m \rightarrow 2y'' + 50y = 0$

$\rightarrow y'' + \underbrace{25}_{\omega_0^2} y = 0 \longrightarrow y = c_1 \cos(5t) + c_2 \sin(5t)$

$y(0) = -.25$

$y'(0) = -1$



$y(0) = -\frac{1}{4} = c_1$

$y'(0) = -1 = 5c_2 \quad$ so $\quad c_1 = -1/4 \quad c_2 = -1/5$

$y = -\frac{1}{4}\cos(5t) - \frac{1}{5}\sin(5t) \qquad R = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2}$

$= \sqrt{\frac{1}{16} + \frac{1}{25}}$

$= \sqrt{\frac{41}{16 \cdot 25}} = \frac{\sqrt{41}}{20}$

$\boxed{\text{Amplitude} = \frac{\sqrt{41}}{20}}$

$\boxed{\text{frequency} = 5} \quad \boxed{\text{period} = \frac{2\pi}{5}}$

$\delta = \pi + \arctan\left(\frac{4}{5}\right) = \pi + \arctan(4/5), \quad y = \frac{\sqrt{41}}{20}\cos(5t - \delta)$

Then crosses equilibrium when $5t_{cross} - \delta = \frac{\pi(2k+1)}{2}$ and $t_{cross} > 0$

ie. $t_{cross} = \dfrac{\delta + \frac{\pi}{2}(2k+1)}{5} = \dfrac{3\pi + 2\pi k}{10} + \dfrac{\arctan(4/5)}{5}$

smallest positive t_{cross} is $k = -1 \longrightarrow \boxed{t_{cross} = \frac{\pi}{10} + \frac{1}{5}\arctan(4/5)}$

Problem 3. (20 points). Consider the equation

$$y'' - 2xy' + 2\lambda y = 0$$

where $\lambda \in \mathbb{R}$ is a free parameter.

(i) (10 points). Find the recurrence relation between a_{n+2} and a_n.

(ii) (4 points). Find the general expression for the coefficients a_n in the cases where when n is even and odd respectively. These will correspond to series solutions of the form $y_{even}(x)$ and $y_{odd}(x)$.

(iii) (1 point). Write down the two series $y_{even}(x)$ and $y_{odd}(x)$.

(iv) (5 points). For which values of λ, are the series solutions, $y_{even}(x)$ and $y_{odd}(x)$ of the above differential equation simple polynomials? (A simple polynomial of degree m is $P_n(x) = a_0 + a_1 x + \cdots + a_m x^m$). Justify your answer.

$x=0$ ordinary pt $\rightarrow y = \sum\limits_{k\geq 0}' a_k x^k \qquad y' = \sum\limits_{k\geq 0}' k a_k x^{k-1}$

$y'' = \sum\limits_{k\geq 0}' k(k-1) a_k x^{k-2}$

$\rightarrow y'' - 2xy' + 2\lambda y = \sum\limits_{k\geq 0}' \left(k(k-1) a_k x^{k-2} - 2k a_k x^k + 2\lambda a_k x^k\right)$

$= \sum\limits_{k\geq 0}' \left(k(k-1) a_k x^{k-2} + 2(\lambda - k) a_k x^k\right)$

$= \sum\limits_{k\geq 0}' \left[(k+2)(k+1) a_{k+2} + 2(\lambda-k) a_k\right] x^k$

so $\boxed{a_{k+2} = -\dfrac{2(\lambda-k)\, a_k}{(k+2)(k+1)}}$

$a_2 = -\dfrac{2\lambda a_0}{2} = -\lambda a_0$

$a_4 = -\dfrac{2(\lambda-2)}{4\cdot 3} a_2 = \dfrac{\lambda\, 2(\lambda-2)\, a_0}{4\cdot 3}$

$a_6 = -\dfrac{2(\lambda-4)\, a_4}{6\cdot 5} = -\dfrac{2^2(\lambda-4)(\lambda-2)\lambda a_0}{6\cdot 5\cdot 4\cdot 3}$

(extra paper for problem 3)

so $$a_{2k} = (-1)^k \frac{2^k(\lambda-2k+2)(\lambda-2k+4)\cdots(\lambda-2)\lambda\, a_0}{(2k)!}$$

$$a_3 = -\frac{2(\lambda-1)a_1}{3\cdot 2} = -\frac{(\lambda-1)a_1}{3}$$

$$a_5 = -\frac{2(\lambda-3)}{5\cdot 4}a_3 = \frac{2(\lambda-3)(\lambda-1)a_1}{5\cdot 4\cdot 3}$$

$$a_7 = -\frac{2(\lambda-5)a_5}{7\cdot 6} = -\frac{2^2(\lambda-5)(\lambda-3)(\lambda-1)a_1}{7\cdot 6\cdot 5\cdot \cancel{4}\cdot 3\cdot}$$

$$a_{2k+1} = \frac{(-1)^k 2^k(\lambda-(2k-1))(\lambda-(2k-3))\cdots(\lambda-1)\,a_1}{(2k+1)!}$$

$$y_{even}(x) = \sum_{k\geq 0} \frac{(-1)^k 2^k(\lambda-2k+2)\cdots(\lambda-2)\lambda x^{2k}}{(2k)!}$$

$$y_{odd}(x) = \sum_{k\geq 0} \frac{(-1)^k 2^k(\lambda-(2k-1))\cdots(\lambda-3)(\lambda-1)x^{2k+1}}{(2k+1)!}$$

if $\lambda = 0, 2, 4, \ldots$ then y_{even} is polynomial

if $\lambda = 1, 3, 5, \ldots$ then y_{odd} is polynomial

Problem 4. (15 points). Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & x \in [1,2) \end{cases}$$

(i) (10 points). Find the Fourier Cosine Series of f.

(ii) (5 points). Sketch the graph of the function to which the Fourier Cosine Series converges over three full periods.

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx \qquad k>0, \quad L=2$$

$$\text{so} \quad a_k = \int_0^2 f(x)\cos\frac{k\pi x}{2}\,dx = \int_0^1 \cos\frac{k\pi x}{2}\,dx$$

$$= \frac{2}{k\pi}\sin\frac{k\pi x}{2}\Big|_0^1$$

$$= \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) = \frac{2}{k\pi}\begin{cases} 0 & k \text{ odd} \\ 1 & k=1,5,9,\dots \\ -1 & k=3,7,11,\dots \end{cases}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}$$

$$\text{so} \quad f(x) \sim \frac{1}{4} + \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right) + \frac{2}{5\pi}\cos\left(\frac{5\pi x}{2}\right) - \cdots$$