FACULTY OF ARTS & SCIENCES
University of Toronto

MAT234: Differential Equations

Midterm Exam #1, February 14, 2018

Duration: 50 minutes

Total: 65 points

Family Name: Answers

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

DO NOT REMOVE any pages from the test booklet. If you happen to need to use the last page CLEARLY INDICATE on whatever question you're working on that it is continued on the extra page.

Do NOT begin until you are instructed to do so.

When you are told the test has ended you MUST stop writing at once. Failure to do so is an academic offence.

Electronic Aids of any kind are Forbidden.

Problem 1 (25 points total). This question has 3 parts, the first of which is independent of the others. Consider the following ODE

$$M(x,y)dx + N(x,y)dy = 0. \tag{1}$$

(i) (5 points). If $x \neq 0$ and $xM(x,y) + yN(x,y) = 0$, then find the general solution to the above ODE (1). *(Hint: Your answer should not depend on the functions M or N.)*

$x \neq 0 \rightarrow M = -\frac{y}{x} N \rightarrow M\,dx + N\,dy = 0$

$\iff -\frac{y}{x} N\,dx + N\,dy = 0 \iff N\left(-\frac{y}{x} + y'\right) = 0$

$\iff y' = \frac{y}{x} \iff \frac{dy}{y} = \frac{dx}{x} \iff \ln y = \ln x + C$

i.e. $y = cx$ for arbitrary $c \in \mathbb{R}$ $\boxtimes$

(ii) (10 points). Let $M(x,y) = y$ and $N(x,y) = 2x - ye^y$. Is the ODE (1) exact? If it is not exact, find an integrating factor, μ, which will make it exact. *(Hint: your solution to parts (ii) and (iii) shouldn't depend on your answer to part (i))*

$M_y = 1 \quad N_x = 2 \quad M_y \neq N_x \rightarrow$ not exact.

Want μ such that $(\mu M)_y = (\mu N)_x$ i.e. $\mu_y M + \mu M_y = \mu_x N + \mu N_x$

if $\mu_y = 0$ gives $\frac{d}{dx} \ln \mu = \frac{M_y - N_x}{N} = -\frac{1}{2x - ye^y} \neq$ function of x.

So try $\mu_x = 0 \rightarrow \frac{d}{dy} \ln \mu = \frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}$

$\rightarrow \ln|\mu| = \ln|y| \longrightarrow \mu = y$ works.

(iii) (10 points). Find the general solution of the ODE in part (ii).

$$\psi_x = \mu M = y^2 \longrightarrow \psi(x,y) = \cancel{xy^2} + \cancel{f(y)}$$

$$\psi_y = \mu N = 2xy - y^2 e^y \longrightarrow \psi(x,y) = xy^2 - \int y^2 e^y \, dy$$

$$\int y^2 e^y \, dy = y^2 e^y - \int 2y e^y \, dy$$

$$= y^2 e^y - 2\left[y e^y - \int e^y \, dy \right]$$

$$= y^2 e^y - 2y e^y + 2e^y + C$$

$$\text{So } \psi = xy^2 - y^2 e^y + 2y e^y - 2e^y + g(x)$$

$$\longrightarrow \boxed{xy^2 - y^2 e^y + 2y e^y - 2e^y = C}$$

Problem 2 (20 points). The rate of melting for a snowball is such that its diameter, $d(t)$, varies proportionally to its surface area $A(t)$.

(i) (18 points). If the snowball was initially 4 cm in diameter and after 4 minutes its diameter is 2 cm, determine when its diameter will be 1 cm.

(ii) (2 points). Mathematically speaking, when will the snowball disappear? *(Hint: from geometry we know $A(t) = \pi d(t)^2$).*

(i) $d'(t) = kA(t) = k\pi d^2(t) \longrightarrow \frac{d(d)}{d^2} = k\pi\, dt$

$\longrightarrow -\frac{1}{d} = k\pi t + C \longrightarrow d(t) = -\frac{1}{C + k\pi t}$

$d(0) = 4 \Longrightarrow 4 = -\frac{1}{C+0}$ so $d(t) = -\frac{1}{-\frac{1}{4} + k\pi t} = \frac{1}{\frac{1}{4} - k\pi t}$

$d(4) = 2 = \frac{1}{\frac{1}{4} - k\pi 4}$ i.e. $\frac{1}{2} = \frac{1}{4} - 4k\pi$.

$\longrightarrow k = -\frac{1}{16\pi}$

so $d(t) = \frac{1}{\frac{1}{4} + \frac{t}{16}} = \frac{16}{4+t}$

$d(\tilde{t}) = 1 = \frac{16}{4+\tilde{t}} \longrightarrow 4 + \tilde{t} = 16 \longrightarrow \boxed{\tilde{t} = 12 \text{ min}}$

(ii) $d(t) \longrightarrow 0$ as $t \longrightarrow \infty$ so snowball disappears at $t = \infty$

(*extra paper*)

Problem 3. (20 points). Consider the following system

$$\begin{cases} x'(t) = -2x(t) + y(t) + 1 \\ y'(t) = -5x(t) + 4y(t) + 2 \end{cases}$$

(i) (4 points). Find any critical points of the above autonomous system.

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{x}_{crit}$ solves $\begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{x}_{crit} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

$\vec{x}_{crit} = -\begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}_{crit} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

(ii) (2 points). Transform the inhomogeneous system to a homogeneous system.

$\underline{A} = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}$ $\qquad \tilde{x} = \vec{x} - \vec{x}_{crit}$

$\tilde{x}' = \vec{x}' - \vec{x}'_{crit} = \vec{x}'$

$= \underline{A}\vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \underline{A}(\tilde{x} + \vec{x}_{crit}) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \underline{A}\tilde{x} + \left(\underline{A}\vec{x}_{crit} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

$= A\tilde{x}$

so $\boxed{\tilde{x}' = \underline{A}\tilde{x} \qquad \tilde{x} = \vec{x} - \vec{x}_{crit}}$

(iii) (6 points). Find the general solution of the homogeneous system in part (ii).

Characteristic Polynomial is $c_A(\lambda) = \lambda^2 - 2\lambda - 3$
$= (\lambda - 3)(\lambda + 1)$

so $\lambda_1 = 3 \quad \lambda_2 = -1$

Then $\begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{v} = 3\vec{v} \rightarrow -2v_1 + v_2 = 3v_1$
$v_2 = 5v_1$
$\rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{v} = -\vec{v} \rightarrow -2v_1 + v_2 = -v_1$
$v_2 = v_1 \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

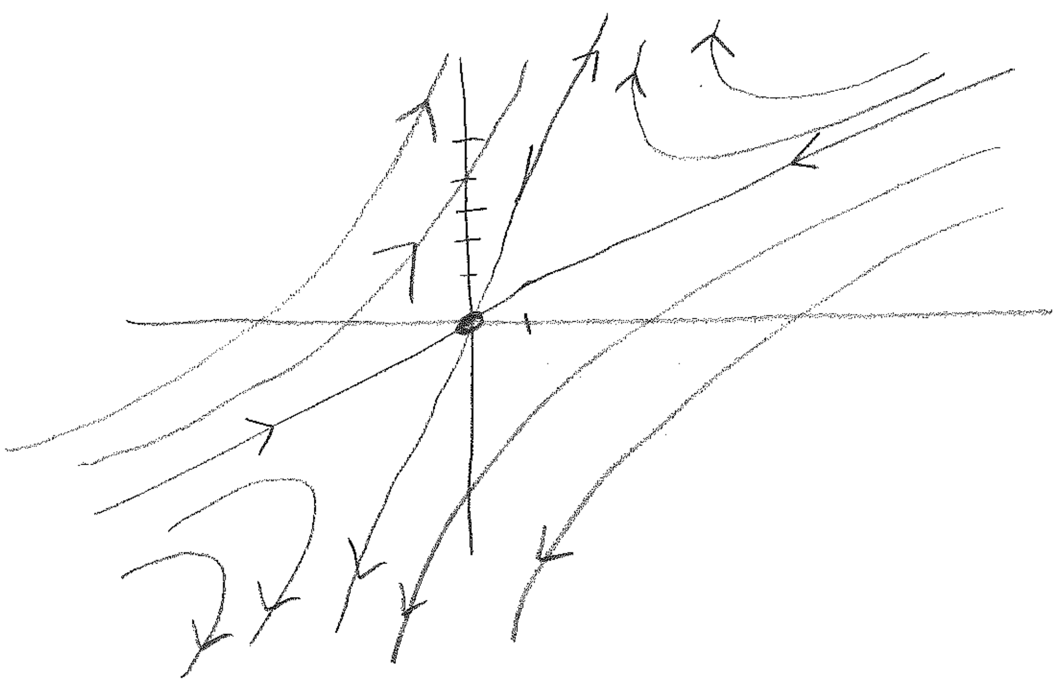
Thus $\boxed{\tilde{X}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(iv) (2 points). Find the general solution of the original inhomogeneous system.

$\vec{x} = \tilde{x} + \vec{x}_{cut}$

so $\boxed{\vec{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\5\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2/3\\1/3\end{bmatrix}}$

(v) (4 points). Draw the phase portrait of the homogeneous system in part (ii).



(vi) (2 points). Classify the equilibrium solution for the homogeneous system from part (ii) (i.e. give its type as well as its stability).

It is an unstable saddle point

(*extra paper*)