Topics Covered so Far

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Below is a list of topics considered "fair game" for the upcoming midterm on October 5. A few of these you may have not seen yet but you will by the test date. In a perfect world (for me, not you) I'd have enough time to test you on all of these but for obvious lack of time I have to select concepts I want to focus on. The list below just recaps a lot of what we've covered to help jog your memory and help you study.

1. **MAT223:** MAT223 is a prerequisite for a reason. We expect you to be able to handle any question from that course in this one, as it can be a useful tool to help solve certain problems in MAT224.

2. **Vector Spaces:** What they are and related vocabulary. Recall that the only way we know whether something is a real vector space is by checking the **vector space axioms** hold. We've seen lots of examples, some of which indicating that the additive identities or additive inverse may not conform to what we might naively expect when the definitions of vector addition and/or scalar multiplication are modified.

3. **Subspaces:** These are sets closed under the two algebraic operations given in a vector space: addition and scalar multiplication.

4. **Linear combinations:** Given a vector space, we can look at creating new vectors by applying our algebraic rules to vectors in the space - namely adding and scalar multiplying vectors. This produces linear combinations, one of the fundamental constructions available to us.

5. **Subspace Sums:** We've overloaded the + symbol in multiple ways already (after all + in $\mathbb{R}^n$ and + in $P_2(\mathbb{R})$ aren't the same are they?). Here we go one further and use the symbol + as a binary operation on *subspaces themselves*. The sum of subspaces is a subspace and it's the smallest subspace containing the union of the summand subspaces.

6. **Linear Independence:** We learned about the fact that if a set of vectors is linearly dependent then any linear combination of the vectors can be reduced to a linear combination of a subset of the vectors. So in this way, a linearly independent set of vectors doesn't have "redundancy". You should know the meaning of linear independence, how to check for it, and have a sense of why it's important.

7. **Bases:** As basis is simply a linearly independent spanning set. Put differently it's a maximal set of linearly independent vectors or a minimal set of spanning vectors (make sure you see why!). Choosing a basis for a vector space allows us to make concrete the former abstractions by giving us a way to represent vectors in the space explicitly. In a way, without having a basis I don't feel I really understand a vector space.

8. **Dimension:** Since all bases in a given vector space must have the same size (prove this!) we can tag each vector space with that number - the number of vectors in any basis for the space. That number is called the dimension and it serves as sort of a proxy for the "size" of a vector space, in that "larger" spaces should have larger dimensions. Another way I like to talk about it is in terms of "information". Dimension captures the number of degrees of freedom in a space and that's essentially the amount of information one can pack into it. [1]

9. **Linear Transformations:** You'll need to know what a linear transformation is, how to verify if something is or isn't one and some basic properties of linear transformations. Linear transformations are those maps between two vector spaces which preserve the algebraic structure, and in this way they allow for a type of correspondence or "communication" between two vector spaces. We've also seen that $\mathcal{L}(V, W)$, the space of linear transformations between vectors space V and vector space W, **is itself a vector space** when equipped with the standard version of addition of transformations and scalar multiplication. Viewing the set of maps between spaces as a space in itself moves us one small rung up the abstraction hierarchy in the world of abstract vector spaces[2].

[1] Here I'm speaking loosely and metaphorically, of course

[2] And, of course, the procedure of "vector spacing" $\mathcal{L}(V, W)$ is trivially replicable on that space itself. So one can produce an infinite hierarchy of abstract spaces where each new space is made up of maps of two spaces of the prior construction ad infinitum.