Topics Covered so Far

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Below is a list of topics considered “fair game" for the upcoming midterm. A few of these you may have not seen yet but you will by the test date. In a perfect world (for me, not you) I'd have enough time to test you on all of these but for obvious lack of time I have to select concepts I want to focus on. The list below just recaps a lot of what we've covered to help jog your memory and help you study.

Topics Covered Before First Test

1. **Vector Spaces:** What they are and related vocabulary. Recall that the only way we know whether something is a real vector space is by checking the **vector space axioms** hold. We've seen lots of examples, some of which indicating that the additive identities or additive inverse may not conform to what we might naively expect when the definitions of vector addition and/or scalar multiplication are modified.

2. **Subspaces:** These are sets closed under the two algebraic operations given in a vector space: addition and scalar multiplication.

3. **Linear combinations:** Given a vector space, we can look at creating new vectors by applying our algebraic rules to vectors in the space - namely adding and scalar multiplying vectors. This produces linear combinations, one of the fundamental constructions available to us.

4. **Subspace Sums:** We've overloaded the + symbol in multiple ways already (after all + in $\mathbb{R}^n$ and + in $P_2(\mathbb{R})$ aren't the same are they?). Here we go one further and use the symbol + as a binary operation on *subspaces themselves*. The sum of subspaces is a subspace and it's the smallest subspace containing the union of the summand subspaces.

5. **Linear Independence:** We learned about the fact that if a set of vectors is linearly dependent then any linear combination of the vectors can be reduced to a linear combination of a subset of the vectors. So in this way, a linearly independent set of vectors doesn't have “redundancy". You should know the meaning of linear independence, how to check for it, and have a sense of why it's important.

6. **Bases:** As basis is simply a linearly independent spanning set. Put differently it's a maximal set of linearly independent vectors or a minimal set of spanning vectors (make sure you see why!). Choosing a basis for a vector space allows us to make concrete the former abstractions by giving us a way to represent vectors in the space explicitly. In a way, without having a basis I don't feel I really understand a vector space.

7. **Dimension:** Since all bases in a given vector space must have the same size (prove this!) we can tag each vector space with that number - the number of vectors in any basis for the space. That number is called the dimension and it serves as sort of a proxy for the "size" of a vector space, in that "larger" spaces should have larger dimensions. Another way I like to talk about it is in terms of "information". Dimension captures the number of degrees of freedom in a space and that's essentially the amount of information one can pack into it. [1]

8. **Basics of Linear Transformations:** You'll need to know what a linear transformation is, how to verify if something is or isn't one and some basic properties of linear transformations. Linear transformations are those maps between two vector spaces which preserve the algebraic structure, and in this way they allow for a type of correspondence or "communication" between two vector spaces. We've also seen that $\mathcal{L}(V, W)$, the space of linear transformations between vectors space V and vector space W, **is itself a vector space** when equipped with the standard version of addition of transformations and scalar multiplication. Viewing the set of maps between spaces as a space in itself moves us one small rung up the abstraction hierarchy in the world of abstract vector spaces[2].

Topics Since Test 1

9. **Properties of Linear Transformations:** Things like *injectivity, surjectivity, bijectivity* and how to check for these.

10. **Representations of Transformations:** You should be completely, absolutely comfortable with the notation and calculation of things like $[T]_\alpha^\beta$ where α, β are bases of finite-dimensional vector spaces on which T acts. Moreover, you need to know what things like $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ mean and why. The **notation** takes some getting used to but once you master it your life will be much easier.

11. **Fundamental Subspaces:** You need to be solid on the fundamental subspaces associated to a linear operator T, namely the **kernel** $\ker T$ and the **image** $Im(T)$. How

[1] Here I'm speaking loosely and metaphorically, of course

[2] And, of course, the procedure of "vector spacing" $\mathcal{L}(V, W)$ is trivially replicable on that space itself. So one can produce an infinite hierarchy of abstract spaces where each new space is made up of maps of two spaces of the prior construction ad infinitum.

can their dimensions be used to classify the properties of a given transformation? These are obvious generalizations of the subspaces $Nul(A)$ and $col(A)$ which we encountered in MAT223. Make sure you understand how/why.

12. **The Dimension Theorem:** In MAT223 we knew this theorem as the Rank-Nullity Theorem. It is a tool which allows us to answer a lot of questions simply by appealing to numerical values. It can help to solve lots of problems, and is the main tool relating the fundamental subspaces of operators on finite dimensional vector spaces. The fact that the textbook devotes an entire section on tis one theorem is telling. I likened it in my lectures to a "conservation of information" equation; make sure you can understand how/why.

13. **Compositions of Operators:** Given $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V)$ you need to know what's meant by ST and how to represent it as a matrix, and how that representation depends on the representations of S and T.

14. **Inverses of Operators and Isomorphisms:** For $T \in \mathcal{L}(V, V)$ you need to know what's meant by T^{-1}. Invertible linear maps are known as *vector space isomorphisms* and you need to know when a given transformation is an isomorphism. We know that finite dimensional spaces with the same dimension are isomorphic (i.e. there's an isomorphism between them).

15. **Change of Basis:** This is a foundational idea - understanding how representations of objects depends on the basis one uses and how to relate two different representations. The change of basis matrices $[I]_{\alpha}^{\beta}$ provide a kind of "dictionary" linking two different representations. Or, if you prefer, it acts as a machine transforming vectors in the α basis to the same vectors in the β basis.

16. **Similarity and Conjugacy:** Representations of linear transformations in different choices of bases are related by **similarity**, i.e. they will be **similar matrices**. Given a square matrix A, the set of all matrices similar to A is a so-called **conjugacy class**. Within the conjugacy class, similarity between matrices defines an equivalence relation. Thus, the space of square matrices is partitioned into conjugacy classes. It follows from the change of basis formulae that a linear operator's representation is always a member of one and only one conjugacy class. Hopefully these concepts hold some intuitive power.

17. **Determinants:** You should know how to calculate determinants of $n \times n$ matrices. You should also be comfortable with the basic properties e.g. $\det(AB) = \det(A)\det(B)$, $\det(cA) = c^n \det(A)$, etc.

18. **Eigenvalues and Eigenvectors:** Given a square matrix A, you should know what eigenvectors and eigenvalues are and how to find them. What are eigenvectors geometrically? You should understand **geometric and algebraic multiplicity** of an eigenvalue. You should also know about the **characteristic equation** (what it is and

how to find it) of a transformation, as well as **invariant subspaces** (what they are and how to find them).

19. **Eigenspaces:** Make sure you know what an eigenspace is and how to find one. Are all elements of an eigenspace an eigenvector? Is E_λ a subspace for all $\lambda \in \mathbb{R}$? Are they *ever* subspaces?

20. **Diagonalization:** You need to know what's mean by diagonalizability, and how to check whether a linear transformation is or isn't diagonalizable. What are some **equivalent conditions** for a matrix being diagonalizable? If a matrix is diagonalizable how do you diagonalize it?

21. **MAT223:** You've taken MAT223 and are assumed to have done well in it. So we expect you to be able to handle questions a MAT223 student could solve. Period.[3]

[3] Not only is MAT223 a formal prerequisite, it is a *conceptual* prerequisite. MAT223 should be viewed simply as a *special case* of the topics listed above, so this shouldn't be seen as additional content.