Problem 1 (10 points total, 5 points each).

(i) Let V be a finite dimensional real vector space. Give a **definition** of what it means for a linear transformation $T : V \to V$ to be diagonalizable. **ANS:** See textbook.

(ii) Give an example of a non-diagonalizable real matrix. Justify your answer.**ANS:** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Notice $\lambda = 1$ is eigenvalue with multiplicity 2 but $Nul(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ implies $\dim E_1 = 1 < 2$ so the matrix isn't diagonalizable.

Problem 2 (22 points, 2 points if left completely blank). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(a, b) = (2a+b, a-3b)$ and let $\alpha = \{(1, 1), (1, 2)\}$. Find $[T]^\alpha_\alpha$. Show your work. (*Hint: You may find it helpful to use the fact that* $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$).**ANS:** $T(a, b) = (2a + b, a - 3b) \implies [T]^E_E = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ where $E = \{e_1, e_2\}$. Then $[T]^\alpha_\alpha = [I]^\alpha_E [T]^E_E [I]^E_\alpha = ([I]^E_\alpha)^{-1} [T]^E_E [I]^E_\alpha = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ -5 & -9 \end{bmatrix}$.

Problem 3 (20 points total, 2 points if left completely blank). For this question we consider the vector space $V = M_{2\times 2}(\mathbb{R})$ with the usual notion of addition and scalar multiplication.

(i) (2 points). Write down any basis for V. No explanation is needed.
ANS: $\alpha = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.

(ii) (18 points). Using the basis you wrote down in part (i) what is the matrix representation of the transpose mapping $T : V \to V$ given by $T(A) = A^T$? **ANS:** $[T]^\alpha_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 4 (30 points total, 3 points if left completely blank). This question has 3 parts. Let Q be an invertible $n \times n$ real matrix. Define $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ via $T(X) = Q^{-1}XQ$.

(i) (15 points, 2 points if left completely blank). Prove that T linear and is an isomorphism. **ANS:** $T(aX + bY) = Q^{-1}(aX + bY)Q = aQ^{-1}XQ + bQ^{-1}YQ = aT(X) + bT(Y)$ holds for all $a, b \in \mathbb{R}$, $X, Y \in M_{n\times n}(\mathbb{R})$ so T is linear. Notice that if $T(X) = 0$ then $Q^{-1}XQ = 0 \implies XQ = 0 \implies X = 0$ since Q is invertible. Thus $\ker T = \{0\}$ so T is injective. Since domain and codomain have the same dimension injectivity implies bijectivity so T is an isomorphism.

(ii) (10 points, 1 point if left completely blank). If possible, find any eigenvalue and associated eigenvector of T. If this is not possible, explain why not. **ANS:** Obviously $T(I) = Q^{-1}Q = I$ so $\lambda = 1$ is an eigenvalue with an eigenvector of I.

(iii) (5 points). If $n = 1$ is T diagonalizable? Explain why or why not. **ANS:** Yes, by the previous question we know that $\lambda = 1$ is always an eigenvalue so when $n = 1$ we have $M_{1\times 1}(\mathbb{R}) = E_1$ proving T is diagonalizable.

Problem 5. (18 points total, 3 points for each correct answer, -2 points for each incorrect answer and 0 points for blank answers for a minimum of 0 points or a maximum of 18 points.) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

Note: In all of the following questions, V denotes a real vector space.

(i) True/False: Let $T : V \to V$ be linear with 0 as an eigenvalue. Then $E_0 = \ker T$. **ANS:** True, by definition.

(ii) True/False: If $T : \mathbb{R}^{223} \to \mathbb{R}^{224}$ and T is linear, then T is not surjective. **ANS:** True, since domain has a smaller dimension than codomain we cannot have surjectivity.

(iii) True/False: Let $T : V \to V$. If T^{224} is invertible, then T is invertible. **ANS:** True, since $0 \neq \det(T^{224}) = (\det T)^{224} \implies \det T \neq 0$.

(iv) True/False: $S : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $S(p(x)) = p(x+1)$ is an isomorphism. **ANS:** True, since only need to check injectivity. But if $a(x+1)^2 + b(x+1) + c = 0 \implies a = 2a + b = c + b + a = 0 \implies a = b = c = 0$ so $S(p) = 0 \implies p = 0$ and S is injective.

(v) True/False: Finite dimensional isomorphic vector spaces must have the same dimension. **ANS:** True, see Proposition 2.6.7

(vi) True/False: Let α, β denote bases for finite dimensional V and let $I : V \to V$ denote the identity transformation. Then $[I]_\alpha^\beta$ is the identity matrix in $M_{\dim V \times \dim V}(\mathbb{R})$. **ANS:** False, we saw a counterexample is Problem 2.