MAT 224 Parting Document

Contents

1 Topics

Below is a list of topics considered “fair game" for the upcoming final exam. You should be prepared to answer technical questions on any of these topics. The exam is written in such a way as to try to find your areas of weakness, so if you think you can squeeze by with only superficial understanding of any of the topics, we are going to try to catch it.

1. **MAT223:** You've taken MAT223 and are assumed to have done well in it. So we expect you to be able to handle questions a MAT223 student could solve. Period.[1]

2. **Vector Spaces:** What they are and related vocabulary. Recall that the only way we know whether something is a real vector space is by checking the **vector space axioms** hold. We've seen lots of examples, some of which indicating that the additive identities or additive inverse may not conform to what we might naively expect when the definitions of vector addition and/or scalar multiplication are modified.

3. **Subspaces:** These are sets closed under the two algebraic operations given in a vector space: addition and scalar multiplication.

4. **Linear combinations:** Given a vector space, we can look at creating new vectors by applying our algebraic rules to vectors in the space - namely adding and scalar multiplying vectors. This produces linear combinations, one of the fundamental constructions available to us.

[1] Not only is MAT223 a formal prerequisite, it is a *conceptual* prerequisite. MAT223 should be viewed simply as a *special case* of the topics listed below, so this shouldn't be seen as additional content.

5. **Subspace Sums:** We've overloaded the + symbol in multiple ways already (after all + in $\mathbb{R}^n$ and + in $P_2(\mathbb{R})$ aren't the same are they?). Here we go one further and use the symbol + as a binary operation on *subspaces themselves*. The sum of subspaces is a subspace and it's the smallest subspace containing the union of the summand subspaces.

6. **Linear Independence:** We learned about the fact that if a set of vectors is linearly dependent then any linear combination of the vectors can be reduced to a linear combination of a subset of the vectors. So in this way, a linearly independent set of vectors doesn't have "redundancy". You should know the meaning of linear independence, how to check for it, and have a sense of why it's important.

7. **Bases:** As basis is simply a linearly independent spanning set. Put differently it's a maximal set of linearly independent vectors or a minimal set of spanning vectors (make sure you see why!). Choosing a basis for a vector space allows us to make concrete the former abstractions by giving us a way to represent vectors in the space explicitly. In a way, without having a basis I don't feel I really understand a vector space.

8. **Dimension:** Since all bases in a given vector space must have the same size (prove this!) we can tag each vector space with that number - the number of vectors in any basis for the space. That number is called the dimension and it serves as sort of a proxy for the "size" of a vector space, in that "larger" spaces should have larger dimensions. Another way I like to talk about it is in terms of "information". Dimension captures the number of degrees of freedom in a space and that's essentially the amount of information one can pack into it. [2]

9. **Basics of Linear Transformations:** You'll need to know what a linear transformation is, how to verify if something is or isn't one and some basic properties of linear transformations. Linear transformations are those maps between two vector spaces which preserve the algebraic structure, and in this way they allow for a type of correspondence or "communication" between two vector spaces. We've also seen that $\mathcal{L}(V, W)$, the space of linear transformations between vectors space V and vector space W, **is itself a vector space** when equipped with the standard version of addition of transformations and scalar multiplication. Viewing the set of maps between spaces as a space in itself moves us one small rung up the abstraction hierarchy in the world of abstract vector spaces[3].

10. **Properties of Linear Transformations:** Things like *injectivity, surjectivity, bijectivity* and how to check for these.

11. **Representations of Transformations:** You should be completely, absolutely comfortable with the notation and calculation of things like $[T]_\alpha^\beta$ where α, β are bases of finite-dimensional vector spaces on which T acts. Moreover, you need to know what

[2] Here I'm speaking loosely and metaphorically, of course

[3] And, of course, the procedure of "vector spacing" $\mathcal{L}(V, W)$ is trivially replicable on that space itself. So one can produce an infinite hierarchy of abstract spaces where each new space is made up of maps of two spaces of the prior construction ad infinitum.

things like $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ mean and why. The **notation** takes some getting used to but once you master it your life will be much easier.

12. **Fundamental Subspaces:** You need to be solid on the fundamental subspaces associated to a linear operator T, namely the **kernel** $\ker T$ and the **image** $Im(T)$. How can their dimensions be used to classify the properties of a given transformation? These are obvious generalizations of the subspaces $Nul(A)$ and $col(A)$ which we encountered in MAT223. Make sure you understand how/why.

13. **The Dimension Theorem:** In MAT223 we knew this theorem as the Rank-Nullity Theorem. It is a tool which allows us to answer a lot of questions simply by appealing to numerical values. It can help to solve lots of problems, and is the main tool relating the fundamental subspaces of operators on finite dimensional vector spaces. The fact that the textbook devotes an entire section on tis one theorem is telling. I likened it in my lectures to a "conservation of information" equation; make sure you can understand how/why.

14. **Compositions of Operators:** Given $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V)$ you need to know what's meant by ST and how to represent it as a matrix, and how that representation depends on the representations of S and T.

15. **Inverses of Operators and Isomorphisms:** For $T \in \mathcal{L}(V, V)$ you need to know what's meant by T^{-1}. Invertible linear maps are known as *vector space isomorphisms* and you need to know when a given transformation is an isomorphism. We know that finite dimensional spaces with the same dimension are isomorphic (i.e. there's an isomorphism between them).

16. **Change of Basis:** This is a foundational idea - understanding how representations of objects depends on the basis one uses and how to relate two different representations. The change of basis matrices $[I]_\alpha^\beta$ provide a kind of "dictionary" linking two different representations. Or, if you prefer, it acts as a machine transforming vectors in the α basis to the same vectors in the β basis.

17. **Similarity and Conjugacy:** Representations of linear transformations in different choices of bases are related by **similarity**, i.e. they will be **similar matrices**. Given a square matrix A, the set of all matrices similar to A is a so-called **conjugacy class**. Within the conjugacy class, similarity between matrices defines an equivalence relation. Thus, the space of square matrices is partitioned into conjugacy classes. It follows from the change of basis formulae that a linear operator's representation is always a member of one and only one conjugacy class. Hopefully these concepts hold some intuitive power.

18. **Determinants:** You should know how to calculate determinants of $n \times n$ matrices. You should also be comfortable with the basic properties e.g. $\det(AB) = \det(A)\det(B)$, $\det(cA) = c^n \det(A)$, etc.

19. **Eigenvalues and Eigenvectors:** Given a square matrix A, you should know what eigenvectors and eigenvalues are and how to find them. What are eigenvectors geometrically? You should understand **geometric and algebraic multiplicity** of an eigen-

value. You should also know about the **characteristic equation** (what it is and how to find it) of a transformation, as well as **invariant subspaces** (what they are and how to find them).

20. **Eigenspaces:** Make sure you know what an eigenspace is and how to find one. Are all elements of an eigenspace an eigenvector? Is E_λ a subspace for all $\lambda \in \mathbb{R}$? Are they *ever* subspaces?

21. **Diagonalization:** You need to know what's mean by diagonalizability, and how to check whether a linear transformation is or isn't diagonalizable. What are some **equivalent conditions** for a matrix being diagonalizable? If a matrix is diagonalizable how do you diagonalize it?

22. **Polynomials:** We've exhaustively gone over what polynomials are, and you should not only know basic properties like the **Fundamental Theorem of Algebra**[4], you should know about polynomial division and polynomial factorization, especially the connection of the splitting of polynomials has to do with the field over which they split.

23. **Fields and Complex numbers:** $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ and others are examples of a *field*, which is a formal, axiomatically defined algebraic object. Field elements we just call "numbers", usually. We've also seen some **finite fields**. Complex numbers, thanks to the fundamental theorem of algebra, are a more natural space to perform arithmetic and algebra[5]

24. **Geometry:** We've covered **inner products** on $\mathbb{R}^n$ together with more general **inner product spaces** (like Hermitian inner product spaces). You'll obviously need to be aware of the Cauchy-Schwartz and triangle inequalities as well as the general Pythagorean equality. This geometry also of course includes the idea of **orthogonality**, a crucial component of the later sections.

25. **Symmetry:** The idea of a symmetric operator is reliant upon the particular inner product one has. But you need to know that if, say, $\langle Tv, w\rangle = \langle v, Sw\rangle$ holds for all v, w in an inner product space, then $S = T^T$. Namely, this algebraic identity uniquely defines the **transpose of an operator.** Once we have the idea of transpose, the definition of symmetry is then obvious. We care about symmetric operators for many, many reasons, not the least of which is that they are the ones most commonly encountered in applications.[6]

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[4]This theorem, which can be stated in many equivalent ways (I'll state it as saying that the complex field is algebraically closed) has a rich history. The first proof is often attributed to Gauss who at age 22 claimed a complete proof. By modern standards, his proof was (fatally) incomplete and, even Gauss himself noted in a footnote regarding his unproven assumptions used in the proof that "As far as I know, nobody has raised any doubts about this. However, should someone demand it then I will undertake to give a proof that is not subject to any doubt, on some other occasion". No one demanded and so he did not ever provide that "other occasion". Gauss gave other proofs of the theorem later on. It seems the first truly rigorous proof came about 7 years afterwards and was given by a French amateur mathematician who was employed as a bookstore manager. It's rare for hobbyists to give profound contributions to the field.

[5]But, notice that *unlike* $\mathbb{R}$, $\mathbb{C}$ is not an *ordered field*. By this I mean $\mathbb{C}$ doesn't have a natural order relation $<$ with the property that for all $w, z \in \mathbb{C}$ either $w < z$, $z < w$ or $w = z$.

[6]For example, the covariance matrix in statistics is symmetric. Further, given a matrix X whose rows are *examples* of observed vectors, the symmetric matrix $X^T X$ is often nicer to work with than X (note that this is

26. **Projections:** Linear operators $P : V \to S$, where $S \subset V$ which are symmetric and satisfy $P^2 = P$ are called orthogonal projections and are sort of like the foundational elements in the space of symmetric linear transformations. You need to know what they are, how to prove things with them, and the role they play in the larger story of linear operators on vector spaces.

27. **Gram-Schmidt Procedure:** Given a basis for a subspace of an inner product space there's an algorithm to build an orthonormal basis for the same subspace. That algorithm is the Gram-Schmidt one. We prefer working with orthonormal bases since coordinate expansions are much nicer than in non-orthonormal bases.

28. **Hermitian Operators and Spaces:** We extended inner product space to its complex cousin, the *Hermitian* inner product spaces. With this we had to define the **adjoint** of a transformation $T : V \to V$, denoteed T^* or $T^\dagger$, which is similar to the usual transpose. Namely, if $\langle \cdot, \cdot \rangle$ is a Hermitian inner product on complex vetor space V then T^* is the unique operator S satisfying $\langle Tv, w \rangle = \langle v, Sw \rangle$ for all $v, w \in V$. Hermitian operators are those operators for which $T = T^*$ and are the complex analogue of symmetric operators.[7]In matrix form $[T]_\beta^\beta = \overline{([T]_\beta^\beta)^T}$, i.e. the representation of the adjoint is the **conjugate transpose** of the representation of T. We care a lot about Hermitian operators, since not only do they abstract the important features of symmetric operators, but they are encountered in applications very often. [8]

29. **The Spectral Theorem:** You should know the spectral theorem, both what it says and how it gets applied. We encountered it in its real form, and then later in it's complex form, in the context of Hermitian linear operators. This theorem is one of the major landmarks in the MAT224 vistas.

30. **Vector Spaces over Fields:** Having defined vector spaces in a fairly easily-abstracted way, we looked at extending the definition so that the scalars in our vector spaces are coming from $\mathbb{F}$, where $\mathbb{F}$ is often just $\mathbb{R}$ or $\mathbb{C}$. This allowed us to look at things like the vector space $\mathbb{F}^n$ and $P_n(\mathbb{F})$ with the "usual" definitions of addition and scalar multiplication.

31. **Triangularization:** This is our first foray into abstraction. We discussed general shapes which representations of linear operators can admit, and saw that triangular form is connected with properties of the underlying field (namely we want the characteristic polynomial to split over the base field). A representation takes on a triangular form if and only if there's an increasing sequence of invariant subspaces (see following). So the abstract *decomposition of the ambient vector space into non-interacting subspaces on which the operator acts invariantly* is the high-level concept in this section.

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actually just a rescaled covariance matrix for the observations). This kind of symmetric matrix is a fundamental object in modern machine learning. Very loosely speaking, since algebra is about equality and analysis is about inequality, this makes $\mathbb{C}$ a natural algebraic space but a more complex (pun intended) analytic space.

[7]These generalize symmetric operators and thus include symmetric operators as a special case.

[8]For example, one can take *as an axiom of the natural world* that physical observables are Hermitian operators in an (infinite-dimensional) inner product space (called a **Hilbert space**). This unusual axiom has the defect of being awkward to state but the advantage of being true and carrying lots of implications for experimentalists.

32. **Invariant Subspaces:** One of the major pursuits in advanced modern mathematics[9] is the study of *invariants*. For us, the abstraction of vectors is vector spaces, and the study of vector spaces involves analysis of operators on these spaces. The study of operators is abstracted by the study of what abstract items those operators "preserve". That's what the invariant subspace material is about.

33. **Nilpotent Operators:** You should know what nilpotent operators are and basic consequences of nilpotency. For instance, if N is nilpotent, what can we say about it's characteristic polynomial? Are nilpotent operators always triangularizable or only in complex vector spaces? Nilpotent operators are a major component of Jordan form so without understanding nilpotency it will be impossible to fully master Jordan form. You need to know about the **canonical form of nilpotent operators**.

34. **Cycle Tableau and Cyclic Subspaces:** You should know how to construct cycle tableaux for a given nilpotent operator and, given a cycle tableaux, what the corresponding canonical form of the nilpotent operator will be.

35. **Generalized Eigenspaces and Generalized Eigenvectors** - Given an eigenvalue λ (namely, a root of the characteristic polynomial) of $T : V \to V$ of multiplicity m we can define the corresponding generalized eigenspace $K_\lambda = \ker(T - \lambda I)^m$. Then $(T - \lambda I)\mid_{K_\lambda}$ will be a nilpotent operator, with index of nilpotency at most m. Further, you should be able to see that K_λ are T-invariant subspaces. There were no assumptions about the operator T, so this means that *for arbitrary linear operators on a complex vector space we can construct T-invariant spaces corresponding to these nilpotent operators.*

36. **Jordan Form:** Given a linear operator on a finite dimensional complex vector space, one can consider its associated *Jordan Canonical Form*, which consists of a block-diagonal matrix, whose blocks are $m_i \times m_i$ matrices of form $\lambda_i I + N_i$ where N_i is an $m_i \times m_i$ nilpotent operator in it's canonical form, and λ_i is an eigenvalue with multiplicity m_i. To calculate the canonical form you have to first find the distinct eigenvalues and their multiplicities. Then, you construct, for each distinct eigenvalue, the nilpotent operator $N_i = (T - \lambda_i I)\mid_{K_{\lambda_i}}$ and it's cycle tableau. The cycle tableau for each such operator gives you the corresponding geometry of the relevant **Jordan Block** appearing in the Jordan canonical form. The book gives several examples, as well as examples of how to calculate a **Jordan basis**, namely a basis β, with respect to which $[T]_\beta^\beta$ takes on a Jordan canonical form. Jordan canonical form and nilpotent canonical form are special cases of triangularization. And triangularization is a special case of **change of basis** we've seen earlier. Namely, the Jordan canonical form theorem can just as well be stated as: *Given $T : V \to V$ linear operator on complex, finite dimensional vector space V, there exists a basis β and an invertible matrix P such that*

$$[T]_\beta^\beta = PJP^{-1}$$

holds, where, in the above, J is a block diagonal matrix with the eigenvalues of T on the diagonal and, possibly, 1's appearing in some of the super-diagonal entries (i.e.

[9] Actually, I would say this is one of the major reconceptualizations in 20th century physics as well. Modern particle physics is often heavily motivated by invariants in the form of so-called "gauge symmetries".

those in the $(i, i+1)$*'th entries for some i).* In other words, **Jordan canonical form is just another matrix factorization**.[10]

2 Meta-Topics

Here are things which we may have only briefly mentioned or implicitly used but for which you are still expected to know.

1. **Induction:** You should be familiar with following inductive proofs as well as constructing your own. This was part of the material in the Appendix of the book (assigned as your homework for week 0 prior to the start of term). If you are rusty on it, get unrusty quick!

2. **Sets:** We expect absolutely zero sloppiness and no abuse of set notation. The notation in this course is very inflexible and you've had three months to master the construction of basic sets. We expect complete perfection in this regard.

3. **Grammar:** I tell my sections that **this is a course on reading comprehension**, since much of the course is on looking at a statement about formal, abstract things and taking a careful, thoughtful approach to interpretation. Unlike in other hermeneutical areas the complete rigidity of our notation means *the things you write are read completely literally*. Think about the following snippet "Let $x = a_1x_1 + \cdots + a_nx_n = S + X$". Read literally, it must be the case that whatever kind of object x is, $S + X$ must be the same kind of object. So, is $S + X$ a vector or a set? Are the $x_1, ..., x_n$'s vectors or something else? If they are vectors, in which space do they belong? What about the $a_1, ..., a_n$'s? Are they numbers, matrices, vectors, or something else? What you write will be read literally and scrupulous attention will be paid to these kinds of grammatical ambiguities or contradictions. Remember, **the graders only know what you're thinking based on what you write!** So if you are having the correct idea about the problem but express it sloppily, they will not be able to follow the logic and you may lose points.

3 Tips on How to Prepare

Here are a few thoughtlets on best practices.

1. Unfortunately, the best preparation is to have stayed on top the entire semester and not have fallen behind at all. It is extremely hard to get back on top if you've fallen behind in a course like this since the abstract material requires significant time to absorb properly.

[10] In MAT223 you've seen the LU factorization of matrices and, in this class we've seen diagonalization and triangularizations as matrix factorizations. There are plenty more. One that comes up very often in numerous applications is called singular value decomposition (SVD). Beyond that, there are plenty of other common factorizations (Schur factorization, QR factorization, QS factorization, Polar decomposition to name a few of the common ones). Breaking complicated things apart into simpler pieces is a very general idea, and that's what each of these factorizations achieves. Which one is best is like asking which tool is the best for woodworking: it depends entirely on the task at hand.

2. I recommend having gone through literally every single recommended problem from the textbook. If you've done that, I consider you well-prepared.

3. Please get lots of sleep the night before the exam. Brains operate better when they are well-rested.

4. I generally recommend getting aerobic exercise the day of the exam (go for a long walk, a swim, a run, etc). Not only does this give you a nice, quiet time to think, but it can help your body relax a bit so that if you're prone to anxiety, your body will naturally have a stronger defence.

5. Remember that **three hours is a long time**. The test is 3 hours and there's no need to rush. You have plenty of time to answer each question thoughtfully and demonstrate you've learned the material well. This should help those who may have felt rushed during the other tests.

6. You'll want to be so solid on the material that you aren't wasting lots of time on things that a well-prepared student could answer quickly. In other words, you'll want the luxury of being able to think more deeply about the things for which deep thought is beneficial as opposed to wasting time on things we expect you to go through quickly. So make sure that any computational type of question you can get done quick. Calculations should be the easiest things we can ask.

7. **Don't think that we won't test you on topic X.** We've created the test with the goal of trying to find where you are weakest, and if there's something you're weak on we'll be hoping to find it. So just make sure you aren't leaving any areas of study out.

8. Decide in advance a strategy for how to deal with true/false questions you aren't sure about. Since there are deductions for incorrect answers, and since exams can be a high stress environment, it's good to think through how certain you want to be in order to guess an answer, prior to test day. Obviously, the best case scenario is that you are confident with the T/F answers (and even better if this confidence correlates with being right!) but, failing that, if you are, say, 70% sure of an answer, should you put down the guess? How about 65% sure? 51% sure? Everyone will have different thresholds and different attitudes towards risk, so there's no best answer here. But you shouldn't be making ad hoc decisions during the exam, you should know how you plan to deal with things like this beforehand.

9. I recommend[11] beginning the test by taking a quick look at each question, to get a "lay of the land". When perusing the questions, I suggest coming up with numbers for each problem. These numbers are your estimate of the amount of time you imagine that problem might take. When working on the test, I would recommend trying to stick to not going over the time limits you've given yourself for a given problem before moving on to the next one you haven't solved and coming back to the one you're stuck on later.

[11]And these really are just *recommendations*. I don't have a one-size-fits-all algorithm for how to write a final exam. These are just things that I would do if it were me writing the final.

10. I also recommend working on what you feel are the easiest questions first. Notice that the “easiest" may not mean “earliest". Perhaps you find T/F questions to be easiest. Or, perhaps you prefer calculation style questions. Whatever your preferences, I would suggest working on those problems first since that way you can secure a nice blanket of points right away. Then I would suggest working on problems in order of increasing estimated difficulty. This way you are gathering as many confident points as possible with the least amount of time.

11. Remember, **we only know what's in your head by what makes it onto the page**. And we aren't mind-readers. If you have an idea for connecting ideas in a proof, say, and you aren't clear in your explanation, then the reader cannot follow the argument. Just be very precise and clear in the way you indicate your thoughts because otherwise there will be ambiguities which can lead to loss of points.

4 Structure of the Exam

1. The exam has 12 questions. Many of the questions have multiple parts.

2. The exam is out of 200 points.

3. The last question is a selection of true/false questions. **Unlike on the term tests** you can end up with negative points on the T/F question page. If your score is, say, -6 on that page, **points will be deducted from other problems.**

4. There is an optional **bonus problem** at the end of the exam. It is (in my opinion) not at all an easy question. It is worth an additional 30 points.[12] Frankly, since we are **not going to be generous** with partial credit on the bonus question, *I recommend only attempting it if you feel confident about your work on the main exam questions*. I don't want people to devote tons of valuable time on this harder question in vain. It's there purely as a bonus and as a reward for those who were well-prepared enough to be able to spend time working on it.

5. Some of the questions will offer a nominal amount of points for leaving the question blank, as was done on the second midterm. Again, this is there discourage people from writing completely aimless things in the desperate hope of getting some kind of partial credit. The amount offered will be clearly stated before the problem itself is stated.

6. Although the exam is done on Crowdmark, you won't receive the Crowdmark link for quite a while, and I **NEVER**[13] respond to emails about the final exam, so don't bother. Every year I say this, and every year, without fail, some students somehow imagine this not applying to them. It applies to you, dear reader.

[12] In other words, some students could conceivably score, say, 230/200 on this final exam.
[13] Like, actually, truly, I never have. And I'm not going to be breaking my streak.

5 Other Courses

Several students have asked about follow-up courses to MAT224, and what they should take if they enjoyed the material in this term. Here are my thoughts.

1. **MAT245:** This is the course I created and designed in 2017 on "Mathematical Methods in Data Science". It's mostly taken by specialists in mathematics and computer science. Here is the syllabus from this past term `http://www.math.toronto.edu/nhoell/MAT245/MAT245_syllabus.pdf` It's a notoriously demanding course (not for the faint of heart!), but the students who've completed it have consistently rated it as the most valuable course they took. We use a lot of advanced linear algebra extensively, and cover the ways linear algebra is used in machine learning and other forms of inference.

2. **MAT301:** This is a course on "Groups and Symmetries". I've not taught it so I don't know the syllabus but the material should be an extension of things we've done in this course. Namely, the course should cover more general algebraic objects than vector spaces called groups. The set of invertible $n \times n$ matrices are an example of a group[14], called GL_n which is presumably something arising in that course.

3. **MAT347:** This is a course on "Groups, Rings and Fields". I've not taught it and don't know the syllabus but I imagine it's a more advanced version of MAT301. We've seen examples of fields in this course, and (without calling them groups) we've seen examples of groups as well. We haven't discussed rings, and we also haven't discussed the ways these abstract things are all related. This course, I presume, covers these algebraic objects in detail.

4. **MAT401:** This course is on "Polynomial Equations and Fields". It covers topics relating to polynomials[15] which we simply took for granted in MAT224. It covers the relationships between polynomials and the number fields over which they have solutions.

5. **APM421:** This course is on "Mathematical Foundation of Quantum Mechanics". The formalism behind atomic physics is, essentially, linear operators on (infinite dimensional) vector spaces. I've not taught the course and don't know the syllabus, but it certainly must cover the basics of matrix algebras and their representations on state spaces in quantum mechanics. You would certainly encounter more generalized versions of what we've discussed in MAT224, especially in that you'd be working with linear operators on infinite dimensional spaces.

6. **Any Course in Partial Differential Equations:** There are several options here and I'm not going to list them. The more mathematical options may introduce you to the infinite-dimensional version of the spectral theorem. Even if they don't you'd be exposed to adjoint theory and infinite-dimensional inner product spaces. The prerequisites for courses like this are generally MAT244 or its equivalent. I've taught courses

[14] This is only one simple example, groups are very general and complicated. But, invertible matrices with determinant one 1, say, is a very common and extremely important example of a group with deep applications in physics.

[15] Like the fundamental theorem of algebra.

like this many times and we tend to cover quite a bit of material which extends stuff we've done in MAT224.

7. **MAT436:** This is a course on "Linear Operators" which I think is a cross-listed graduate course. I think it's a demanding course. But, it is also the most obvious one on the list as candidate for "more advanced version of MAT224". Basically, working out theorems on infinite-dimensional vector spaces is called **functional analysis** and I think this is UofT's course on functional analysis. If you like MAT224, and you have taken some kinds of ϵ, δ advanced analysis, this might be a course to consider. But I believe it's taught at a high level.