MAT224 Proof Clinics for Test 2

We will be holding proof clinics to help prepare you for the upcoming test. The sessions are on:

Date	Time	Location
Thursday, Mar. 7	10:00 am - 12:00 pm	BI 131
Friday, Mar. 8	10:00 am - 12:00 pm	MP 134
Saturday, Mar. 9	10:00 am - 12:00 pm	SS 1072
Sunday, Mar. 10	10:00 am - 12:00 pm	SS 1072
Monday, Mar. 11	11:00 am - 1:00 pm	SF 2202
Tuesday, Mar. 12	10:00 am - 2:00 pm	FE 24

Like last time, these clinics are your opportunities to get feedback from a TA regarding your proof writing technique so attendance is encouraged. Six exercises are given below and additional exercises will be provided during the sessions to simulate an exam environment. If you choose to attend, we strongly recommend showing up with written proofs to the exercises below.

In the problems below V is a finite-dimensional real vector space.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear map such that $T^2 = 0$. Show that there exists a basis $\beta \subseteq \mathbb{R}^2$ of $\mathbb{R}^2$ such that

$$[T]_\beta^\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Suppose $T\colon V \to V$ is an invertible linear transformation and let $v \in V$ be an eigenvector of T. Show that v is also an eigenvector of T^{-1}.

3. Below is a true statement and an incorrect proof of it. Explain why the "proof" is wrong, and provide a correct version.

   Fact. If $T : V \to V$ is an invertible linear transformation and $W \subseteq V$ is a subspace such that $T(w) \in W$ for all $w \in W$, then for all $w \in W$, $T^{-1}(w) \in W$.

   Fake proof. Let $v \in W$. Then the vector $w = T(v)$ is in W, and by construction we also see that $T^{-1}(w) = T^{-1}(T(v)) = v$ is in W. Since v and w were arbitrary, we are done. □

   Hint/pun: For the correct proof, you should *restrict* your attention to what's useful.

4. Let $T\colon V \to V$ be a linear transformation and let $v_1, v_2, v_3 \in V$ be eigenvectors of T with distinct eigenvalues. Show that $\{v_1, v_2, v_3\}$ is linearly independent.

5. Let $S, T\colon V \to V$ be linear transformations such that $ST = TS$. Show that if $v \in V$ is an eigenvector of T and $S(v) \neq 0$ then $S(v)$ is also an eigenvector of T.

6. Let $n \geq 2$ and define the transformation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $T(A) = A^\top$.

   (a) Show that T is a linear transformation.

   (b) Show that $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if $\lambda \in \{-1, 1\}$.