MAT334, Complex variables

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This document includes lecture notes, homework sets and solutions, term test and final assessment solutions, and background review material. It was typed up as the course progressed and has not been subsequently modified, so should be considered a rough draft. Comments or corrections can be sent to the author at ncarruth@math.toronto.edu.

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Summary:
- We give a description of the complex number system.
- We then give a description of the complex plane and indicate why it is something which might be useful.

I. INTRODUCTION TO THE COMPLEX PLANE

1. Complex numbers. We probably saw complex numbers for the first time when we learned how to solve quadratic equations. For example, the equation

$$x^2 = -1$$

has no solution over the real numbers. It turns out to be useful in algebra, and even more in analysis, to *extend* our number system by including an extra quantity, written i, which behaves exactly like a real number except that it has the property

$$i^2 = -1. \tag{1}$$

A general number in our new number system can be written in the form $a + bi$, where a and b are arbitrary *real* numbers,[1] and we require that these numbers satisfy all of the standard rules of algebra, augmented by equation (1). Thus, for example, the product of two complex numbers is given by

$$(a + bi)(c + di) = ac + bi \cdot c + a \cdot di + bi \cdot di = ac + bci + adi + bdi^2 = ac - bd + (bc + ad)i.$$

(As shown here, whenever we write out a complex number we always combine real and imaginary terms when possible.)

We generally use the letters z and w to denote complex numbers, and x and y to denote real numbers. We let $\mathbf{C}$ denote the set of all complex numbers. If $z = a + bi$ is a complex number, we call a the *real part* of z and b the *imaginary part* of z, and write $a = \operatorname{Re} z$, $b = \operatorname{Im} z$. Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if their real and imaginary parts are equal.[2]

To every complex number $a+bi$ there corresponds another complex number known as its *conjugate* and given by $a - bi$.[3] If z is any complex number, we write $\overline{z}$ for its conjugate. The conjugate will be seen later to have many uses, but for the moment we note its use in finding inverses. First, note that if $z = a+bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Thus if $a + bi \neq 0$, then

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2},$$

which is defined since $a + bi \neq 0$ implies that at least one of a and b is nonzero, so $a^2 + b^2 > 0$. This is the desired formula for the inverse of a complex number.

See Goursat, §1.

2. The complex plane. If z is any complex number, it determines two real numbers $\operatorname{Re} z$ and $\operatorname{Im} z$, and is in turn uniquely determined by these two numbers. This suggests that, just as we may think of arbitrary real numbers as points on the *real number line*, we may think of arbitrary complex numbers as points in the *complex plane*. Specifically, given a plane with perpendicular axes which we call x and y, we associate with any complex number z the point in this plane whose x-coordinate is $\operatorname{Re} z$ and whose y-coordinate is $\operatorname{Im} z$. While complex numbers are *per se* abstract objects without any direct concrete significance, this association allows us to think and speak of them as points in the plane. We shall do this whenever it seems

[1] Whenever we write an arbitrary complex number as $a + bi$, it will always be assumed that a and b are real.

[2] This means that the set $\{1, i\}$ is a basis for $\mathbf{C}$ considered as a real vector space.

[3] We remind the reader that the presence of a + or − in front of a quantity does not guarantee the resulting sign; in other words, $+b$ can be negative and $-b$ can be positive, and both will be respectively when b is negative.

convenient; thus we shall speak of "the point $a+bi$", etc., when more carefully we should say "the point corresponding to the complex number $a+bi$".

Given the foregoing, it is clear that the point corresponding to the conjugate of a complex number $a+bi$ is simply the reflection in the x-axis of the point corresponding to $a+bi$.

The foregoing connection between complex numbers and points in a plane, while it may be interesting, would not be particularly useful if the *geometric* properties inherent in the Euclidean plane were not somehow related to *algebraic* or *analytic* properties of the complex numbers its points represent. We shall see throughout this course that there are in fact many and deep connections between the geometry of the plane on the one hand and the algebraic and analytic properties of complex numbers on the other. Here we shall indicate one example.

EXAMPLES. One simple example is as follows. Suppose that $z = a+bi$ and $w = c+di$ are any two complex numbers. Then clearly

$$\overline{z}w = (a-bi)(c+di) = ac+bd+i(ad-bc).$$

Now if we think of the vectors (corresponding to the points) corresponding to $a+bi$ and $c+di$, i.e., $\mathbf{v} = a\mathbf{i}+b\mathbf{j}$, $\mathbf{u} = c\mathbf{i}+d\mathbf{j}$, we see that their dot product is $\mathbf{v}\bullet\mathbf{u} = ac+bd$ while their cross product is $\mathbf{v}\times\mathbf{u} = (ad-bc)\mathbf{k}$; in other words, roughly, the real part of $\overline{z}w$ is the dot product of the vectors corresponding to z and w, while the imaginary part is their cross product.[4] We shall see some other relations of this sort when we talk about derivatives of functions of a complex variable; it turns out that, when viewed as a vector field, the derivative of the conjugate of such a function essentially encodes the divergence and curl of the vector field.[5]

As another, more interesting, example, let $a+bi$ be any complex number, and consider the corresponding point in the plane. This point has polar coordinates (r,θ), where r is the distance from the origin to the point and θ is the angle from the positive x-axis to the ray from the origin passing through the point. In symbols, this becomes

$$r = \sqrt{a^2+b^2}, \qquad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$
$$a = r\cos\theta, b = r\sin\theta$$

Note that θ is only defined up to a multiple of 2π: the two polar coordinate expressions (r,θ) and $(r,\theta+2\pi)$ determine exactly the same point in the plane. We shall see shortly that for many important functions to be continuous (in an appropriate sense) on the complex plane, there is no way around this ambiguity: it is simply something which must be dealt with.

Now suppose that $c+di$ is any other complex number which satisfies $c^2+d^2 = 1$: this means that the point corresponding to $c+di$ lies on the unit circle. If we let (r_0,θ_0) denote the polar coordinates of this point, then we have $r_0 = 1$, while θ_0 satisfies $\cos\theta_0 = c$, $\sin\theta_0 = d$.[6] Now applying basic trigonometric identities, we obtain

$$\begin{aligned}(a+bi)(c+di) &= ac-bd+i(ad+bc)\\ &= r\cos\theta\cos\theta_0 - r\sin\theta\sin\theta_0 + i(r\cos\theta\sin\theta_0 + r\sin\theta\cos\theta_0)\\ &= r\cos(\theta+\theta_0) + ir\sin(\theta+\theta_0)\\ &= r\left[\cos(\theta+\theta_0) + i\sin(\theta+\theta_0)\right],\end{aligned}$$

from which it is evident that the point corresponding to the product $(a+bi)(c+di)$ is simply that corresponding to $a+bi$ rotated counterclockwise by the angle θ_0!

[4] It turns out that there is a *four-dimensional* extension of the real numbers called the *quaternions*, which contain the complex numbers, and which in some sense generalises results of this sort to full three-dimensional vectors. We shall not deal with these in this course, though, except for a few asides like this one.

[5] While interesting, these examples are somewhat tangential to the main content of this course.

[6] If you are familiar with De Moivre's theorem, it is useful to note that this means that $c+di = \cos\theta_0 + i\sin\theta_0$.

Summary:

- We discuss another geometric interpretation of complex multiplication.
- We then discuss taking powers and roots of complex numbers, and the geometric interpretation of these operations.

We have just observed that multiplying a complex number by another complex number of unit modulus is equivalent to rotating the original complex number by an angle equal to that of the second complex number. It turns out that multiplication by a general complex number can be viewed as the composition of a rotation and an isotropic scaling. Let us see how this works. Suppose that we have two complex numbers,

$$z = r(\cos\theta + i\sin\theta), \qquad w = r'(\cos\theta' + i\sin\theta').$$

Then their product comes out to be (the angular part is exactly analogous to what we saw at the end of the notes of May 5)

$$\begin{aligned} zw &= rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= rr'(\cos\theta\cos\theta' - \sin\theta\sin\theta' + i[\sin\theta\cos\theta' + \cos\theta\sin\theta']) \\ &= rr'[\cos(\theta+\theta') + i\sin(\theta+\theta')]; \end{aligned} \tag{1}$$

in other words, the point corresponding to zw is exactly the point corresponding to z, rotated by θ' and scaled by r'. This is the sense in which multiplication by a complex number is just a rotation and a scaling. (This is related to some of the problems on the review sheet!)

3. Exponentiation. We have seen that the affect of multiplication on the angular part of a complex number is just a rotation. What happens under exponentiation? Let $z = r(\cos\theta + i\sin\theta)$; then we see that, by the formula in (1) above,

$$\begin{aligned} z^2 &= z\cdot z = r^2(\cos 2\theta + i\sin 2\theta), \\ z^3 &= z\cdot z^2 = r(\cos\theta + i\sin\theta)\cdot r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta), \end{aligned}$$

and so on, so that it is evident that for any positive integer m we have

$$z^m = r^m(\cos m\theta + i\sin m\theta).$$

To try to get some sense of what this means geometrically, let us first consider the case $r = 1$; then $r^m = 1$ for all m and we have simply

$$z^m = \cos m\theta + i\sin m\theta.$$

Now any complex number of unit modulus is represented in the complex plane by a point on the unit circle, and completely determined by the angle between a ray drawn from the origin to that point and the positive x-axis, measured in a counterclockwise direction: this is just the number θ above. This formula then tells us that the point corresponding to z^m is also on the unit circle, but with an angle from the positive x-axis equal to m times that of the point corresponding to z. In other words, if we must traverse an angle θ to arrive at z, we must traverse an angle of $m\theta$ to arrive at z^m.

Suppose now that we consider the affect of exponentiation on not just a single point on the unit circle but rather an *arc*, say from $\theta = 0$ to $\theta = \theta_0$ for some $\theta_0 > 0$. The point corresponding to θ_0, namely $\cos\theta_0 + i\sin\theta_0$, will be mapped by this exponentiation to $\cos m\theta_0 + i\sin m\theta_0$; and it is clear that every point with $\theta \in [0, \theta_0]$ will be mapped to a point with $\theta \in [0, m\theta_0]$. Thus exponentiation simply stretches out the original arc.

With this in mind, let us consider the affect of exponentiation on an angular wedge, namely on the set of all points (of whatever modulus) whose angle with the positive x-axis lies between 0 and θ_0. Such a point can be written in the form $z = r(\cos\theta + i\sin\theta)$, where $\theta \in [0, \theta_0]$, and $z^m = r^m(\cos m\theta + i\sin m\theta)$; from the foregoing, then, it is clear that this point will lie inside a 'wedge' (it may have an angle greater than π and hence not really be a proper 'wedge' anymore) extending from 0 to $m\theta_0$.

Now there is no particular reason to restrict the lower angular bound on the wedge to be 0; we may as well consider a wedge $[\theta_1, \theta_2]$. The same logic shows that this will be mapped to a wedge $[m\theta_1, m\theta_2]$.

In particular, if we consider the wedge from 0 to π and let $m = 2$, we see that the image under exponentiation is the 'wedge' from 0 to 2π, i.e., the entire complex plane. The same is true if we consider the wedge from 0 to $\frac{2\pi}{3}$ and let $m = 3$, and in general, if m is any positive integer, then the wedge from 0 to $\frac{2\pi}{m}$ will be mapped to the entire complex plane by the map $z \mapsto z^m$. Similarly, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will also be mapped to the entire complex plane, and so will the wedges from $\frac{2n\pi}{m}$ to $\frac{2(n+1)\pi}{m}$ for any $n = 0, 1, \dots, m-1$.

While we do not quite have all of the necessary tools to make the following picture precise, it provides much useful intuition and I think is simple enough to understand. We may think of exponentiation by a positive integer as an endpoint in a process that starts with exponentiation by 1 (i.e., doing nothing!) and then slowly increases the exponent *through all real numbers* until it reaches m. Under this kind of a map, the wedge from 0 to $\frac{2\pi}{m}$ (say) will be slowly stretched out (with the bottom edge, i.e., that along the x-axis, remaining fixed) until the outer edge finally reaches the x-axis. Under the same map, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will behave slightly differently: the lower edge $\frac{2\pi}{m}$ also moves until it reaches the positive x-axis, while the upper edge $\frac{4\pi}{m}$ moves even faster so that by that point it has travelled one full 2π past the positive x-axis. Similar things can be said about the additional wedges.

What all of this means is that under exponentiation by a positive integer, the wedges $\frac{2\pi n}{m}$ to $\frac{2\pi(n+1)}{m}$ are each rotated and stretched in such a way as to cover the entire complex plane exactly once.[7] This means that each complex number is the image under the exponentiation map of exactly one point from each of these wedges. A little thought shows that this means that each complex number (except 0) has exactly m mth roots.

More precisely, suppose that $z = r(\cos\theta + i\sin\theta)$ is some complex number. Now for each positive real number r there is exactly one positive real number R satisfying $R^m = r$, and we denote this unique positive real mth root by $r^{\frac{1}{m}}$. Given this, for $n = 0, 1, \dots, m-1$, let $w_n = r^{\frac{1}{m}}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$; then clearly

$$\begin{aligned} w_n^m &= \left(r^{\frac{1}{m}}\right)^m (\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n)) \\ &= r(\cos\theta + i\sin\theta) = z, \end{aligned}$$

so that each of the w_n is an mth root of z. More specifically, if we assume that $\theta \in [0, 2\pi]$, then it is clear that w_n is in the nth of the above wedges. We note that $w_m = w_0$, and in general $w_{n+km} = w_n$ for any positive integer k. It can be shown that the w_n are the only complex mth roots of z, and that z therefore has exactly m distinct mth roots, as claimed. [The proof is not that hard: suppose that $w = r'(\cos\theta' + i\sin\theta')$ is any mth root of z, i.e., that $w^m = z$; this means that

$$r'^m(\cos m\theta' + i\sin m\theta') = r(\cos\theta + i\sin\theta),$$

which means that $r'^m = r$, i.e., $r' = r^{\frac{1}{m}}$, and that there is an integer n such that $m\theta' = \theta + 2n\pi$, which gives $\theta' = \frac{\theta}{m} + \frac{2n\pi}{m}$ for some integer n. Now dividing n by m we can find integers q and r such that $n = qm + r$ and $r \in \{0, 1, 2, \dots, m-1\}$; thus $\theta' = \frac{\theta}{m} + \frac{2(qm+r)\pi}{m} = \frac{\theta}{m} + 2q\pi + \frac{2r\pi}{m}$ and this w is equal to w_r.]

4. Complex derivatives. Cauchy-Riemann equations In first-year calculus we learned that the derivative of a real-valued function of a single real variable, if it exists, is given by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

In multivariable calculus, we learned about taking *partial derivatives*, which are derivatives in a single direction at a time; we couldn't take the derivative 'with respect to a vector' since we had no way of dividing by a vector.[8] Those of you who have seen how derivatives of functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be viewed as linear operators between those spaces will still recall that the components of the matrix representations of those operators are still calculated as partial derivatives, i.e., even in that case we reduce back to the case of a single function of a single variable.

[7] Well, *almost* exactly once. To be precise we should only include one of the two edges, restricting the angle to lie in a half-open interval.

[8] This is not *entirely* correct and there is in fact a nice way in which the gradient can be viewed as a derivative $\frac{df}{d\mathbf{r}}$. But that is probably more of a notational shorthand than anything fundamental, unlike what we are about to do with complex numbers.

In complex analysis, though, we can go further since we have a well-defined way of dividing by complex numbers even though they are two-dimensional quantities (at least over $\mathbf{R}$!). Let $f : \mathbf{C} \to \mathbf{C}$ be a complex-valued function of a complex variable, and consider the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

where now h is allowed to be a *complex* number. Since h is complex, this means that we are taking a two-dimensional limit. As we have learned in multivariable calculus, a two-dimensional limit can only exist if directional limits from different directions exist and are equal (and it may fail to exist even then). Let us consider what the above limit looks like in the two cases where we restrict h to go to zero along the real and imaginary numbers (in terms of the complex plane, this means that h goes to zero along the horizontal and vertical axes, respectively). First, let us write out f explicitly in terms of its real and imaginary parts as (writing $z = x + iy$)

$$f(x+iy) = P(x,y) + iQ(x,y),$$

and assume that all partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ exist. If $h = \Delta x$ is real, the quotient inside the limit becomes

$$\begin{aligned}\frac{f(x+iy+\Delta x) - f(x+iy)}{\Delta x} &= \frac{P(x+\Delta x, y) + iQ(x+\Delta x, y) - [P(x,y) + iQ(x,y)]}{\Delta x} \\ &= \frac{[P(x+\Delta x, y) - P(x,y)] + i[Q(x+\Delta x, y) - Q(x,y)]}{\Delta x}.\end{aligned}$$

Since the partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial x}$ exist, in the limit as Δx goes to zero this becomes

$$\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

This gives the original (two-dimensional) limit along the real axis. To find the limit along the imaginary axis, let $h = i\Delta y$ (note the i!); then we obtain

$$\begin{aligned}\frac{f(x+iy+i\Delta y) - f(x+iy)}{i\Delta y} &= \frac{P(x, y+\Delta y) + iQ(x, y+\Delta y) - [P(x,y) + iQ(x,y)]}{i\Delta y} \\ &= -i\left\{\frac{[P(x, y+\Delta y) - P(x,y)] + i[Q(x, y+\Delta y) - Q(x,y)]}{\Delta y}\right\},\end{aligned}$$

so that since the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial y}$ exist this becomes

$$-i\left\{\frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y}\right\} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

For the full two-dimensional limit to exist, this must equal the limit along the real axis; thus we must have

$$\frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x},$$

which gives the celebrated *Cauchy-Riemann equations*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

To sum up: for a function f of a complex variable to have a derivative at a point, its real and imaginary components P and Q must have partial derivatives at that point and those partial derivatives must satisfy the Cauchy-Riemann equations. It can be shown (see Goursat, §3) that if the partial derivatives of P and Q are also continuous at the point in question, then these conditions are sufficient in that f is then guaranteed to have a derivative at that point. Functions whose real and imaginary parts satisfy the Cauchy-Riemann equations but which do not have a derivative shall not concern us much in this course.

When f has a derivative at a certain point, by the foregoing that derivative is given by either of the expressions

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

Other equivalent expressions can also be derived; see Goursat, §3, equation (2).

Let us consider a specific example of the foregoing.

EXAMPLES. Let us consider a very simple function:

$$f(z) = z^2.$$

To find its real and imaginary parts, let $z = x + iy$; then

$$f(z) = f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy),$$

whence we see that its real and imaginary parts are, respectively,

$$P(x,y) = x^2 - y^2, \qquad Q(x,y) = 2xy.$$

We leave it as a worthwhile exercise to the reader to show that these do in fact satisfy the Cauchy-Riemann equations. Since they certainly have continuous partial derivatives, we see that f must have a derivative at any point z. The formulas above give this derivative as

$$f'(z) = f'(x+iy) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = 2x + 2iy = 2z.$$

(This should not be a surprise, since we know from real-variable calculus that the derivative of x^2 is $2x$.) In this case, we can also derive this result directly, as follows:

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} \frac{z^2 + 2zh + h^2 - z^2}{h} = \lim_{h\to 0} \frac{2zh + h^2}{h} = \lim_{h\to 0}(2z + h) = 2z. \end{aligned}$$

This result turns out to be typical: most of the standard functions we are familiar with from calculus which have derivatives as functions of a real variable *also* have derivatives as functions of a *complex* variable, and the derivatives are the same. (There is a very good reason for this, which will become clearer throughout the course: it is tied up with the fact that most of the functions we deal with in calculus do not just have a single derivative but are rather *real analytic*, i.e., are equal to their Taylor series expansions. Such functions always extend to differentiable functions of a complex variable, and this is one of the major links from real to complex variable theory.)

As a still elementary but slightly more complicated example, let us show that the power rule of elementary calculus holds for functions of a complex variable, if we restrict ourselves to positive integer exponents. (It holds for more general exponents, too, at least away from $z = 0$, but that will require a separate treatment.) Thus let m be a positive integer, and define $f(z) = z^m$. Then we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{(z+h)^m - z^m}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{m} \binom{m}{k} z^{m-k}h^k - z^m\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(mz^{m-1}h + \frac{m(m-1)}{2}z^{m-2}h^2 + \cdots\right) \\ &= \lim_{h\to 0}\left(mz^{m-1} + \frac{m(m-1)}{2}z^{m-2}h + \cdots\right) = mz^{m-1}, \end{aligned}$$

since all terms in $\cdots$ have at least an h^2 in them and hence must go to zero as h does. Thus we have $f'(z) = mz^{m-1}$, exactly as in the real-variable case.

Summary:

- We wrap up some loose ends from last time.
- We discuss how differentiation rules from elementary calculus can be extended to the current setting.
- We discuss multiple-valued functions and give a brief introduction to the notion of *branch cut*.

5. Harmonic functions. If a function $f'(z)$ has a derivative throughout a region, we say that it is *analytic* in that region.[9] From last time, we know that if we write f as

$$f(x+iy) = P(x,y) + iQ(x,y),$$

then, assuming that P and Q possess continuous first-order partial derivatives, f will be analytic if P and Q satisfy the Cauchy-Riemann equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

It turns out that these equations impose a very strong condition on P and Q, namely that they be *harmonic*, i.e., that they satisfy Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Assuming that P and Q possess continuous second-order partial derivatives, this can be shown easily as follows:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y}\left[-\frac{\partial Q}{\partial x}\right] = \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} = 0,$$

since under the above assumption the mixed partial derivatives of Q commute. The calculation for Q is similar and we leave it to the reader as an exercise.

To summarise, then, we have the implication

$$f \text{ analytic} \Longrightarrow \operatorname{Re} f, \operatorname{Im} f \text{ harmonic}.$$

Note that the reverse implication is false, since if P and Q are two harmonic functions there is in general no reason at all to expect them to satisfy the Cauchy-Riemann equations. Note also that for us the term *harmonic* is applied only to real-valued functions of real variables; we do not speak of a function f of a complex variable being harmonic. (We could define *analytic* for functions of a real variable – it is simply that the function have a convergent power series representation – but we have not done so as we shall have no particular need for this concept by itself.)

Harmonic functions are very important in many areas of physics and science, as they can be used to describe temperature distributions, static electric fields, and steady-state fluid flows, for example. We shall see later that one major application of complex variable theory lies in the use of analytic functions *qua* conformal maps to find solutions to Laplace's equation in nontrivial geometries.

Given a harmonic function P, there is a harmonic function Q, unique up to an additive constant, such that $f(x+iy) = P(x,y) + iQ(x,y)$ is analytic. This is discussed in Goursat, §3, and also in §9 below.

6. Differentiation rules. We have already seen one example (at the end of §4 from last time) where a differentiation rule from elementary calculus carried across essentially unchanged to the current setting. It turns out that almost all of the differentiation rules from elementary calculus do also carry over to functions of a complex variable: for example, the product rule and quotient rule do, since the proofs of those two

[9] The word *analytic*, when applied to a real-valued function of a real variable, means that the function can be extended in a power series, i.e., that the Taylor series of the function converges to the function on some interval. We shall show later that, for functions of a complex variable, existence of the derivative throughout an appropriate region allows us to conclude that the function has derivatives of all orders, and that the Taylor series about each point converges to the function on some disc. Thus our terminology is consistent with the real-variable case.

rules work equally well for complex independent variables as they do for real. This means that derivatives of rational functions (quotients of polynomials) can be found exactly as for functions of a real variable.

The chain rule also carries over to the current setting, as can be seen as follows. Suppose that f and g are analytic functions, and let $z \in \operatorname{dom} f$ be such that $f(z) \in \operatorname{dom} g$. Then since f and g are analytic we have

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z), \qquad \lim_{h'\to 0} \frac{g(f(z)+h')-g(f(z))}{h'} = g'(f(z)).$$

Now the first relation can be rewritten in the following way:

$$\lim_{h\to 0} \frac{f(z+h)-f(z)-f'(z)h}{h} = 0.$$

Let us write $\epsilon(h) = f(z+h) - f(z) - f'(z)h$, so that this result becomes $\lim_{h\to 0} \frac{\epsilon(h)}{h} = 0$. Similarly let us write $\epsilon'(h') = g(f(z)+h') - g(f(z)) - g'(f(z))h'$.[1]0 Then we note that

$$\begin{aligned}
\lim_{h\to 0} \frac{g(f(z+h))-g(f(z))}{h} &= \lim_{h\to 0} \frac{g(f(z)+f'(z)h+\epsilon(h))-g(f(z))}{h} \\
&= \lim_{h\to 0} \frac{g'(f(z))[f'(z)h+\epsilon(h)]+\epsilon'(f'(z)h+\epsilon(h))}{h} \\
&= g'(f(z))f'(z) + \lim_{h\to 0}\left[g'(f(z))\frac{\epsilon(h)}{h} + \frac{\epsilon'(f'(z)h+\epsilon(h))}{h}\right];
\end{aligned}$$

but the limit of the first fraction is zero by what we know about $\epsilon(h)$, while the limit of the second is also zero by what we know about $\epsilon(h)$ and $\epsilon'(h)$. Thus we have

$$\frac{d}{dz} g(f(z)) = g'(f(z))f'(z),$$

exactly as we do in elementary calculus.

We shall see shortly that, given appropriate extensions of the elementary transcendental functions of calculus (the trigonometric, exponential, and logarithmic functions), the derivatives of all of these functions are also what one would expect from calculus.

7. Roots and branch cuts. There is one class of functions which we have already extended to all complex numbers but whose derivatives we have not yet discussed, namely the roots. It turns out that a study of these functions reveals a subtlety in functions of a complex variable which is not visible in functions of a real variable. Let us fix some positive integer m and consider mth roots. Recall that if $z = r(\cos\theta + i\sin\theta)$ is any complex number, then the m complex numbers

$$w_n = r^{1/m}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$$

all satisfy $w_n^m = z$. Now a function must have a unique value at a given point; thus if we wish to define an mth root *function* we must have some way of choosing just one of these values for each point. At first sight it would appear that we could just take w_0 and be done, but a bit more thought reveals that the situation is not quite that simple: for example, should $z = r$, for r a positive real number, be represented as

$$z = r(\cos 0 + i\sin 0), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m},$$

or as

$$z = r(\cos 2\pi + i\sin 2\pi), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m}\left(\cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}\right)?$$

[1]0 For those who have seen this notation, we note that this is equivalent to saying that $\epsilon(h) = o(h)$ and $\epsilon'(h') = o(h')$.

If we are interested only in defining a function we may just choose one of these and be done. The problem with that method, though, is that the resulting function will not be continuous across the real axis. For suppose that we make the requirement that $\theta \in [0, 2\pi)$, which corresponds to choosing the first of these two expressions. Let us consider the two limits

$$\lim_{h\to 0^+}(\cos h + i\sin h)^{1/m} \qquad \text{and} \qquad \lim_{h\to 0^-}(\cos h + i\sin h)^{1/m}.$$

For our mth root function to be continuous these two limits must be equal. But since we have required the angle θ to lie in the interval $[0, 2\pi)$, we must rewrite the second number as

$$\cos(2\pi + h) + i\sin(2\pi + h)$$

(remember that here h is negative so $2\pi + h < 2\pi$!), which means that the two limits become

$$\begin{aligned}\lim_{h\to 0^+}(\cos h + i\sin h)^{1/m} &= \lim_{h\to 0^+}\left(\cos\frac{h}{m} + i\sin\frac{h}{m}\right)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0^-}(\cos(2\pi + h) + i\sin(2\pi + h))^{1/m} &= \lim_{h\to 0^-}\left(\cos\frac{2\pi + h}{m} + i\sin\frac{2\pi + h}{m}\right)\\ &= \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m},\end{aligned}$$

and these two expressions are clearly not equal unless $m = 1$ (when everything is quite trivial). A similar problem would happen if we made the second choice above.

It turns out that the above difficulty is not just a result of our lack of cleverness: there is in fact no way to define an mth root function which is single-valued and continuous on the entire complex plane. The basic idea is already contained in the foregoing. Suppose that $f : \mathbf{C} \to \mathbf{C}$ were a function of a complex variable satisfying everywhere on $\mathbf{C}$ the formula

$$[f(z)]^m = z,$$

and such that $f(z)$ were continuous everywhere on $\mathbf{C}$. Let us consider how f behaves on the unit circle. By our study of roots above, we know that there must be integer $n \in \{0, 1, 2, \cdots, m-1\}$ such that $f(1) = \cos\frac{2\pi n}{m} + i\sin\frac{2\pi n}{m}$. Since f is continuous, for θ close to zero we must also have

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m}.$$

Now let us consider what happens when we gradually increase θ more and more. Clearly we must always still have

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m},$$

since otherwise there would be a point where we would need to switch to a different value of n, and this would lead to a discontinuity in f (this could be shown analogously to how we argued above about discontinuity across the real axis). Thus we can keep on going up until we get close to 2π. But if θ is very close to 2π the above result gives

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m};$$

but since we can consider $\theta < 0$ as well as $\theta > 0$, we also have

$$f(\cos\theta + i\sin\theta) = f(\cos(\theta - 2\pi) + i\sin(\theta - 2\pi)) = \cos\frac{\theta + 2\pi(n-1)}{m} + \sin\frac{\theta + 2\pi(n-1)}{m},$$

a contradiction.

Let us sum up what we have shown: No matter which choice of mth root we choose, if we continue it along a curve which encloses the origin, it will come back as a different root when we come back to the original point. This phenomenon is actually quite common in the study of functions of a complex variable, and the origin is what is called a *branch point* of the mth root function. Far from being a failure of the theory, it actually leads to very interesting new mathematical structures called *Riemann surfaces*, which we discuss momentarily.

It turns out that if we wish to define an mth root function, there are two distinct ways to proceed. First of all, we could restrict the domain by removing (say) a ray from the origin to infinity from the domain of the function; for example, if we remove the positive real axis together with the origin, it is clear that we may make any single choice of n and get a continuous mth root function on the remaining set. The same is true if we remove any other ray from the origin to infinity. In this setting, the ray we remove from the domain of f is termed a *branch cut*. See Goursat, §6, especially the discussion around Figure 5; see also some additional discussion in §8 herein, below.

Goursat's discussion of cutting the plane relates to the notion of a Riemann surface, which is part of the second possible route out of our difficulties, namely extending the domain to a so-called *m-sheeted cover* of the complex plane.[1]1 This is rather complicated and we shall only sketch it. The idea is to consider the point 1 on the real axis as distinct from the point obtained by rotating it around the origin once, twice, thrice, …, $m-1$ times, but as *the same* as what one gets by rotating m times.[1]2 This gives m different 'sheets' – in some sense, m different 'copies' of the complex plane – which are joined onto each other in some fashion (think of a spiral staircase which somehow ends up where it started); and we can then define the mth root function by choosing root n on the nth of the sheets.

[1]1 I don't suppose anyone has studied covering spaces, but in case anyone has, let me just note that this corresponds to the m-sheeted cover of the unit circle by itself. The universal cover of the unit circle will show up when we talk about the logarithm.

[1]2 If any of you have some familiarity with the notions of particle physics, you may recall that certain elementary particles, such as the electron, are said to have *spin-1/2*, in that they must be turned around *twice* to look the same (a most peculiar property!); that is exactly the same as what is going on here except that for mth roots we must 'turn around' m times to look the same. While it has been too long since I studied the Dirac equation to be sure of myself here, I doubt this is entirely a coincidence, as those of you who have studied the Dirac equation will probably recall that it arises as a *square-root* of the Klein-Gordon equation.

Summary:

- We clarify some matters related to branch cuts.
- We then fill in some points from the last set of lecture notes.
- Finally, we introduce power series and discuss how to extend the exponential and logarithm to complex numbers.

8. Roots and branch cuts, II. In the lecture notes from Tuesday, §7, we demonstrated that it is impossible to make a continuous choice of root on the entire complex plane, so that we either need to remove a part of the plane (make a *branch cut*) or embed the complex plane into a much larger set (the so-called *Riemann surface* of the function) in order to get a well-defined, continuous, single-valued function. In this section we will step back a bit to consider what all of this means, and why we are discussing it.

First of all, a philosophical point which will be useful to keep in mind at many other points in the course also. In mathematics there are some results or concepts which we study because they can be immediately used to solve problems, and there are other results or concepts which we study because they help deepen our understanding, even if they are not directly (or at least immediately) applicable to solving problems. In elementary calculus, for example, the product rule is of the first kind, as is the first derivative test; while the notion of a continuous function, or the extreme value theorem, are more of the second kind. In this class, methods for calculating residues, which we shall study later, are of the first kind; while branch cuts, which we are studying now, are of the second kind. We study them not so much because we need them immediately for applications, or because we can immediately solve problems about them, but because they help deepen our understanding of what an analytic function of a complex variable *is*, and how it might behave.[1]3

With this in mind, let us go back and investigate exactly *why* we needed a branch cut in the first place. The most immediate answer is that we needed a branch cut to make sure we could keep our function continuous and single-valued. Why did it become multiple-valued in the first place?

Let z be any nonzero complex number, and suppose that $z = r(\cos\theta + i\sin\theta)$ is a polar form of z. Then clearly so is $r\left[\cos(\theta+2\pi n) + \sin(\theta+2\pi n)\right]$. Now consider the following diagram; the block on the left is to be read top to bottom, then left to right, and we use the abbreviation $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$ (we will see very soon that $\operatorname{cis}\theta = e^{i\theta}$, of course):

$$\underbrace{\left.\begin{matrix} \cdots, & r\operatorname{cis}(\theta-2\pi m), & r\operatorname{cis}\theta, & r\operatorname{cis}(\theta+2\pi m), & \cdots \\ \cdots, & r\operatorname{cis}(\theta-2\pi(m-1)), & r\operatorname{cis}(\theta+2\pi), & r\operatorname{cis}(\theta+2\pi(m+1)), & \cdots \\ \cdots, & r\operatorname{cis}(\theta-2\pi(m-2)), & r\operatorname{cis}(\theta+4\pi), & r\operatorname{cis}(\theta+2\pi(m+2)), & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots, & r\operatorname{cis}(\theta-2\pi), & r\operatorname{cis}(\theta+2\pi(m-1)), & r\operatorname{cis}(\theta+2\pi(2m-1)), & \cdots \end{matrix}\right\}}_{z} \overset{z\mapsto z^{1/m}}{\longrightarrow} \left\{\begin{matrix} r^{\frac{1}{m}}\operatorname{cis}\frac{\theta}{m} \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi}{m} \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+4\pi}{m} \\ \vdots \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi(m-1)}{m} \end{matrix}\right.$$

where each quantity on the left is equal to z, and where each line on the left maps under the mth root function to a single value on the right. The issue is that while each of the quantities on the left is a polar representation of the *same* complex number z, the m quantities on the right represent *distinct* complex numbers – namely, the m possible mth roots of z. This diagram indicates one way of looking at the issue: the mth root function is most naturally considered as acting on the polar representation of a complex number z, but it takes representations of the *same* complex number to representations of *distinct* complex numbers. The point of a branch cut is to allow us to single out a preferred choice of polar representation for z in such a way that the resulting mth root is uniquely defined. (In terms of the above diagram, such a choice corresponds to picking a specific row.)

For example, suppose that we take our branch cut along the positive real axis: then we may require the angle in any polar representation of z to lie in the interval $(0, 2\pi)$. Now suppose that we are given the complex number $z = -1$. Since the point corresponding to this number makes an angle of π radians with the positive real axis, we can write it as $z = \operatorname{cis}\pi$. Now we could equally well write $z = \operatorname{cis}(2k+1)\pi$ for

[1]3 I read recently somewhere – unfortunately I have forgotten where – that functions of a complex variable are essentially defined by their singularities. Of the three kinds of singularities we shall see in this course, namely poles, essential singularities, and branch points, branch points are the hardest to deal with; in other words, as far as singularities are concerned anyway, things get simpler from here on out!

any integer k; but our choice of interval $(0, 2\pi)$ for the angle requires us to use $z = \operatorname{cis}\pi$. The mth root we get in this case is then $z^{1/m} = \operatorname{cis}\pi/m$.

It is not hard to find other examples; we give two just to demonstrate the point. Suppose that we choose the same branch cut but now require the angle to lie in the interval $(2\pi, 4\pi)$; there is no reason why we can't do this. Then the point $z = -1$ will be represented as $z = \operatorname{cis} 3\pi$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis} 3\pi/m$.

Finally, suppose that we choose a different ray as our branch cut, say the positive *imaginary* axis. Our possible choices of intervals are different now: instead of avoiding the positive real axis, which has angle 0, we now need to avoid the positive imaginary axis, which has angle $\pi/2$. Thus we may choose an interval of the form $(-3\pi/2, \pi/2)$ (for example). In this case, the polar representation of z will be $z = \operatorname{cis}(-\pi)$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis}(-\pi/m)$.

To sum up: a branch cut determines the possible different choices of representation for z, and a selection of one of these makes the root function (or whatever other function we happen to be studying) single-valued.

Before moving on, I would like to emphasise again that the point of learning about branch cuts at this point is not because we are going to use them right away to solve problems (though we will see that they do come up in practical problems later on in the course), nor is it because we are going to immediately be able to go off and determine where functions have branch points. (Another, more involved, example of branch cuts is however given in the second part of §6 of Goursat.) Rather it is to be given an introduction to a particular feature of certain functions of a complex variable which we shall study more later.

– See §§5 and 6 above –

9. Conjugate harmonic functions [continuing §5]. Recall that we have shown in §5 above that the Cauchy-Riemann equations imply that the real and imaginary parts of an analytic function f must satisfy Laplace's equation[14] $\Delta u = 0$. However, the Cauchy-Riemann equations have more information than just this, as they give also a relationship between the real and imaginary parts. Thus we may consider the following question: Suppose that $P(x, y)$ is a real-valued function of two real variables which satisfies Laplace's equation; is there a function $Q(x, y)$ of two real variables such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is an analytic function? It is not too hard to see that the answer is actually yes, at least if we stick to simply-connected regions. Let us write out the Cauchy-Riemann equations and see if we can solve them for Q:

$$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x}. \tag{1}$$

Probably the most direct way to treat these equations is to use a bit of vector calculus. Let us define a vector field

$$\mathbf{F} = -\frac{\partial P}{\partial y}\mathbf{i} + \frac{\partial P}{\partial x}\mathbf{j};$$

then since P is harmonic we have

$$\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right) = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0,$$

which means that, as long as we stick to simply-connected regions (recall that these are regions 'without holes'; generally these are introduced when one studies Green's theorem), there must be a function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. In other words, there must be a function f such that

$$\frac{\partial f}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial f}{\partial y} = \frac{\partial P}{\partial x}.$$

[14] At least, assuming that they have continuous second-order partial derivatives. We shall see shortly that if a function f is analytic throughout a region – as opposed to at a single point – then this condition is always satisfied. As far as I know, functions which are analytic at isolated points are of interest only as mathematical curiousities, and have no particular use in applications, so we shall not generally worry about them.

But these are exactly the equations we wanted Q to satisfy; in other words, what we know from vector calculus shows us that there must be a solution Q to the equations (1). It is unique up to an additive constant.

To be more specific, recall that we also know from vector calculus that the function f can be written as

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{x} + C,$$

where (x_0, y_0) is any point in the domain of P, the integral is a line integral along any path joining the two points (it will not depend on this path because $\operatorname{curl} \mathbf{F} = 0$ implies that $\mathbf{F}$ is conservative) and C is any constant. (In vector calculus, of course, we take C to be a real constant. Here C can be any complex constant.) This allows us to write

$$Q(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C,$$

and finally

$$f(x+iy) = P(x,y) + i \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C.$$

10. Power series. Let us recall a few facts about power series over the real numbers. A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k, \tag{2}$$

where $\{a_k\}$ is a sequence of coefficients, x_0 is some real number, and we consider x as a variable real number. The series will be *absolutely convergent* (meaning that the sum of the absolute values of its terms will be finite)[1]5 on some interval of the form $(x_0 - R, x_0 + R)$, called the *interval of convergence*, where $R > 0$ is called the *radius of convergence* and can be calculated from

$$\frac{1}{R} = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|,$$

when this limit exists.

Suppose now that we allow the numbers in the series in (2) to become complex. Now it turns out that, just as for real numbers, a series of complex numbers which is absolutely convergent is also convergent, so we may begin by asking where this series is absolutely convergent, which means that we must consider the series

$$\sum_{k=0}^{\infty} |a_k| \, |z - z_0|^k .$$

But this is just a power series of *real* numbers with coefficients $|a_k|$, and must therefore converge when $|z - z_0| < R$, where R is given as before. From this we can draw two conclusions:

1. Power series over the complex numbers converge in *discs*;
2. In the case that the coefficients a_k are all real, the radius of the disc of convergence is equal to the radius of the interval of convergence.

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[1]5 The notion of absolute convergence is very important in more theoretical parts of analysis. Since a series of positive terms converges if and only if it has an upper bound, and since in most spaces in which these concepts make sense – and in particular, for real and complex numbers – an absolutely convergent series is convergent, we are to reduce a question of convergence of a series – which is hard – to the question of finding an upper bound for a series, which is generally simpler. We shall probably not have much need to use these concepts and results directly, however.

Point 2 in particular makes the term *radius* of convergence much more sensible!

Just as with real power series, power series of complex numbers can be added, multiplied (though that becomes messy very quickly, as anyone who has attempted such a procedure can surely attest!), and differentiated term-by-term. This means, *inter alia*, that power series represent analytic functions where they converge. Also as with real power series, a power series converges inside its disc of convergence and diverges outside; on the boundary, as with real power series, it may converge or diverge, depending on the point and the situation.[1]6 Our main interest with power series right now is that they provide a convenient way to extend the elementary transcendental functions (the exponential, trigonometric, and logarithmic functions) to complex numbers, which we take up now.

11. Exponentials and logarithms of a complex variable. Recall that the exponential function e^x has the power series representation

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k,$$

and that this series converges for all real numbers x. By our discussion above, this shows that the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k,$$

where z is now a *complex* variable, must converge for all complex numbers z. It is clearly equal to e^x when $z = x$ is a real number. Now it can be shown (and we shall probably be able to show this in the second half of the course) that analytic functions are incredibly *rigid*: roughly, if they are equal on any set which is not somehow 'discrete', they must be equal everywhere. (We shall make this more precise later as it is not exactly true as it stands.)[1]7 This suggests that the above power series of complex numbers, which as we have seen defines a function which is analytic everywhere on the complex plane, is the *unique* function analytic everywhere on the complex plane which is equal to the ordinary exponential function on the real axis. We thus define, for any complex number z, the complex exponential

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

When convenient for typographical reasons we may write $\exp z$ instead of e^z. The standard properties of exponential functions can be shown to follow from this expansion; for example, if z_1 and z_2 are any complex numbers, we have

$$\begin{aligned} e^{z_1} e^{z_2} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} z_2^\ell \right) \\ &= \sum_{k,\ell=0}^{\infty} \frac{1}{k!\ell!} z_1^k z_2^\ell \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} z_1^k z_2^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n = e^{z_1+z_2}, \end{aligned}$$

[1]6 We note that it is possible to find a function which is analytic everywhere inside a disc but at no point of the boundary.

[1]7 For those who know enough topology to understand the following, we note that two analytic functions which agree on a set with at least one accumulation point must agree on the connected component of the intersection of their domains containing that set.

where in the third line we have introduced the variable $n = k + \ell$.

We know that on the real axis e^z agrees with the ordinary exponential function; what happens on the imaginary axis? Let $z = iy$; then we have

$$\begin{aligned} e^z = e^{iy} &= \sum_{k=0}^{\infty} \frac{1}{k!}(iy)^k \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(iy)^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(iy)^{2m+1} \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(-1)^{\ell} y^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i(-1)^m y^{2m+1} \\ &= \cos y + i \sin y, \end{aligned}$$

probably one of the most fascinating results in mathematics. This formula makes much of our work with powers and roots far more transparent: for example, the result

$$[r(\cos\theta + i\sin\theta)]^{1/m} = r^{1/m}(\cos\theta/m + i\sin\theta/m)$$

(where we have chosen just one particular mth root for simplicity) becomes now

$$\left[re^{i\theta}\right]^{1/m} = r^{1/m} e^{i\theta/m},$$

which is exactly what we would expect were the standard rules of exponents applicable to the complex exponential function.

Having now defined the exponential function for all complex numbers, we proceed to consider the logarithm. From what we have just seen, an arbitrary nonzero complex number z can be written in the form

$$z = re^{i\theta}$$

for some real number $r > 0$ ($r \neq 0$ since z is nonzero) and some real number θ. But since $r > 0$ we have

$$r = e^{\log r},$$

where here log represents the ordinary logarithm of positive real numbers; thus we can write

$$z = e^{\log r + i\theta}.$$

Now the defining property of the logarithm on real numbers is, that it is the inverse of the exponential function; if we wish to define the logarithm of a complex number the same way, the above formula suggests that we should define it to be $\log r + i\theta$. But here we run into the same problem we found when we discussed roots: θ is only defined up to an integer multiple of 2π. Thus for complex numbers we must evidently define the logarithm to be a multi-valued function. With this in mind, we define the logarithm of a nonzero complex number z, which we write $\operatorname{Log} z$, to be the collection of numbers

$$\operatorname{Log} z = \log r + i\theta,$$

where $r = |z|$ is the modulus of z and θ is any value of the argument of z. As with roots, this means that the logarithm has a branch point at the origin, and we must make a branch cut in order to get a single-valued continuous logarithm.

With these functions now defined, we may define exponents of any (nonzero) complex base and any complex power. First we recall that if $x_1 > 0$ and x_2 are two real numbers, we may write, by rules of exponents and logarithms (here log denotes the ordinary logarithm of positive real numbers)

$$e^{x_2 \log x_1} = e^{\log x_1^{x_2}} = x_1^{x_2}.$$

Now if we use the complex logarithm Log defined above, we can compute the left-hand side of the above equation for all *complex* numbers z_1 and z_2, as long as $z_1 \neq 0$. Thus, let $z_1 \neq 0$ and z_2 be two complex numbers; then we define

$$z_1^{z_2} = e^{z_2 \mathrm{Log}\, z_1}.$$

Note though that, since Log is multivalued, this definition in general makes $z_1^{z_2}$ a multivalued function as well. This leads to some rather amusing results. Let us give some examples.

EXAMPLES. 1. Before giving the amusing examples, let us first see how this definition fits in with the exponents we have already studied, namely integer powers and roots. If m is a positive integer and $z = r(\cos\theta + i\sin\theta)$ is any nonzero complex number, the above definition gives

$$z^m = e^{m\mathrm{Log}\, z} = \exp\left(m[\log r + i\theta]\right) = \exp\left(m\log r + im\theta\right) = e^{m\log r}e^{im\theta} = r^m(\cos m\theta + i\sin m\theta),$$

exactly in accord with our previous definition. Note that in this particular case the exponential function is *single*-valued, since if θ' is any other value of the argument of z, we would have $\theta' - \theta = 2\pi k$ for some integer k, and the above formula would give

$$r^m(\cos m\theta' + i\sin m\theta') = r^m(\cos m(2\pi k + \theta) + i\sin m(2\pi k + \theta)) = r^m(\cos m\theta + i\sin m\theta)$$

as before.

Let us now consider roots. Thus, again, let m be a positive integer and $z = r(\cos\theta + i\sin\theta)$ a nonzero complex number; then we have

$$z^{\frac{1}{m}} = \exp\left(\frac{1}{m}[\log r + i\theta]\right) = \exp\left(\frac{1}{m}\log r\right)\exp\left(i\frac{\theta}{m}\right) = r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right),$$

exactly in accord with our original definition of mth roots. Recall that here θ represents *any* possible argument value for z, so that this expression represents all possible mth roots and is, as usual, multivalued for $m \neq 1$.

More generally, if $z' = \frac{k}{m}$ where k and m are relatively prime integers (meaning that they have no common divisors; this restriction is for convenience only), then we have for any complex number $z = re^{i\theta}$

$$z^{z'} = z^{k/m} = \exp\left(\frac{k}{m}[\log r + i\theta]\right) = \exp\left(\frac{k}{m}\log r + i\frac{k\theta}{m}\right) = r^{k/m}\left(\cos\frac{k\theta}{m} + \sin\frac{k\theta}{m}\right).$$

2. Now for some amusing examples. Let us recall that the exponential for real numbers is only defined for *positive* bases. We now have a means of defining it for arbitrary *complex* bases, but in particular for *negative* real bases; what does it give us? In particular, what is say -1 raised to an irrational power, say $\sqrt{2}$? To find this, we write $-1 = \cos(2n+1)\pi = e^{(2n+1)\pi i}$, where n is any integer; then we have

$$\begin{aligned} -1^{\sqrt{2}} &= \exp\left(\sqrt{2}\,\mathrm{Log}\,(-1)\right) = \exp\left(\sqrt{2}(2n+1)\pi i\right) \\ &= \cos\left(\sqrt{2}(2n+1)\pi\right) + i\sin\left(\sqrt{2}(2n+1)\pi\right). \end{aligned}$$

What does this set of numbers look like? It turns out that this set is actually *infinite*; this is because $\sqrt{2}$ is irrational: if the set were finite, we would have integers $n \neq m$ and k such that

$$\sqrt{2}(2n+1)\pi = \sqrt{2}(2m+1)\pi + 2k\pi,$$

which would give $\sqrt{2} = \frac{k}{n-m}$, contradicting irrationality of $\sqrt{2}$. It is also clear that all of these numbers lie on the unit circle; thus we have an infinite set of numbers on the unit circle, which means that they cannot be 'evenly spaced' in any meaningful sense. (For those who are familiar with the concept of density, we note that this set is in fact *dense* in the unit circle.)

Even more bizarre things happen when we look at *complex* bases. For example, let us consider i^i. Writing $i = \exp i\left(\frac{\pi}{2} + 2n\pi\right)$, we have

$$i^i = \exp\left(i\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right) = \exp\left(-\frac{\pi}{2} - 2n\pi\right),$$

i.e., the number i^i is an infinite sequence of *real* numbers!

(We hasten to note that these examples are more amusing than indicative, and while it is important to keep in mind that exponentials like $z_1^{z_2}$ can be very ill-behaved compared with their real counterparts, this behaviour will not generally concern us in the remainder of the course.)

Summary:

- We discuss the branches of the logarithm function defined previously and show how to differentiate them.
- We introduce the extension of the trigonometric functions to the complex plane, and relate them to the ordinary trigonometric and hyperbolic trigonometric functions of a real variable.
- We show how the inverse trigonometric functions can be determined in terms of roots and logarithms, and calculate their derivatives.
- Finally, we give a slightly more careful description of the kind of region we assume our functions are defined; then we give an introduction to *conformal mappings* and show that analytic functions are conformal.

12. Differentiation of Log. Recall that we have defined the complex logarithm as a multi-valued function as follows. If z is any nonzero complex number and $re^{i\theta}$ is any polar representation of z, then we define

$$\mathrm{Log}\, z = \log r + i(\theta + 2n\pi), \qquad n \in \mathbf{Z},$$

where here log denotes the ordinary real logarithm of a positive real number. (Note that this definition allows us to extend the logarithm to negative real numbers but *not* to zero. Since even over the complex plane the exponential is never 0, there is no way to extend the logarithm to zero.) As for the root functions we studied previously, a single-valued, continuous logarithm can only be defined on a cut plane. Let us see how this works in practice. Suppose that we cut the plane along the ray $\theta = \theta_0$, i.e., that we define the logarithm only on complex numbers with polar representation $z = re^{i\theta}$ where $\theta \in (\theta_0, \theta_0 + 2\pi)$, and that we consider only this polar representation in defining the logarithm. (Note that, while related, these are two distinct points.) Then we have

$$\mathrm{Log}\, z = \log r + i\theta.$$

We note that this function is continuous on the cut plane; an outline of a proof is given in the appendix. Some examples related to this are given in the problem set.

Let us now see whether these branches of Log are analytic functions. Specifically, let us take the above branch, obtained by cutting the plane along $\theta = \theta_0$. We shall denote this particular branch by $\mathrm{Log}\, z$ in the following, for convenience. We must determine whether the limit

$$\lim_{h \to 0} \frac{\mathrm{Log}\,(z+h) - \mathrm{Log}\,(z)}{h}$$

exists. This limit may clearly be written as

$$\lim_{z' \to z} \frac{\mathrm{Log}\, z' - \mathrm{Log}\, z}{z' - z}.$$

Now if $z = re^{i\theta}$, where $\theta \in (\theta_0, \theta_0 + 2\pi)$, then as long as z' is close enough to z[18] we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and also θ' is close to θ. Let us now define

$$w = \mathrm{Log}\, z = \log r + i\theta, \qquad w' = \mathrm{Log}\, z' = \log r' + i\theta'.$$

Then

$$\frac{\mathrm{Log}\, z' - \mathrm{Log}\, z}{z' - z} = \frac{w' - w}{e^{w'} - e^{w}}.$$

Now as $z' \to z$, we have clearly (by continuity of the logarithm) $\mathrm{Log}\, z' \to \mathrm{Log}\, z$, i.e., $w' \to w$; and in this limit the above fraction becomes

$$\lim_{w' \to w} \frac{w' - w}{e^{w'} - e^{w}} = \lim_{w' \to w} \frac{1}{\frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{\lim_{w' \to w} \frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{e^{w}},$$

[18] Specifically, we need the angle between them to be less than the smaller of $\theta - \theta_0$ and $\theta_0 + 2\pi - \theta$.

since the exponential function is analytic and is equal to its own derivative. But recall that

$$e^w = e^{\operatorname{Log} z} = z,$$

so that we have shown that

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Note that this final result does not depend on the choice of branch cut; in other words, each branch of Log has the same derivative. This accords with what we know about derivatives from ordinary calculus, since the various branches of Log differ only by constants.

To sum up, we have shown that each branch of Log is an analytic function on its domain, and all of the branches have the same derivative, namely $1/z$.

Appendix I. Continuity of Log. Let us show that each branch of the logarithm, as outlined at the start of the section above, is in fact continuous. We shall give a formal ϵ-δ argument, but provide intuitive commentary to hopefully make the ideas clear to those who do not have much background in such things. Thus let $z = re^{i\theta}$ be an element of the cut plane, with $\theta \in (\theta_0, \theta_0 + 2\pi)$, and let $\epsilon > 0$. We may assume that $\epsilon < \frac{\pi}{4}$. Since log is continuous on the positive real line, there must be a $\delta' > 0$ such that

$$|\log r - \log r'| < \frac{1}{2}\epsilon \qquad \text{if} \qquad |r - r'| < \delta';$$

in other words, if r' is close to r then $\log r'$ is close to $\log r$. Further, it can be shown that the function $z \mapsto |z|$ is continuous; thus there is a $\delta'' > 0$ such that

$$||z| - |z'|| < \delta' \qquad \text{if} \qquad |z - z'| < \delta'';$$

in other words, $|z|$ is close to $|z'|$ if z is close to z' (clearly a reasonable statement geometrically!). Dealing with the angular part of z and z' is slightly messy; intuitively though the result is clear: if z' is sufficiently close to z, then we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and θ' is close to θ. To prove what we need carefully, though, let us set

$$\delta''' = \begin{cases} \frac{1}{2} r \sin(\theta - \theta_0), & \theta \in (\theta_0, \theta_0 + \pi/2) \cup (\theta_0 + 3\pi/2, \theta_0 + 2\pi), \\ \frac{1}{2} r, & \text{otherwise.} \end{cases}$$

Since $2\delta'''$ is simply the distance from z to the cut (draw a picture!), it is clear that $|z - z'| < \delta'''$ means that z' is on the same side of the cut as z, and hence can be written in the above form. Now let δ be the smaller of δ', δ'', δ''', and $\sin(\epsilon/2)$, and suppose that

$$|z - z'| < \delta.$$

By the foregoing, then,

$$||z| - |z'|| < \delta', \qquad \text{so} \qquad |\log|z| - \log|z'|| < \frac{1}{2}\epsilon;$$

furthermore, writing $z' = r'e^{i\theta'}$, $\theta' \in (\theta_0, \theta_0 + 2\pi)$, it is clear geometrically (again, draw a picture!) that the angle between z and z' is no greater than $\arcsin \delta$, which is bounded by $\epsilon/2$, so that $|\theta - \theta'| < \epsilon/2$. Thus finally

$$|\operatorname{Log} z - \operatorname{Log} z'| = |\log r + i\theta - \log r' + i\theta'| \le |\log|z| - \log|z'|| + |\theta - \theta'| < \epsilon,$$

proving continuity of Log, as desired.

13. Trigonometric functions. To extend the trigonometric functions to the complex plane, we shall proceed in the same way we did with the exponential function. Recall that on the real line we have the power series expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k x^{2k+1}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k x^{2k}.$$

Since the radius of convergence of both of these series is infinite, they must converge on the entire complex plane as well; thus we may define

$$\sin z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1}, \qquad \cos z = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k},$$

where now z is any complex number. Moreover, as we mentioned in our discussion of the exponential function in section 11 above, these power series are the *unique* way of extending sin and cos to the complex plane as analytic functions.

The standard identities of trigonometry can be shown to hold over the complex numbers as well; in particular, we have

$$\cos^2 a + \sin^2 a = 1,$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b, \qquad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$
$$\sin 2a = 2 \sin a \cos a, \qquad \cos 2a = \cos^2 a - \sin^2 a,$$

and so forth, where now a and b can be any complex numbers. Moreover, sin is odd ($\sin(-z) = -\sin z$) while cos is even ($\cos(-z) = \cos z$), as with real numbers. Further, the differentiation formulæ for sin and cos also hold. This can be shown by differentiating the above series:[1]9

$$\frac{d}{dz}\sin z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \cos z,$$
$$\frac{d}{dz}\cos z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!}(-1)^k z^{2k-1} = -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = -\sin z,$$

where we have set the lower index to 1 in the second series on the second line since the constant term in the series for $\cos z$ differentiates to zero, and we have adjusted the index in the last equality.

Now recall that, by substituting in to the power series expression for e^z, we found that when y is real

$$e^{iy} = \cos y + i \sin y.$$

Now there is nothing in this derivation which requires y to be a real number; thus with the above definitions for sin and cos, we find that for all *complex* numbers z that

$$e^{iz} = \cos z + i \sin z.$$

Using the fact that cos is odd and sin is even, we see that

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z;$$

adding and subtracting these two equations, we obtain the results

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

This allows us to derive expressions for the real and imaginary parts of $\cos z$ and $\sin z$. First of all, note that if y is real (actually for all complex y if we define cosh and sinh in the usual way, but we are only interested in real y for the moment)

$$\cos iy = \frac{e^{-y} + e^{y}}{2} = \cosh y, \qquad \sin iy = \frac{e^{-y} - e^{y}}{2i} = i \sinh y,$$

[1]9 As noted previously, convergent power series can be differentiated term-by-term on their discs of convergence.

where as usual

$$\cosh y = \frac{e^y + e^{-y}}{2}, \qquad \sinh y = \frac{e^y + e^{-y}}{2}.$$

Thus if $z = x + iy$,

$$\begin{aligned}
\cos z &= \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y, \\
\sin z &= \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y.
\end{aligned}$$

Now since cosh and sinh are unbounded, this means in particular that cos and sin are unbounded along the imaginary direction. In particular, the inequalities $|\cos x| \leq 1$, $|\sin x| \leq 1$, which are true for real x, do *not* hold for complex numbers.

Similar results can be derived for the other trigonometric functions (tangent, cotangent, secant, and cosecant) but we shall not go into that here.

14. Inverse trigonometric functions. Let us see what we can find about the inverse trigonometric functions, given the foregoing. Let us first consider $\sin z$; or, since we are interested in finding its inverse, $\sin w$, where w is another complex variable. We have the relation

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Now let us set $z = \sin w$ and see whether we can solve for w. We have

$$\begin{aligned}
\frac{e^{iw} - e^{-iw}}{2i} &= z \\
e^{iw} - e^{-iw} &= 2iz \\
e^{2iw} - 1 &= 2ize^{iw} \\
e^{2iw} - 2ize^{iw} - 1 &= 0 \\
e^{iw} &= \frac{1}{2}\left(2iz + \left(4(iz)^2 + 4\right)^{1/2}\right) \\
&= iz + \left(1 - z^2\right)^{1/2},
\end{aligned}$$

where we have dispensed with the $\pm$ usually present in the quadratic formula since $\left(1 - z^2\right)^{1/2}$ is defined to mean both square roots. Thus we may write

$$w = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

In other words, whenever w is any of the (infinitely many) complex numbers indicated by the right-hand side of this equation, we must have $\sin w = z$. We thus define

$$\arcsin z = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

Note that there are, in general, *two* distinct sources of multi-valuedness in the above expression, one from the square root (when $z \neq \pm 1$) and the other from the log. This is in good accord with our understanding of the graph of $\sin x$ on the real line: as long as $y_0 \neq \pm 1$, the graph of $y = \sin x$ will intersect the line $y = y_0$ twice per interval of length 2π.

Similar expressions can be derived for arccos and arctan but we pass over them for the moment.

The above expression may be differentiated, assuming that we are using appropriate branches:

$$\begin{aligned}
\frac{d}{dz}\frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right] &= \frac{1}{i}\frac{1}{iz + (1-z^2)^{1/2}}\left(i - \frac{z}{(1-z^2)^{1/2}}\right) \\
&= \frac{1}{iz + (1-z^2)^{1/2}}\frac{\left(1-z^2\right)^{1/2} + iz}{(1-z^2)^{1/2}} = \frac{1}{(1-z^2)^{1/2}},
\end{aligned}$$

in accord with what we know from real-variable calculus (except recall that here the square root means *both* square roots, i.e., it has a sign ambiguity).

15. Regions; conformal mappings. We have mentioned that we are principally interested in functions which are analytic in some *region*, rather than at a single point. We have however not defined what kind of region we are interested in. We are interested in the first place in functions which are analytic everywhere inside a so-called *simple closed curve*, i.e., a closed curve which does not intersect itself; such a region is *simply-connected* in the sense in which that word is typically used in discussions of Green's theorem, namely, it does not have any *holes*.[20] Later we shall also consider functions which are analytic on a set which has a finite number of holes, i.e., whose boundary is a finite number of simple closed curves, which moreover do not intersect each other. Whenever we speak of an analytic function, we are assuming that the function is analytic throughout a region of this form.

We shall now introduce so-called *conformal mappings*. It will turn out that all analytic functions on the complex plane are conformal mappings whenever they have nonzero derivative, but the definition of a conformal mapping does not require any use of complex numbers. A map

$$f : \mathbf{R}^2 \to \mathbf{R}^2$$

is said to be *conformal* at a point p when it preserves angles at that point; in other words, it $\gamma_1(t)$ and $\gamma_2(t)$ are any two curves which intersect at p, which for convenience and without loss of generality we may take to be $t = 0$ for both curves, then the angle between $\gamma_1(t)$ and $\gamma_2(t)$ at $t = 0$ is equal to the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $t = 0$, in both magnitude and sign (i.e., we measure it in the same direction, either clockwise or counterclockwise).[21] (See figures $9a$ and $9b$ in Goursat for an illustration.) Note that, in general, a map must be at least differentiable (in the sense of real functions on the plane!) for the angle of the image curves to make sense. Some examples immediately come to mind.

EXAMPLES. 1. Since translations and rotations of the plane preserve distances, they also preserve angles, and hence give conformal transformations.

2. So-called *isotropic scalings* of the plane, i.e., maps

$$(x, y) \mapsto (ax, ay),$$

where $a = 0$, are also conformal maps. This will follow from our general result below.

The main application we shall make of conformal mappings is to find solutions of Laplace's equation, which we shall take up probably in the second half of the course. The main example of conformal maps for us is given by the following result:

If f is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.

This may be shown as follows. (Here we first give the derivation given in the lecture, and supplement it to fill in a hole; we follow this with a slightly more concise demonstration.) For convenience we treat complex numbers as though they were their corresponding points in the plane. Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth curves which satisfy $\gamma_1(0) = \gamma_2(0) = z_0$. Then they have tangent vectors there

$$\mathbf{T}_1 = \gamma_1'(0), \qquad \mathbf{T}_2 = \gamma_2'(0),$$

and hence make an angle θ which satisfies

$$\cos\theta = \frac{\mathbf{T}_1 \bullet \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|},$$

[20] For those who have seen something of general topology, the main point is that we are interested in functions which are analytic on some connected, simply-connected open set.

[21] For those of you who know something of modern differential geometry, the curves $\gamma_1(t)$ and $\gamma_2(t)$ here are being used as proxies for tangent vectors.

where $\bullet$ denotes the dot product. Now since f is analytic, it is in particular differentiable (in the real-variable sense) as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, and thus the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are also smooth; moreover they have tangent vectors

$$\mathbf{S}_1 = f'(z_0) \cdot \gamma_1'(0), \qquad \mathbf{S}_2 = f'(z_0) \cdot \gamma_2'(0),$$

where we treat γ_1 and γ_2 as though they were complex-valued, and $\cdot$ denotes multiplication of complex numbers. (The foregoing is a simple extension of the chain rule.) Thus the angle between these image curves, say θ', satisfies

$$\cos\theta' = \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|}.$$

Now recall (see the first example in §2, notes of May 5, above) that if z and w are any two complex numbers, then the dot product of the vectors corresponding to z and w is equal to $\operatorname{Re}\overline{z}w$. Thus we may compute as follows:

$$\begin{aligned}\mathbf{S}_1 \bullet \mathbf{S}_2 &= \operatorname{Re}\overline{f'(z_0)\mathbf{T}_1} f'(z_0)\mathbf{T}_2 = \operatorname{Re}\overline{f'(z_0)} f'(z_0)\overline{\mathbf{T}_1}\mathbf{T}_2 \\ &= |f'(z_0)|^2 \operatorname{Re}\overline{\mathbf{T}_1}\mathbf{T}_2 = |f'(z_0)|^2 \mathbf{T}_1 \cdot \mathbf{T}_2.\end{aligned}$$

Since $|\mathbf{S}_1|$ can be computed in terms of a dot product, we see that

$$\begin{aligned}\cos\theta' &= \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|} = \frac{|f'(z_0)|^2 \mathbf{T}_1 \cdot \mathbf{T}_2}{|f'(z_0)|\,|\mathbf{T}_1|\,|f'(z_0)|\,|\mathbf{T}_2|} \\ &= \frac{\mathbf{T}_1 \cdot \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|} = \cos\theta.\end{aligned}$$

This shows that θ and θ' have the same cosine. However this of course does not mean that they are equal. (This point was not mentioned in the lecture.) To show that they are actually equal, we recall also that if z and w are any two complex numbers, the *cross product* (more carefully, the $\mathbf{k}$ component of the cross product) of z and w is equal to $\operatorname{Im}\overline{z}w$. Now recall from vector calculus that the cross product in this case is also given by $|z||w|\sin\phi$, where ϕ is the angle between the vectors corresponding to z and w. The foregoing calculation shows, replacing Re by Im everywhere, that we must have $\sin\theta = \sin\theta'$. Since two angles which have the same sine and cosine must be equal up to some integer multiple of 2π, and this means for our purposes that they are the same angle, this shows that f must be conformal at z_0, as claimed.

A slightly more concise demonstration may be given as follows. (Those of you who are familiar with derivatives considered as linear maps can skip straight to the appendix where an even more concise proof is given.) Let $t > 0$ be small. Then the tangent vectors to γ_1 and γ_2 at $t = 0$, i.e., at z_0, can be approximated by

$$\frac{\gamma_1(t) - z_0}{t}, \qquad \frac{\gamma_2(t) - z_0}{t}.$$

Similarly, the tangent vectors to $f(\gamma_1(t))$ and $f(\gamma_2(t))$ can be approximated by

$$\frac{f(\gamma_1(t)) - f(z_0)}{t}, \qquad \frac{f(\gamma_2(t)) - f(z_0)}{t}.$$

Now for z near z_0 we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0),$$

where $o(z - z_0)$ denotes a quantity which vanishes faster than $z - z_0$ as the latter goes to zero; i.e.,

$$\lim_{z \to z_0} \frac{o(z - z_0)}{z - z_0} = 0.$$

Thus we have

$$f(\gamma_k(t)) = f(z_0) + f'(z_0)(\gamma_k(t) - z_0) + o(\gamma_k(t) - z_0),$$

so

$$\frac{f(\gamma_k(t)) - f(z_0)}{t} = f'(z_0)\frac{\gamma_k(t) - z_0}{t} + \frac{o(\gamma_k(t) - z_0)}{t}.$$

Now in the limit $t \to 0$ we have similarly $\gamma_k(t) = \gamma_k(0) + \gamma_k'(0)t + o(t) = z_0 + \gamma_k'(0)t + o(t)$, so that in this limit the last quantity on the right-hand side above vanishes and we find that the tangent vector to the curves $\gamma_k(t)$ are given by

$$f'(z_0)\gamma_k'(0),$$

where as before the multiplication is to be considered as multiplication of complex numbers. Now suppose that we have

$$\gamma_k'(0) = r_k e^{i\theta_k},$$

and that

$$f'(z_0) = re^{i\theta};$$

then the tangent vectors to the image curves are given by

$$f'(z_0)\gamma_k'(0) = rr_k e^{i(\theta_k+\theta)};$$

in other words, the effect of an analytic map f on tangent vectors to smooth curves is to scale and rotate, which clearly preserves angles. This shows that f is conformal at z_0, as claimed.

Appendix I. Abstract derivation. Let us consider f as a map of the real plane. Then its derivative $f'(z_0)$ is a linear map from the plane to itself which satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|),$$

where here $f'(z_0)$ is considered as a linear map and $z - z_0$ as a vector, and the 'product' above is the application of this linear map to this vector. Evidently, $f'(z_0)$ may be considered to be multiplication by the complex derivative also denoted $f'(z_0)$. Now abstractly the derivative as a linear map takes tangent vectors to tangent vectors; in other words, two tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ (say) at the point z_0 are taken by the map f to the vectors $f'(z_0)\mathbf{T}_1$ and $f'(z_0)\mathbf{T}_2$. By the discussion in the last few lines of the section above, the angle between these vectors must be that between $\mathbf{T}_1$ and $\mathbf{T}_2$.

(I admit that this is a little bit hand-wavy. The reason for this is that the definition of 'conformal' given above is somewhat informal. The argument just given can be made entirely rigorous if we define 'preserves angles at a point' to mean that its derivative preserves angles as a map of tangent vectors, which is more or less equivalent to the definition in terms of curves given above.)

Summary:

- We continue to discuss conformal mappings and expand on a couple examples from Goursat.

16. Examples of conformal maps. (a) (See Goursat, §19, Example 2.) Consider the map on the punctured plane $\mathbf{R}^2\backslash\{(0,0)\}$ which is given in complex notation by

$$f(z) = \frac{1}{z}.$$

Since this function is analytic on the punctured plane, it must be conformal at every point other than the origin. Let us consider how it behaves with respect to the unit circle. We have the following properties:

$$\begin{aligned}
\text{If } |z| = 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} = 1. \\
\text{If } |z| > 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} < 1. \\
\text{If } |z| < 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} > 1.
\end{aligned}$$

This means that the map f takes the unit circle to itself, while it takes the region *outside* the unit circle to the region *inside* the unit circle, and vice versa. See Fig. 1. It is worth noting that on the unit circle

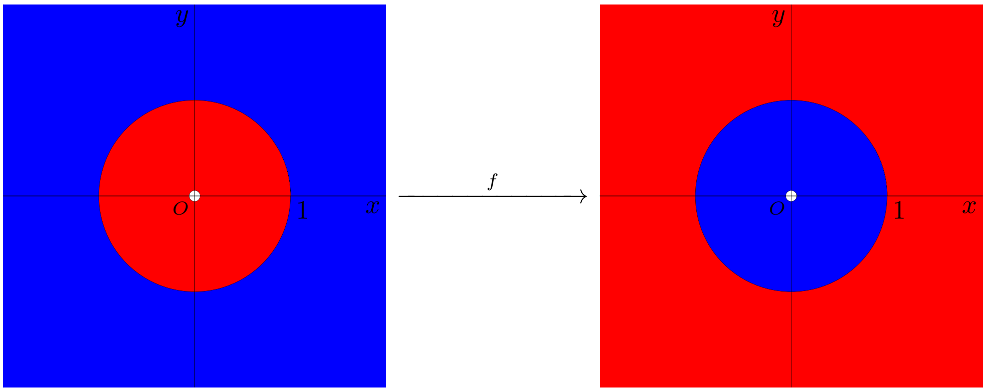


FIG. 1

$$f(z) = \frac{1}{z} = \overline{z}.$$

Note though that $\overline{z}$ is *not* an analytic function in general! It does turn out to be (almost) conformal though (it preserves magnitudes of angles but reverses their sense); and it can be shown (see §21 of Goursat, noting that replacing Q by $-Q$ is equivalent to taking the complex conjugate of f) that every sufficiently smooth conformal map is either an analytic function or the conjugate of an analytic function (which is the same thing as an analytic function of $\overline{z}$, as is apparent if one thinks of a Taylor expansion: but that is a bit beyond what we have technically covered so far).

(b) (See Goursat, §22, Example 2.) Let us now consider the function on the entire plane given in complex notation by

$$f(x+iy) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

This function is analytic everywhere, and will be conformal everywhere that its derivative is nonzero. (We pause for a moment to clarify a point which the author fumbled during lecture. The derivative of $\cos z$ is $-\sin z$, which means that $\cos z$ will be conformal at every point where $\sin z$ is nonzero. Now

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y;$$

and for this to be zero, we see first of all that we must have $\sin x = 0$ (since $\cosh y \geq 1$ for all real y), and since this means that $\cos x \neq 0$, we must have $\sinh y = 0$, or $y = 0$. Thus the zeros of $\sin z$ over the complex plane are the same as those of $\sin x$ over the real line, i.e., $n\pi$, $n \in \mathbf{Z}$.) Thus f will be conformal at every point inside the strip $\{x+iy|0<x<\pi, y>0\}$. Let us consider how f maps straight lines within this strip. Let us consider first a horizontal line, say $y = y_0 > 0$. On such a line, f is equal to

$$f(x+iy_0) = \cos x \cosh y_0 - i \sin x \sinh y_0,$$

where $x \in (0,\pi)$. Now this is just another way of writing the parametric curve

$$t \mapsto (\cosh y_0 \cos t, -\sinh y_0 \sin t), \qquad t \in (0,\pi).$$

If we denote this curve by $(x(t), y(t))$ (where unfortunately here $x(t)$ and $y(t)$ are completely distinct from the real and imaginary parts of z), then we have

$$\left(\frac{x(t)}{\cosh y_0}\right)^2 + \left(\frac{y(t)}{\sinh y_0}\right)^2 = 1,$$

i.e., the curve must lie on an ellipse with major axis $\cosh y_0$ along the horizontal axis and minor axis $\sinh y_0$ along the vertical axis, and centred at the origin. Now since $y_0 > 0$, $\sinh y_0 > 0$, so $-\sinh y_0 < 0$ and $y(t) < 0$ for all $t \in (0,\pi)$, while $x(t)$ takes on all values from $\cosh y_0$ to $-\cosh y_0$. Thus we obtain the lower half of this ellipse.

Now let us consider a vertical line, say $x = x_0 \in (0,\pi)$. Working as before, we see that on this line

$$f(x_0+iy) = \cos x_0 \cosh y - i \sin x_0 \sinh y.$$

If $x_0 = \pi/2$ then $\cos x_0 = 0$ and this is simply a parametrisation of the negative imaginary axis. Otherwise, we again write

$$(x(t), y(t)) = (\cos x_0 \cosh t, -i \sin x_0 \sinh t), \qquad t \in (0,\pi)$$

and note that (this follows from the basic identity $\cosh^2 x - \sinh^2 x = 1$)

$$\left(\frac{x(t)}{\cos x_0}\right)^2 - \left(\frac{y(t)}{\sin x_0}\right)^2 = 1,$$

which means that the curve lies on a hyperbola opening along the real axis with intercept $\pm\cos x_0$ and with asymptotes having slope $\pm\tan x_0$. Now we note that $y(t) < 0$ for all t, while $x(t) > 0$ for $t \in (0,\pi/2)$ and $x(t) < 0$ for $t \in (\pi/2,\pi)$; thus in the first case we have the lower right-hand portion of the hyperbola, while in the second case we have the lower left-hand portion. See Fig. 2. Note especially how the blue and red

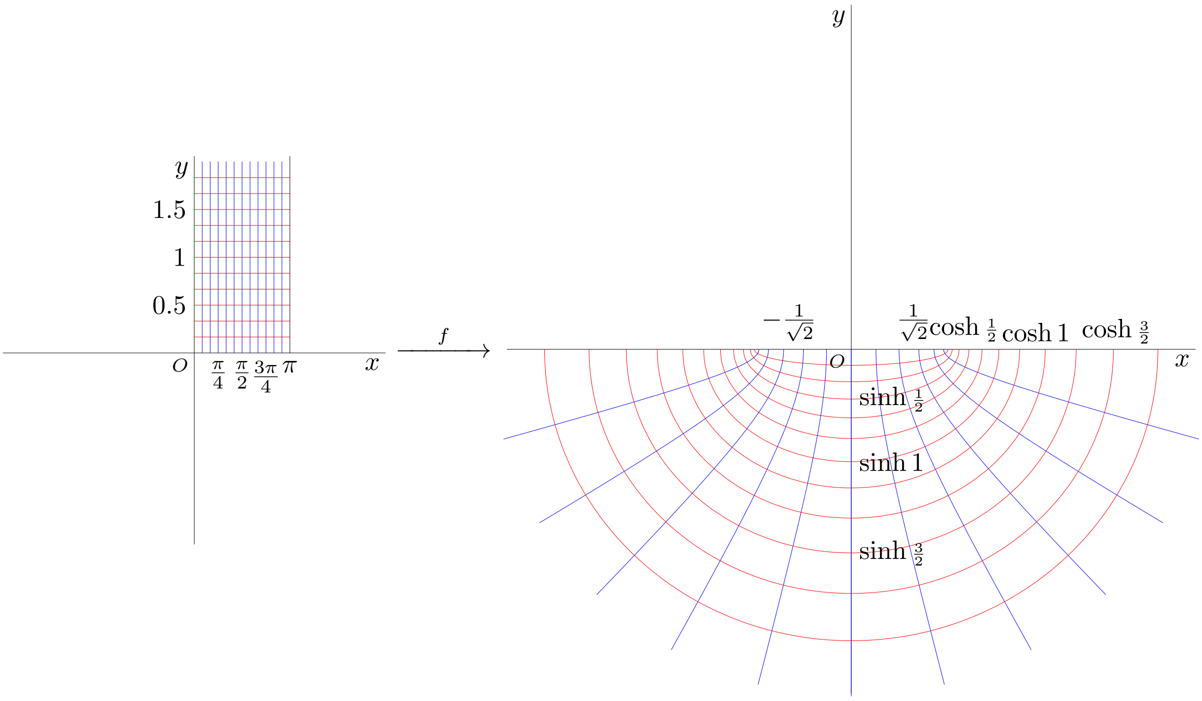


FIG. 2

curves on the right intersect at right angles, exactly like those on the left.

Summary:
- We outline a particular application of conformal maps.
- We then define and investigate integrals of complex functions over curves in the complex plane.

(Goursat, §§24, 25 – 26, 32.)

17. Application of conformal maps to harmonic functions. In fields as varied as electrostatics, heat flow, and fluid mechanics (and probably others) one is often interested in solving problems of the following form: we are given a particular region U in the plane[2]2 with boundary curve C, and some particular function g on the boundary curve C, and we wish to find a function P on U which satisfies

$$\Delta P = 0 \,\text{on}\, U, \qquad P|_C = g.$$

This problem in full generality is a topic for a course in partial differential equations, but there are specific cases which can be treated by using complex variable techniques to replace the region U by another one for which the problem is more tractable. Let us see how this works. (Here we shall simply outline the idea; we shall go over it in more detail later on in the course. Thus what follows is meant to be more of a conceptual introduction than a careful exposition.) Suppose that we have another region U' with boundary curve C' and a conformal map $f : U' \to U$ which maps C' onto C and is also analytic with an analytic inverse $f^{-1} : U \to U'$ (I accidentally forgot about this restriction in the lecture; there are probably ways of treating the problem without it, but we shall leave a detailed discussion of the matter for another time). Then we may consider instead the problem

$$\Delta P' = 0 \,\text{on}\, U', \qquad P'|_{C'} = g \circ f.$$

For a suitable choice of U' and f, this problem may be easier to solve than the original one. Suppose that we are able to find a solution to this modified problem. Then we claim that $P' \circ f^{-1}$ is a solution to the original problem. To see this, let $z_0 = x_0 + iy_0 \in U'$ be some particular point and define

$$Q' = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P'}{\partial y}\, dx + \frac{\partial P'}{\partial x}\, dy,$$

i.e., the conjugate harmonic function to P'; then the function

$$F(x + iy) = P'(x, y) + iQ'(x, y)$$

will be analytic, at least on any simply connected subset of U' containing (x_0, y_0). Since f^{-1} is also analytic, this means that $F \circ f^{-1}$ is also analytic, and hence that its real part

$$P' \circ f^{-1}$$

is harmonic.[2]3 Further, on C

$$\left(P' \circ f^{-1}\right)\big|_C = P'|_{C'} \circ f^{-1} = g \circ f \circ f^{-1} = g,$$

so that the boundary condition is satisfied as well. Thus $P' \circ f^{-1}$ is indeed a solution to the original problem, as claimed.

(In the lecture I actually showed the *opposite* implication, namely that if P is a solution to the original problem, then $P \circ f$ is a solution to the modified problem. The argument is identical to that here, replacing f^{-1} by f and P' by P as appropriate.)

[2]2 While not strictly necessary, we can assume that U is simply connected below to avoid some technical complications which are not important at the moment.

[2]3 Note that we can afford to be vague about the region here since the property of being analytic and – especially – harmonic is really a pointwise property; or if we want to be a bit more careful, it is a property we only need to consider on small disks, which are always simply connected.

Appendix I. Formal definition of simple connectedness. Informally, a region which is simply connected is one which has no 'holes'. As mentioned in lecture, if the region is given to us pictorially this is about all we could go on to determine whether it is simply connected. More carefully, though, a region is simply connected if any closed curve can be continuously shrunk to a point within the region. But what do we mean by 'continuously shrunk to a point within the region'?

The precise definition of simply connected, which is valid in any topological space, is as follows. A set U is said to be *simply connected* if for any continuous closed curve $\gamma : [0,1] \to U$ (i.e., γ is continuous and satisfies $\gamma(0) = \gamma(1)$) there is a continuous map $F : [0,1] \times [0,1] \to U$ satisfying the following conditions:

$$\begin{aligned} &F(\cdot, s) : [0,1] \to U \text{ is a closed curve in } U \text{ for each } s \in [0,1] \\ &F(t, 0) = \gamma(t) \text{ for all } t \in [0,1] \\ &F(t, 1) = u_0 \text{ for all } t \in [0,1] \end{aligned}$$

where $F(\cdot, s)$ means the map $t \mapsto F(t, s)$, and $u_0 \in U$ is some point. If we unwrap this definition a bit, what it means is that $F(t, s)$ is a family of continuous, closed curves, where the curves are paremeterised by t and the family by s, such that the first curve in the family (when $s = 0$) is the original curve γ and the final 'curve' in the family (when $s = 1$) is a single point. More informally, F shows us specifically how to continuously deform γ to a single point within the region.

(It is probably worth pointing out here that the definition of simply connected as meaning 'without holes' only works in two dimensions. If we consider a ball, for example, and remove a single point, the resulting set clearly has a whole, but it is also clearly possible to shrink any continuous curve to a point regardless of the hole. (Think about it for a bit if it isn't clear!) These 'higher-dimensional' holes lead to so-called 'higher homotopy groups', or, more tractably, to homology theory – which I think actually originated in the study of functions of a complex variable!)

18. Complex integration. We now enter into one of the core parts of the course, the notion of *contour integrals* in the complex plane. We shall introduce these in the same way as done in Goursat (§25) and then show how they may be computed by reducing to the line integrals one studies in multivariable calculus.

Recall that in one-variable calculus we define the definite integral of a function f between points a and b more or less as follows:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k,$$

where $\Delta x_k = x_k - x_{k-1}$ and $x_k^* \in [x_{k-1}, x_k]$.[2]4 Now when we defined derivatives of functions on the complex plane, we were able to proceed by using essentially the same definition we used in the case of real-variable calculus. Let us see whether the same thing can be done in this case. Thus, let a and b be two *complex* numbers, and consider how we might adapt the limit definition above to this case. First of all we need to determine what is meant by the intermediate points between a and b; evidently we need a set of values z_1, z_2, $\cdots$, z_{n-1}. Now in the real case there is no real point in doing anything except going directly from the initial point to the final point; but in the complex plane there are many different paths which lead from a to b, and from what we have seen so far it is possible that these different paths may lead somehow to different results. Thus we evidently need to pick a path. Suppose that $\gamma(t)$ is a smooth (continuous with continuous derivative) path from a to b, and let $z_0 = a$, z_1, z_2, $\cdots$, z_{n-1}, $z_n = b$ be points along the curve ordered

[2]4 More precisely, for those of you who know something of ϵ–δ definitions, the limit above can be defined as follows: we say it is equal to L if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every *partition* $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$, $x_0 = a < x_1 < \cdots < x_n = b$ satisfying $\max\{|\Delta x_k|\,|k = 1, 2, \cdots, n\} < \delta$ and any set $\{x_k^*|k = 1, \cdots, n\}$ satisfying $x_k^* \in [x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(x_k^*)\Delta x_k - L\right| < \epsilon.$$

(Here, of course, $\Delta x_k = x_k - x_{k-1}$.)

by increasing parameter value. (Cf. Figure 12 in §25 of Goursat.) Then we define $\Delta z_k = z_k - z_{k-1}$, and consider the sum

$$\sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

(where z_k^* is some point along the curve between z_{k-1} and z_k in parameter value). Now suppose that

$$f(x+iy) = P(x,y) + iQ(x,y)$$

and write

$$z_k^* = x_k^* + iy_k^*, \qquad \Delta z_k = \Delta x_k + i\Delta y_k;$$

then working out the above product, we have

$$\begin{aligned}\sum_{k=1}^{n} f(z_k^*)\Delta z_k &= \sum_{k=1}^{n}[P(x_k^*, y_k^*) + iQ(x_k^*, y_k^*)][\Delta x_k + i\Delta y_k] \\ &= \sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k + i[P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k] \\ &= \sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k + i\sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k.\end{aligned}$$

Now we wish to consider the limit of the above sum as $\Delta z_k \to 0$ (in the same sense as elaborated in the footnote above). Since $\Delta z_k = \Delta x_k + i\Delta y_k$, this is the same as the limit as Δx_k and Δy_k go to zero independently. Thus we may write

$$\begin{aligned}\lim_{\Delta z_k \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k = \lim_{(\Delta x_k, \Delta y_k)\to(0,0)} \sum_{k=1}^{n} & P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k \\ & + i\sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k\end{aligned}$$

,

and we recognise these limits as giving ordinary line integrals of the form we have studied previously. In particular, the above limit becomes

$$\int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right];$$

assuming, of course, that all of these integrals exist (as they will if P and Q are both continuous, for example). We take this as our definition and write

$$\int_\gamma f(z)\,dz = \int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right].$$

Now suppose that γ is parameterised as $\gamma(t) = (x(t), y(t))$, $t \in [t_0, t_1]$; then the above can be written

$$\int_{t_0}^{t_1} P(x(t), y(t))x'(t) - Q(x(t), y(t))y'(t) + i[P(x(t), y(t))y'(t) + Q(x(t), y(t))x'(t)]\,dt.$$

Note that this is exactly what we would obtain if we were to replace dz in the integral $\int_\gamma f(z)\,dz$ with $x'(t)dt + iy'(t)dt$ and integrate from t_0 to t_1, i.e., if we were to pretend that the complex integral were simply another line integral with element $x'(t)dt + iy'(t)dt$. While this is not in itself a proof of anything, of course, it is useful for remembering the above formula; and it also suggests another mode of calculation: suppose that the curve γ is written in *complex* form as $z(t) = x(t) + iy(t)$; then the integral $\int_\gamma f(z)\,dz$ is equal to

$$\int_{t_0}^{t_1} f(x(t) + iy(t))(x'(t) + iy'(t))\,dt = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt,$$

where we are useful to use *complex* techniques to determine $z'(t)$ and $f(z(t))$. In other words, this formula does not require us to split f into its real and imaginary parts, which is not convenient in many cases (such as when f is most usefully represented in terms of polar coordinates, for example).

The integral $\int_\gamma f(z)\,dz$ is called a *contour integral.*

19. First glimpse of the Cauchy integral theorem. Let us consider what happens when we integrate an analytic function over a closed curve. More specifically, suppose that we have a function $f(x + iy) = P(x, y) + iQ(x, y)$ which is analytic over a simply connected region U which has boundary curve C, and assume that C is oriented with respect to U as required by Green's theorem. Let us assume furthermore that the real and imaginary parts of f, namely P and Q, have continuous first-order partial derivatives. Then by applying Green's theorem and the Cauchy-Riemann equations, we have

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C P(x, y)\,dx - Q(x, y)\,dy + i\int_C Q(x, y)\,dx + P(x, y)\,dy \\
&= \int_U \frac{\partial}{\partial x}[-Q(x, y)] - \frac{\partial}{\partial y}[P(x, y)]\;dA + \int_U \frac{\partial}{\partial x}[P(x, y)] - \frac{\partial}{\partial y}[Q(x, y)]\;dA \\
&= \int_U -\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,dA + \int_U \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\,dA = 0,
\end{aligned}$$

since the Cauchy-Riemann equations give

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}.$$

Thus under the assumptions above, the contour integral of an analytic function over a closed curve is always 0. This is a central result in complex variable theory.

Unfortunately, the above demonstration, since it requires that the partial derivatives of the real and imaginary parts of f be continuous, is not sufficient for our purposes, since we actually want to use this result to *prove* continuity of those derivatives. Thus we shall soon see another proof of this result from first principles which does not make use of this assumption.

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR MAY 25 – 29 SOLUTIONS

Due Tuesday, June 2, at 12:00 noon EDT.

1. [8 marks] Determine $\operatorname{Log} z$ for each of the following points and branch cuts:

(a) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.
We have $z = e^{i\frac{7\pi}{4}} = e^{-i\frac{\pi}{4}}$, so $\operatorname{Log} z = \log 1 - i\frac{\pi}{4} = -i\frac{\pi}{4}$.

(b) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = 0$, interval $(0, 2\pi)$.
We have $z = e^{i\frac{7\pi}{4}}$, so $\operatorname{Log} z = \log 1 + i\frac{7\pi}{4} = i\frac{7\pi}{4}$.

(c) $z = e$, branch cut along $\theta = \pi/2$, interval $(\pi/2, 5\pi/2)$.
We have $z = e \cdot e^0 = e \cdot e^{2\pi i}$, so $\operatorname{Log} z = \log e + 2\pi i = 1 + 2\pi i$.

(d) $z = e$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.
We have $z = e \cdot e^0$, so $\operatorname{Log} z = \log e + i \cdot 0 = 1$.

Marking: for each part, 1 mark each for the correct real and imaginary part of $\operatorname{Log} z$.

2. [5 marks] Compute the following difference of limits:

$$\lim_{\theta \to 0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2\pi^-} \operatorname{Log} re^{i\theta},$$

where $r > 0$. Does this difference depend on which branch of Log is used? What would happen if we considered instead the difference

$$\lim_{\theta \to \theta_0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to (\theta_0 + 2\pi)^-} \operatorname{Log} re^{i\theta}$$

where θ_0 is any real number?

Strictly speaking, since

$$\operatorname{Log} re^{i\theta} = \log r + i(\theta + 2n\pi), \qquad n \in \mathbf{Z},$$

the first difference above would become a difference of *sets* of values, namely

$$\{\log r + i \cdot 2n\pi | n \in \mathbf{Z}\} - \{\log r + i \cdot (2m+1)\pi | m \in \mathbf{Z}\},$$

which gives simply $\{2ni\pi | n \in \mathbf{Z}\}$. What was intended, though, was that we consider a *particular branch* of the logarithm before taking the limit. Now recall that choosing a branch also involves choosing a particular interval for θ; now we wish this interval to have numbers close to but slightly above 0 as well as numbers close to but slightly below 2π, and since it must be of length 2π, it must be simply the interval $(0, 2\pi)$. If we use this interval, we obtain

$$\lim_{\theta \to 0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2\pi^-} \operatorname{Log} re^{i\theta} = \log r - [\log r + 2\pi i] = -2\pi i.$$

Taken in this sense, there is only one branch for which the limit makes sense, so the second part of the question does not even make any sense. If, however, we consider the limits on θ not strictly as limits on θ but rather as limits on *points*, while still requiring that the full 2π range between the points be included in the θ interval, then we can take other branches, but only those for which the θ interval is $(2n\pi, 2(n+1)\pi)$ for some integer n. For such a branch, the difference in the limits is now (rewriting the limits as noted above to correspond to points)

$$\lim_{\theta \to 2n\pi^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2(n+1)\pi^-} \operatorname{Log} re^{i\theta} = \log r + 2n\pi i - [\log r + 2(n+1)\pi i] = -2\pi i,$$

so that the difference does not depend on the branch chosen.

The second part is very similar. If we consider the full Log function, it is quite easy to see that we will get the same set as the difference:

$$\begin{aligned}\lim_{\theta\to\theta_0^+}\operatorname{Log} re^{i\theta} - \lim_{\theta\to(\theta_0+2\pi)^-}\operatorname{Log} re^{i\theta} &= \{\log r + i(\theta_0+2n\pi)|n\in\mathbf{Z}\} - \{\log r + i(\theta_0+2(m+1)\pi)|m\in\mathbf{Z}\}\\ &= \{2n\pi i|n\in\mathbf{Z}\}.\end{aligned}$$

If we consider a branch cut along $\theta=\theta_0$, and again consider the indicated limits as indicating limits on *points* rather than limits on θ, then we see that we must take the θ range to be of the form $(\theta_0+2n\pi, \theta_0+2(n+1)\pi)$, and the difference in limits will be

$$\lim_{\theta\to(\theta_0+2n\pi)^+}\operatorname{Log} re^{i\theta} - \lim_{\theta\to(\theta_0+2(n+1)\pi)^-}\operatorname{Log} re^{i\theta} = \log r + i(\theta_0+2n\pi) - [\log r + i(\theta_0+2(n+1)\pi)] = -2\pi i,$$

exactly as before. This difference also does not depend on which branch we take.

Marking: Roughly, 2 marks for each difference, 1 mark for making a correct statement about the dependence on the branch.

3. [4 marks] For a given complex number z, use the quadratic formula and the relation

$$\cos w = \frac{e^{iw}+e^{-iw}}{2}$$

to compute all complex numbers w satisfying $\cos w = z$.

Let us write $z=\cos w$; then we may proceed as follows:

$$\begin{aligned} z &= \frac{e^{iw}+e^{-iw}}{2}\\ e^{iw}+e^{-iw} &= 2z\\ e^{2iw}-2ze^{iw}+1 &= 0 && \text{[1 mark]}\\ e^{iw} &= \frac{1}{2}\left(2z+(4z^2-4)^{1/2}\right) = z+(z^2-1)^{1/2} && \text{[1 mark]}\\ iw &= \operatorname{Log}\left(z+(z^2-1)^{1/2}\right) && \text{[1 mark]}\\ w &= \frac{1}{i}\operatorname{Log}\left(z+(z^2-1)^{1/2}\right). && \text{[1 mark]}\end{aligned}$$

4. We know that the exponential function e^z is analytic on the entire complex plane, and hence conformal at each point. Let us see what this map looks like in practice.

(a) [6 marks] Consider straight lines parallel to the real and imaginary axes. What is the image of these lines under the map $z\mapsto e^z$? (For example, if you parameterise the two lines as $\gamma_k(t)$, what kind of curve is $e^{\gamma_k(t)}$?) Sketch a couple representative examples (both the original lines and the image curves).

A straight line parallel to the real axis can be parameterised as

$$\gamma(t) = t+iy$$

for some real number y. Under the map $z\mapsto e^z$, this becomes

$$t\mapsto e^te^{iy} = e^t(\cos y + i\sin y),$$

which is a ray from the origin (but not including the origin) going to infinity along the direction given by $\cos y + i\sin y$. Similarly, a straight line parallel to the imaginary axis can be parameterised as

$$\gamma(t) = x+it,$$

where x is some real number; under the map $z \mapsto e^z$, this becomes

$$t \mapsto e^x e^{it} = e^x(\cos t + i \sin t),$$

which is a circle centred at the origin with radius e^x.

Marking: For both lines, 2 marks for a full and correct description of the image curve. 1 mark for each of the corresponding sketches.

(b) [4 marks] Now consider two lines passing through the origin, making angles θ_1 and θ_2 with the positive real axis. What is the image of these two curves under the map $z \mapsto e^z$? Sketch the image curves for two particular values of θ_1 and θ_2 (neither of which is a multiple of $\pi/2$!).

A curve passing through the origin making an angle θ with the positive real axis can be parameterised as

$$\gamma(t) = t(\cos\theta + i \sin\theta);$$

one way of seeing this is to note that the complex number $\cos\theta + i\sin\theta$ corresponds to a unit vector which makes an angle θ with the positive real axis. Thus the image of the two given curves under the exponential map $z \mapsto e^z$ is

$$e^{t(\cos\theta_1 + i\sin\theta_1)} = e^{t\cos\theta_1}\left(\cos\left[t\sin\theta_1\right] + i\sin\left[t\sin\theta_1\right]\right)$$

and

$$e^{t(\cos\theta_2 + i\sin\theta_2)} = e^{t\cos\theta_2}\left(\cos\left[t\sin\theta_2\right] + i\sin\left[t\sin\theta_2\right]\right)$$

If there weren't the factors of $e^{t\cos\theta_1}$ and $e^{t\cos\theta_2}$, these would be circles centred at the origin; but this leading factor means that we get instead *spirals* – at least as long as $\cos\theta_k \neq 0$!

Marking: 2 marks (total) for determining the image curves; 1 mark for each of the graphs.

(c) [3 marks] How does your work from (a) and (b) exemplify the conformality of e^z?

For part (a), it is clear that rays from the origin and circles centred at the origin intersect at 90° angles, just as do lines parallel to the real and imaginary axes. For part (b), conformality would mean that the angle between the original lines at the origin is equal to the angle between the spirals *at the origin* – if they intersect anywhere else it doesn't matter. The angle between the spirals at the origin can be found by computing their tangent vectors. In complex form, these are

$$\left.\frac{d}{dt}e^{t(\cos\theta_1 + i\sin\theta_1)}\right|_{t=0} = \cos\theta_1 + i\sin\theta_1,$$
$$\left.\frac{d}{dt}e^{t(\cos\theta_2 + i\sin\theta_2)}\right|_{t=0} = \cos\theta_2 + i\sin\theta_2,$$

which are just the direction vectors for the original lines. Thus the angle between the spirals at the origin is the same as the angle between the original lines.

Marking: 2 marks for noting that the rays and circles in (a) intersect at 90° as do the original lines. 1 mark for observing that the spirals in (b) make the same angle at the origin as do the original lines.

5. We know that branches of root functions are analytic on their domains, and hence conformal there. Let us see how this works out in practice.

(a) [8 marks] Choose a particular branch of the square root function $z \mapsto z^{1/2}$. (Make sure you indicate your choice clearly!) Consider straight rays from the origin and circles centred on the origin; what is their image under this map? Derive formulas and sketch a couple representative examples (sketch both original and image curves). How does this exemplify the conformality of your particular branch of $z \mapsto z^{1/2}$?

We choose the branch obtained by making a cut along the negative real axis and requiring θ to lie in $(-\pi, \pi)$; thus we have

$$z^{1/2} = r^{1/2}e^{i\theta/2} \qquad \text{when } z = re^{i\theta},\ \theta \in (-\pi, \pi).$$

Now a straight ray from the origin which makes an angle of θ with the positive real axis can be parameterised as

$$\gamma(t) = t(\cos\theta + i\sin\theta), \qquad t \in (0, +\infty);$$

we may assume that $\theta \in (-\pi, \pi)$, which means that the image of this ray under the chosen branch of the square root function is simply

$$t \mapsto t^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \qquad t \in (0, +\infty)$$

which is still a ray from the origin, but at an angle of $\theta/2$ to the positive real axis instead of θ. Similarly, a circle of radius r centred on the origin can be parameterised as

$$\gamma(t) = r(\cos t + i\sin t), \qquad t \in [-\pi, \pi];$$

in order to find its image we drop the point $-r$, which corresponds to $t = \pm\pi$, and thence obtain the image curve

$$t \mapsto r^{1/2}\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right), \qquad t \in (-\pi, \pi),$$

which is a *semicircle* extending from $-\pi/2$ to $\pi/2$ and with radius $r^{1/2}$, though still centred at the origin.

Since straight rays from the origin intersect both circles and semicircles at 90° angles, we see that the angles between these curves is preserved by the square root function, as expected.

Marking: 2 marks for specifying the branch (both the location of the cut and the θ interval). 2 marks each for deriving an analytic representation of the image curves. 1 mark for both sketches, 1 mark for noting the conformality relation.

(b) [4 marks] Consider two rays from the origin which make an angle of less than π with each other. What is the angle between the images of these lines under the map from (a)? Does this contradict what we know about the relationship between analytic functions and conformal maps? Why or why not?

Note that any two rays from the origin make two angles, one of which is less than (or equal to) π and the other of which is greater than (or equal to) π. Suppose that the angle which is less than π does not include the branch cut, and let $\theta_1, \theta_2 \in (-\pi, \pi)$ denote the angles between the lines and the positive real axis; then $|\theta_1 - \theta_2| < \pi$. We may assume that $\theta_1 > \theta_2$ (just reorder if not!). The two rays may be parameterised as

$$\begin{aligned}\gamma_1(t) &= t(\cos\theta_1 + i\sin\theta_1), \qquad t \in (0, +\infty),\\ \gamma_2(t) &= t(\cos\theta_2 + i\sin\theta_2), \qquad t \in (0, +\infty).\end{aligned}$$

Under the square root map, these two rays will be mapped to the rays

$$\begin{aligned}t &\mapsto t^{1/2}\left(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}\right), \qquad t \in (0, +\infty),\\ t &\mapsto t^{1/2}\left(\cos\frac{\theta_2}{2} + i\sin\frac{\theta_2}{2}\right), \qquad t \in (0, +\infty),\end{aligned}$$

which make an angle of $\frac{1}{2}(\theta_1 - \theta_2)$ with each other – i.e., *half* that of the angle between the original two rays. This does not contradict what we know about the relationship between analytic and conformal maps, though, since the square root function is not analytic at the origin.

Marking: 2 marks for showing (this requires some kind of computation, not simply a picture) that the angle between the image lines is half that between the original lines. (The requirement 'make an angle of less than π with each other' was actually an accidental red herring.) 1 mark for saying that this is not a contradiction, 1 mark for an explanation as to why.

Summary:
- We fill in some holes in the previous exposition.
- We then proceed to give a proof of the Cauchy integral theorem which does not require continuity of the partial derivatives of the real and imaginary parts of the function.
- We show that analytic functions have antiderivatives, at least on simply-connected regions, which are also analytic, and discuss a connection with branch cuts.
- Finally, we discuss an extension of the Cauchy integral theorem to regions which are not simply connected.

(Goursat, §§28 – 31)

20. A few points from previous material. Recall that we have shown that, if m is a positive integer, then the *power rule* for differentiation on the real line applies also to derivatives in the complex plane:

$$\frac{d}{dz}z^m = mz^{m-1}.$$

The same result holds true for *any* complex exponent m, as long as we interpret the left- and right-hand sides appropriately. To see this, recall that if m is any complex number, we define the exponential z^m by

$$z^m = e^{m\operatorname{Log} z},$$

where $\operatorname{Log} z$ represents the full multivalued complex logarithm of the complex number z. As we discussed when we first gave this definition, the right-hand side is multivalued since Log is. Suppose now that we take a particular branch of Log, say by requiring the angle to lie between $(\theta_0, \theta_0 + 2\pi)$ for some $\theta_0 \in \mathbf{R}$.[0] For this particular branch, as in general,

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z},$$

and by the chain rule we have

$$\frac{d}{dz}z^m = \frac{d}{dz}e^{m\operatorname{Log} z} = e^{m\operatorname{Log} z}\frac{d}{dz}m\operatorname{Log} z = me^{m\operatorname{Log} z}\frac{1}{z} = me^{m\operatorname{Log} z - \operatorname{Log} z} = me^{(m-1)\operatorname{Log} z} = mz^{m-1},$$

where z^{m-1} is taken using the same branch of Log as z^m.[1] Thus we do indeed have

$$\frac{d}{dz}z^m = mz^{m-1},$$

as long as the powers on both sides are computed using the same branch of the logarithm.

Since the functions $z \mapsto z^m$, where m is any nonzero integer, are all single-valued, the result above holds without any conditions for (nonzero) integer exponents. It holds for $m = 0$ if we define z^0 to be 1 everywhere, including at 0. (Recall that 0^0 is not defined.)

We now wish to point out another version of the chain rule involving complex numbers. Recall that, if f and g are two complex-valued functions of a complex variable, both of which are analytic, then $f \circ g$ is also analytic where it is defined, and we have

$$\frac{d}{dz}(f \circ g) = f'(g(z))g'(z).$$

Now suppose that f is an analytic function of a complex variable, and that $\gamma : [a, b] \to \mathbf{C}$ is a smooth curve. Then we have also

$$\frac{d}{dt}(f \circ \gamma) = f'(\gamma(t))\gamma'(t).$$

[0] It is worth noting here that, although we specify a branch cut for Log and for the root functions by specifying an interval for the angle θ, a branch cut is a cut of the *entire plane*, not just the unit circle.

[1] Note that it is *always* valid to write $z = e^{\operatorname{Log} z}$ and $\frac{1}{z} = e^{-\operatorname{Log} z}$, regardless of the branch of Log we are using (or even if we are not taking a branch at all). The first follows from the definition of Log as the inverse function of exp, and the second follows from the first by laws of exponents.

This can be shown in the same way that we showed the original chain rule above; briefly, we may write

$$f(\gamma(t+h)) = f(\gamma(t) + \gamma'(t)h + o(h)) = f(\gamma(t)) + f'(\gamma(t))(\gamma'(t)h + o(h)) + o(\gamma'(t)h + o(h)),$$

so if we are willing to accept that $o(\gamma'(t)h + o(h))$ is also $o(h)$, this becomes

$$f(\gamma(t+h)) = f(\gamma(t)) + f'(\gamma(t))\gamma'(t)h + o(h),$$

from which the result follows by computing the difference quotient and taking a limit. (Here, again, by $o(h)$ we mean any function – of a real variable in this case – which satisfies $\lim_{h\to 0} o(h)/h = 0$.)

(It is worth noting the difference between these two chain rules. In the first one, both f and g were functions of a *complex* variable, while in the second one f is a function of a complex variable but γ is a function only of a *real* variable. We have been told many times – and will shortly begin to see for ourselves! – that the requirement that a function of a complex variable have a derivative is far more restrictive than the requirement that a function of a real variable have a derivative: note that the difference is between the *domains*, and *not* the ranges. In other words, the difference is between a function *defined* on the complex numbers, and a function *defined* on the real numbers, and not a function taking values in the complex numbers and a function taking values in the real numbers.)

21. The Cauchy integral theorem, full proof. Recall that in section 19 above we showed that, if f is an analytic function on a simply-connected region, and C is any simple (non self-intersecting) closed curve contained in that region, then *if f has continuous first-order partial derivatives on the region,*

$$\int_C f(z)\,dz = 0.$$

We will now show that this result holds *without* the assumption of continuous first-order partial derivatives, which we will actually be able ultimately (next week) to derive as a consequence. Our treatment follows very closely that given in Goursat, §28.

Thus, let f be an analytic function on some region, and let C be any simple closed curve in that region such that f is analytic everywhere on the interior of C. Let U denote the region bounded by C, which is necessarily simply-connected; then by assumption f is analytic on U and on C. Now suppose that we subdivide U into squares and partial squares by drawing a square grid across it (see Figure 13 in Goursat for an example of what we mean by this). We let γ_k denote the boundary curve – oriented counterclockwise – of the kth full square, and γ'_j denote the boundary curve – again oriented counterclockwise – of the jth partial square. Then we claim that

$$\sum_k \int_{\gamma_k} f(z)\,dz + \sum_j \int_{\gamma'_j} f(z)\,dz = \int_C f(z)\,dz.$$

This is clear after a bit of thought, since the sides of the grid squares appear exactly twice, and in opposite directions, in the sum of integrals on the left, and hence cancel, meaning that we are left only with the integral around the boundary curve, i.e., the right-hand side.

We now claim that each of the integrals in the above sums is small. To see this, note that since the functions $z \mapsto a$ and $z \mapsto a(z-z_0)$ are analytic with continuous partial derivatives, the result from section 19 can be applied to show that around any closed curve they integrate to zero. (Another way of showing this, without applying Green's theorem – which we did in section 19 – is outlined in section 28 of Goursat.) Now consider $\int_{\gamma_k} f(z)\,dz$, and let z_0 be some point either inside or on γ_k. Then since f is analytic on U, we can write, for any point z on γ_k,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \epsilon(z-z_0)(z-z_0),$$

where $\epsilon(z-z_0) \to 0$ as $z \to z - z_0$. (In o notation, $\epsilon(z-z_0) = o(z-z_0)/(z-z_0)$, but we stick with this notation here for consistency with the lecture.) Now the functions $z \mapsto az$ and $z \mapsto a(z-z_1)$, where $a, z_1 \in \mathbf{C}$ are any two constant complex numbers, are both analytic with continuous first-order partials (this is entirely trivial!); thus the result from Section 19 shows that both integrate to zero around any simple closed curve. Hence we may write

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z-z_0) + \epsilon(z-z_0)(z-z_0)\,dz = \int_{\gamma_k} \epsilon(z-z_0)(z-z_0)\,dz,$$

and similarly, letting z_0' denote some point within or on γ_j', and $\epsilon'(z-z_0')$ the corresponding function analogous to $\epsilon(z-z_0)$,

$$\int_{\gamma_j'} f(z)\,dz = \int_{\gamma_j'} \epsilon'(z-z_0')(z-z_0')\,dz.$$

Now recall that, if we have a curve γ of length ℓ and an analytic function f which is bounded by M on γ, then we have the bound

$$\left|\int_\gamma f(z)\,dz\right| \le \ell M.$$

Let us apply this to the two integrals above. Suppose that $\epsilon(z-z_0) \le \eta$ on γ_k, and that γ_k is a square with side lengths ℓ_k; then the total length of γ_k is $4\ell_k$, and moreover the function $z-z_0$ on γ_k is bounded by $\ell_k\sqrt{2}$ (since this is the length of a diagonal of γ_k and that is the farthest apart any two points can be on a square). Thus we may write

$$\left|\int_{\gamma_k} f(z)\,dz\right| \le 4\ell_k \cdot \eta\ell_k\sqrt{2} = 4\sqrt{2}\ell_k^2\eta = 4\sqrt{2}A_k\eta,$$

where $A_k = \ell_k^2$ is the area enclosed by γ_k. Similarly, suppose that $\epsilon'(z-z_0')$ is bounded by some number η' on γ_j'. Now γ_j' consists of parts of four sides of a square, together with some portion of C; thus, if we let ℓ_j' denote its side length and λ_j the length of that portion of C, then the length of γ_j' is bounded by $4\ell_j' + \lambda_j$. (This may be a very bad upper bound, since we may only have a small portion of the square sides, but the point is that it *is* an upper bound, and as we shall see later, it is a sufficiently good upper bound.) Now because we have decomposed the region U along a square grid, the region enclosed by γ_j' is a portion of a square, i.e., it is a region entirely contained in one of these squares; thus as before the function $z-z_0'$ on γ_j' is bounded by $\ell_j'\sqrt{2}$ and we may write

$$\left|\int_{\gamma_j'} f(z)\,dz\right| \le (4\ell_j' + \lambda_j)\cdot\eta'\ell_j'\sqrt{2} = (4A_j' + \ell_j'\lambda_j)\sqrt{2}\eta'.$$

Now we come to a technical point which is addressed in §29 of Goursat but which we shall just touch on without giving a formal proof. We know that as $z \to z_0$, $\epsilon(z-z_0) \to 0$, and similarly that $\epsilon'(z-z_0') \to 0$ as $z \to z_0'$. Similar relations will be true in all of the other squares and partial squares into which we have subdivided U.[2] This means that, by taking each individual square small, we can make the quantities η and η' small. We claim that by taking the entire *grid* arbitrarily fine, i.e., to have squares and partial squares which are arbitrarily small, *all* of the functions $\epsilon(z-z_0)$ and $\epsilon'(z-z_0)$, for *all* indices k and j (respectively), can *simultaneously* be made arbitrarily small. This does not automatically follow from the foregoing, but as it does seem reasonable, and the proof is slightly technical, we shall assume its truth and see how it can be used to derive the result. (As mentioned, an explanation of this result is given in §29 of Goursat for those who are interested.) Thus we assume that, for any $\eta_0 > 0$, by taking the grid sufficiently fine, we may assume that for all k and j, we may take $\eta, \eta' < \eta_0$. Now consider such a sufficiently fine grid, and let L be the side length of the squares in the grid; then we may write

$$\left|\sum_k \int_{\gamma_k} f(z)\,dz\right| \le \sum_k \left|\int_{\gamma_k} f(z)\,dz\right| \le 4\sqrt{2}\eta_0 \sum_k A_k \le 4\sqrt{2}\eta_0 A,$$

[2] Note that the points z_0 and z_0' actually depend on the indices k and j, respectively, but we have chosen not to indicate this in our notation just for simplicity.

where A denotes the area of some circle completely containing U, and such that all squares in the grid which intersect U are completely contained in that circle; similarly, letting λ denote the length of the curve C,

$$\left|\sum_j \int_{\gamma_j'} f(z)\,dz\right| \leq \sum_j \left|\int_{\gamma_j'} f(z)\,dz\right| \leq \sqrt{2}\left[4\sum_j A_j' + L\sum_j \lambda_j\right]\eta_0$$
$$\leq \sqrt{2}(4A + L\lambda)\eta_0.$$

Thus, finally, we have

$$\left|\int_C f(z)\,dz\right| \leq (4\sqrt{2}A + 4\sqrt{2}A + \sqrt{2}L\lambda)\eta_0 = (8\sqrt{2}A + L\sqrt{2}\lambda)\eta_0,$$

where η_0 is an arbitrary positive number. Now if we take any grid *finer* than the one we just considered, clearly L will decrease, while we can use the same A as before; in other words, if $\eta_0' < \eta_0$ and we consider any grid fine enough to have $\eta, \eta' < \eta_0'$, we may still write

$$\left|\int_C f(z)\,dz\right| \leq (8\sqrt{2}A + L\sqrt{2}\lambda)\eta_0',$$

where A and L have the same values as they did before. By taking η_0' arbitrarily small, we see that the left-hand side must be arbitrarily small; since it does not depend on the grid, or η_0', it must actually be zero. This proves that

$$\int_C f(z)\,dz = 0,$$

as claimed.

In the above we have assumed that the function f was defined and analytic on a larger region completely containing the curve C and its interior. It turns out that one only need assume f to be analytic on the interior of C and *continuous* up to the boundary; a brief discussion of this is given in the footnote in Goursat, pp. 48 – 49 (of the typescript; p. 71 of the original).

22. Antiderivatives and branch cuts. Recall that in multivariable calculus we learned that a vector field $\mathbf{F}$ which is *conservative*, in the sense that its integral around any closed curve is zero, has a *potential function*, i.e., that there is a function f such that $\mathbf{F} = \nabla f$. Moreover, f can be constructed as

$$f(x, y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r},$$

where (x_0, y_0) is any fixed point, and the line integral does not depend on the choice of curve from (x_0, y_0) to (x, y) since $\mathbf{F}$ is conservative. We now show a similar result in the case of analytic functions of a complex variable, though as usual the import is quite a bit deeper.

Thus suppose that f is a function analytic on a simply-connected region, pick some point z_0 in that region, and define a function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'.$$

Let us see in what sense this formula defines a function. Recall that a function consists of three things: a domain, a range, and a rule giving an element of the range for any element of the domain. Here the domain can clearly be taken to be the simply-connected region on which f is analytic, and as usual we don't really worry about the range (F will certainly be in $\mathbf{C}$, at any rate). Thus we only need to consider in what sense the function above defines a rule which gives a complex number given any complex number in its domain. In order to evaluate the integral, we need to choose a particular path γ from z_0 to z. Suppose that γ_1 and γ_2 are two distinct paths from z_0 to z. If γ_1 and γ_2 have no intersection points other than their endpoints z_0 and z, then by running γ_1 forwards and γ_2 backwards we obtain a simple closed curve; if we call it γ, then we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz;$$

but by the Cauchy integral theorem, the left-hand side is zero, so that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ and the integral evidently does not depend on the choice of curve in this case. It can be shown that this holds true even if the two curves have other intersection points; thus in the situation we are considering here, $F(z)$ depends only on the endpoints z and z_0, and not on the curve chosen from z_0 to z. It therefore does indeed give a single-valued function on the region.

Let us see whether we can compute its derivative. Thus we consider the quotient $[F(z+h) - F(z)]/h$. Now by choosing the curve used to calculate $F(z+h)$ so that it passes through z, we may write

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(z')\,dz'.$$

Now we note that $\int_z^{z+h} dz' = h$, just as in elementary calculus on the real line (this can be shown by parameterising the line from z to $z+h$, for example); thus this last expression is equal to

$$\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'.$$

But now if h is very small, $f(z') - f(z)$ will be very small for all points on the straight line from z to $z+h$, which means that also $|f(z') - f(z)|$ will also be very small there; if η is any upper bound on this quantity, then we may write

$$\left|\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'\right| \leq \frac{1}{|h|}|h|\eta = \eta,$$

which means that by taking h sufficiently small, the above quantity must be less than η. But if we unravel everything, this means that the limit

$$\lim_{h\to 0}\frac{F(z+h) - F(z)}{h} - f(z)$$

must be zero, which means that F is analytic and $F'(z) = f(z)$, as we might have expected.

It is worth noting that, if z_1 and z_2 are any two complex numbers in the region above, then by taking the curve from z_0 to z_2 to pass through z_1, we may write

$$\int_{z_1}^{z_2} f(z')\,dz' = \int_{z_0}^{z_2} f(z')\,dz' - \int_{z_0}^{z_1} f(z')\,dz' = F(z_2) - F(z_1),$$

which shows that the fundamental theorem of calculus is true in this case as well.

Let us now consider what could have gone wrong if the region on which f was known to be analytic had not been simply connected. For the kinds of regions we are interested in here (essentially, open sets in the plane), the notion of 'simply connected' is a *global* notion, in the sense that it is in general a property of the entire region, not just some portion of the region. Alternatively, any region is *locally* simply connected, since if we consider any point in the region, there is certainly a small disk around that point contained in the region, and that disk will be simply connected. On such a disk, the above logic goes through, and thus we see that, at least near any given point, we can still construct an antiderivative of f in exactly the same fashion as above. What goes wrong is when we try to push this construction further away from the point. Thus suppose for example that f is analytic everywhere except at some point ζ_0, and let $z_0 \neq \zeta_0$; then near z_0 the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

will be well-defined and independent of the curve connecting z_0 and z, and will give an antiderivative of f. But now consider trying to determine this function everywhere on some circle starting at z_0 which encloses the point ζ_0. At z_0 we have $F(z_0) = 0$ by definition. But when we traverse this circle around ζ_0, as we come back close to z_0, $F(z)$ may not be small, since there is no guarantee that the integral around the entire curve will vanish. This means that the limit of $F(z)$ may not equal 0 as $z \to z_0$ along this direction, and hence that it may not be possible to find a single-valued continuous antiderivative of f everywhere on the region. This is, in fact, a generalisation of what we have seen goes wrong when we consider the logarithm: since $\frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z}$, $\mathrm{Log}\, z$ is an antiderivative of $\frac{1}{z}$, and as we try to take its value along some closed curve containing the origin, we know that we run into problems of discontinuity or multivaluedness exactly like those just discussed. One solution to this problem in the general case is to use the solution we used for the logarithm, and take a branch cut starting at ζ_0 and going to infinity; the resulting region will be simply connected, and thus on it we may define a single-valued, continuous antiderivative using the above formula.

The notions above of starting out with an analytic function only defined on a small disk and attempting to extend it further are related to notions of *analytic continuation* which we shall discuss later on in the course.

23. An extension of Cauchy's integral theorem to non-simply connected regions. It turns out that there is a way of extending Cauchy's integral theorem to non-simply connected regions, in quite the same way one extends Green's theorem to such regions, which will be important to our derivation of the Cauchy integral formula and is also noteworthy in its own right. Suppose for definiteness that a function f is analytic everywhere on a region except at two holes (these could be two isolated points, or larger holes), and consider $\int_C f(z)\, dz$, where C is some simple closed curve in this region. As long as C does not enclose either of the holes, this integral will still vanish by the Cauchy integral theorem. Now if C contains just one of the holes, we may shrink it down to either the boundary curve of the hole (if the hole is itself a region) or to an arbitrarily small circle around the hole (if the hole is a point), and the integral of f over this new curve, call it C', will be equal to that of f over C: to see this, think of taking a point on C and joining it to some point on C' by a straight line; if we break this straight line open slightly, and pull the two edges apart, we will get a simple closed curve which does not enclose any singularities of f, and the integral over this curve will therefore vanish; but in the limit as the two lines come together, the integral over this curve is just

$$\int_C f(z')\, dz' - \int_{C'} f(z')\, dz',$$

assuming that we orient both C and C' counterclockwise. Thus these two integrals must be equal, as claimed.

In the case that C is a curve enclosing both holes, we may do something similar except that we will find

$$\int_C f(z')\, dz' = \int_{C_1'} f(z')\, dz' + \int_{C_2'} f(z')\, dz',$$

where C_1' and C_2' are curves enclosing the two holes, as described above. Here we are still assuming that all three curves are oriented counterclockwise. If we instead orient C_1' and C_2' clockwise, and call the resulting curves C_1 and C_2, then the above result becomes

$$\int_C f(z')\, dz' + \int_{C_1} f(z')\, dz' + \int_{C_2} f(z')\, dz' = 0,$$

i.e., the integral of f is still zero as long as we include curves around the singularities of f as well.

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JUNE 1 – 5

Due Tuesday, June 9, at 3:30 PM EDT.

1. [6 marks] Without doing any differentiation, explain why the following functions are harmonic on the indicated regions:

$$\cos\left(\cos x\cosh y\right)\cosh\left(\sin x\sinh y\right), \qquad \text{everywhere on the plane.}$$
$$\frac{1}{2}\log\left(\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right) \qquad \text{on the set } \{(x,y)|x,y>0\}.$$

We recall the following functions, which are analytic on the given regions:

$$\cos(x+iy)=\cos x\cosh y-i\sin x\sinh y, \qquad \text{everywhere on the plane}$$
$$\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y, \qquad \text{everywhere on the plane}$$

moreover, the complex logarithm

$$\operatorname{Log}(x+iy)=\frac{1}{2}\log\left(x^2+y^2\right)+i\theta(x,y)$$

will be analytic on any cut plane, where $\theta(x,y)$ denotes the angle for $x+iy$ corresponding to the particular choice of branch. Thus

$$\begin{aligned}\cos[\cos(x+iy)]&=\cos(\cos x\cosh y)\cosh(-\sin x\sinh y)-i\sin(\cos x\cosh y)\sinh(-\sin x\sinh y)\\&=\cos(\cos x\cosh y)\cosh(\sin x\sinh y)+i\sin(\cos x\cosh y)\sinh(\sin x\sinh y)\end{aligned}$$

will be analytic everywhere on the plane, and its real part

$$\cos(\cos x\cosh y)\cosh(\sin x\sinh y)$$

will thus be harmonic everywhere on the plane, as required. Further, for every point $x+iy\neq 0$

$$\operatorname{Log}\left[\sin(x+iy)\right]=\frac{1}{2}\log\left(\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right)+i\theta(\cos x\sinh y,\sin x\cosh y)$$

will be analytic in some disk around $x+iy$. Here we must take an appropriate branch to ensure single-valuedness; but since the choice of branch only affects the imaginary part, not the real part, we see that the real part will be harmonic at every point where $\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\neq 0$, and this is satisfied on $\{(x,y)|x,y>0\}$, since $\sinh y,\cosh y>0$ on that region and $\sin x,\cos x$ are never simultaneously zero.

Marking: First function: 1 mark for identifying cos, 1 mark for noting that it is $\cos(\cos(x+iy))$ and hence has a harmonic real part. Second function: 1 mark for recognising Log, 1 mark for recognising sin, 1 mark for talking about the branch, 1 mark for concluding that the real part is therefore harmonic.

2. [18 marks] Evaluate the following integrals:

$$\int_\gamma \frac{1}{z}\,dz, \qquad \text{where } \gamma \text{ represents the unit circle, traversed counterclockwise.}$$
$$\int_\gamma \frac{1}{z}\,dz, \qquad \text{where } \gamma \text{ represents the circle of radius one and centre } 2i\text{, traversed counterclockwise.}$$
$$\int_\gamma \frac{1}{z^2}\,dz, \qquad \text{where } \gamma \text{ represents any circle centred at the origin.}$$

Do any of these results contradict the result we derived in class about integrals of analytic functions over closed curves? Do any of them add to that result? Why or why not?

The unit circle may be parameterised as

$$z(t)=\cos t+i\sin t=e^{it}, \text{[1 mark]} \qquad t\in[0,2\pi]\text{[1 mark]};$$

the integral may be evaluated as follows:

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\,dt[\text{1 mark}] = \int_0^{2\pi} e^{-it}\cdot ie^{it}\,dt[\text{1 marks}] = \int_0^{2\pi} i\,dt = 2\pi i.[\text{1 mark}]$$

Alternatively, we may write

$$\begin{aligned}\int_\gamma \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{1}{\cos t + i\sin t}(-\sin t + i\cos t)\,dt = \int_0^{2\pi} \frac{\cos t - i\sin t}{\cos^2 t + \sin^2 t}(-\sin t + i\cos t)\,dt \\ &= \int_0^{2\pi} \frac{-\cos t\sin t + \sin t\cos t + i(\cos^2 t + \sin^2 t)}{\cos^2 t + \sin^2 t}\,dt \\ &= \int_0^{2\pi} i\,dt = 2\pi i.\end{aligned}$$

These two methods are entirely equivalent, though obviously the first is much shorter!

For the second integral, we note that that the circle of radius one and centre $2i$ does not include the origin[1 mark], which means that the function $\frac{1}{z}$ is analytic everywhere on the interior of the curve γ[2 marks], so by the Cauchy integral theorem this integral is zero.[2 marks]

For the final integral, let r denote the radius of the circle, Then we may parameterise γ as

$$z(t) = re^{it}, [\text{1 mark}] \qquad t \in [0, 2\pi], [\text{1 mark}]$$

and the integral is

$$\int_\gamma \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{1}{r^2 e^{2it}} rie^{it}\,dt[\text{1 mark}] = \int_0^{2\pi} \frac{1}{r} ie^{-it}\,dt[\text{1 marks}] = -\left.\frac{1}{r}e^{-it}\right|_0^{2\pi} = 0.[\text{1 mark}]$$

The Cauchy integral theorem states that if a function is analytic everywhere inside a particular simple closed curve, then its integral over that curve must be zero. Thus the second and third integrals do not contradict this theorem since both integrals are zero [1 mark]. The first integral does not contradict it either since the function $\frac{1}{z}$ is not analytic at $z = 0$, which lies inside the curve. [1 mark]The third integral does add something to the Cauchy integral theorem, though, since the integral is zero even though the function is not analytic everywhere inside the curve.[1 mark]

Marking: 5 marks for the first and third integrals, as indicated. The marks are for the following points: (1) a correct parameterisation; (2) correct interval; (3) correctly substituting into the integral; (4) algebra; (5) correct final result. 5 marks for the third integral; for an integral solution, the scheme is the same as for the first and third integrals, while for the above solution marks are as indicated. Remaining three marks for final paragraph as indicated.

Summary:

- We derive the Cauchy integral formula from the Cauchy integral theorem for non-simply connected regions.
- We then proceed to show how it may be applied to derive Taylor and Laurent series expansions, and give a simple example.

(Goursat, §§33, 35, 37.)

24. Cauchy integral formula. Suppose that a function f is analytic everywhere inside a simple closed curve C, and continuous on C. Then from our comment at the end of §21 above it follows that the Cauchy integral theorem applies and we have

$$\int_C f(z)\,dz = 0.$$

Now let us fix some point z_0 in the *interior* of the curve C. Then the function

$$\frac{f(z)}{z - z_0}$$

is clearly analytic everywhere inside C except at the point z_0. If we let C' be a small circle centred at z_0 and contained in the interior of C, say with radius $r > 0$, oriented counterclockwise, then by the discussion and result in §23 above we have

$$\int_C \frac{f(z)}{z - z_0}\,dz = \int_{C'} \frac{f(z)}{z - z_0}\,dz;$$

in other words, we are able to replace the (fairly arbitrary and possibly very complicated) curve C by the (presumably much simpler) curve C'. Now we can make C' as small as we like, and the above result will still hold, since $z = z_0$ is the only point inside C at which the integrand $f(z)/(z - z_0)$ is not analytic. Now f is analytic at z_0, so near z_0 we can write as we have before

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)(z - z_0),$$

where $\epsilon(z - z_0) \to 0$ as $z \to z_0$. Thus we may write

$$\begin{aligned}\int_{C'} \frac{f(z)}{z - z_0}\,dz &= \int_{C'} \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}\,dz \\ &= \int_{C'} f'(z_0) + \epsilon(z - z_0) + \frac{f(z_0)}{z - z_0}\,dz. \qquad (1)\end{aligned}$$

The integral of $f'(z_0)$ over C' is clearly zero since $f'(z_0)$ is a constant; we shall show in a moment that the integral of $\epsilon(z - z_0)$ over C' must be zero also. Thus we consider the integral

$$\int_{C'} \frac{f(z_0)}{z - z_0}\,dz.$$

Now C' is a circle of radius r centred at z_0, and can be parameterised as

$$z(t) = z_0 + re^{it}, \qquad t \in [0, 2\pi],$$

whence the integral above becomes[0]

$$\int_0^{2\pi} \frac{f(z_0)}{re^{it}} rie^{it}\,dt = \int_0^{2\pi} if(z_0) = 2\pi i f(z_0).$$

[0] Note that this is not really just a 'substitution' as used in elementary calculus; most obviously, substitution in elementary calculus was only shown for integrals of functions of a real variable, and here we are dealing with functions of a complex variable. More substantively, though, the process by which we reduce a contour integral to a definite integral in terms of a parameterisation of the curve follows from the *definition* of the contour integral as we showed above. The formal similarity is however obvious and worth noting as an aid to memory, though it should be borne in mind that the two processes are not identical.

Note that this does not depend on the radius r. Now, finally, consider the integral

$$\int_{C'} \epsilon(z - z_0)\, dz.$$

To evaluate it, note that since $\epsilon(z - z_0) \to 0$ as $z \to z_0$, by taking r sufficiently small we may assume that $|\epsilon(z - z_0)| < 1$ on C'; thus the absolute value of the above integral satisfies

$$\left| \int_{C'} \epsilon(z - z_0)\, dz \right| \leq 2\pi r;$$

thus if we take the limit as $r \to 0$ this integral must vanish. Now if we investigate equation (1), we find that $\int_{C'} \epsilon(z - z_0)\, dz$ is the *only* term in the whole equation which could depend on r; thus *it* can't depend on r either, so since its limit as $r \to 0$ must vanish, it must actually be zero for all r (all r sufficiently small that C' lies entirely inside C, anyway!). Putting all this together, we obtain finally

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

or

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz. \tag{2}$$

This is called the *Cauchy integral formula*. Thus the *Cauchy integral theorem* tells us that the integral of an analytic function around a closed curve is 0, while the *Cauchy integral formula* gives us a formula for calculating the *value* of an analytic function inside some curve in terms of an integral around that curve.

Let us expand on this last point for a bit. In equation (2), z_0 is *any* point inside the curve C. Note though that the right-hand side of the equation depends *only* on the values of f *on* the curve C! In other words, what we have here is a formula which will give us the value of a function at any point inside a curve, given only its values on that curve. In the one-variable case, this would be equivalent to saying that the values of a function at the endpoints of an interval determine the function everywhere inside the integral, a claim so patently false as to be silly. For those of you who have seen some partial differential equations, this property should be reminiscent of the solution to boundary-value problems, particularly for Laplace's equation: there, in fact, if one has a Green's function, one can actually produce an integral formula quite reminiscent of (3) for the value of the solution inside a region given only its values on the boundary of the region.[1]

Let us rewrite equation (3) as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz', \tag{3$'$}$$

to emphasise that what we have on the left-hand side is actually a full function rather than a single value. Now it can be shown (see Goursat, §33) that we can differentiate the right-hand side by taking the derivative under the integral sign. In other words, since the point z in (3') must lie *within* C, it cannot lie on C, so that the quantity z' in the integrand is never equal to z and we may therefore write for every z' on C

$$\frac{d}{dz} \frac{1}{z' - z} = \frac{1}{(z' - z)^2},$$

by the power rule and chain rule for differentiating functions of a complex variable. (Note that, while in the integrand we view $1/(z' - z)$ as a function of z', with z fixed, here we view it as a function of z with z' fixed.) Now assuming that we can differentiate under the integral sign, we may write

$$f'(z) = \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz' = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z)^2}\, dz'.$$

[1] Note that there are some connections between these last two sentences. A harmonic function of a single variable would be an f which satisfied the equation $f'' = 0$; the only solutions to this equation are functions $f(x) = ax + b$, where a and b are constants – and a little thought shows that *these* functions actually *do* satisfy the property just stated: in other words, they *are* determined by their values on the endpoints of any interval! The class of harmonic functions on the line, though, is too small to be very interesting.

Assuming that we may again differentiate under the integral sign, we see that the right-hand side also has a derivative and in fact, since

$$\frac{d}{dz}\frac{1}{(z'-z)^2} = \frac{2}{(z'-z)^3},$$

this derivative is

$$\frac{d}{dz}\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-z)^2}\,dz' = \frac{1}{2\pi i}\int_C \frac{2f(z')}{(z'-z)^3}\,dz'.$$

Continuing in the same way, then, we may evidently write

$$\frac{d^n}{dz^n}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Since the integral we are differentiating above is equal to $f(z)$, this shows that $f(z)$ has *arbitrarily many* derivatives, as we have often claimed and never actually proved until now. Note that the only assumption we needed to make was that f be analytic on a certain region; we did *not* need to assume that the derivative of f was continuous, or that the real and imaginary parts of f had continuous partial derivatives. These results now follow as a consequence, since the derivative of f must itself have a derivative, and hence must be analytic, hence continuous, showing that the real and imaginary parts of f do indeed have continuous partial derivatives.

To sum up, then, we have, for any nonnegative integer n, the *Cauchy integral formula*

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Let us give a couple examples.

EXAMPLES. If $f(z) = a$ is some constant, then we have

$$a = f(z) = \frac{1}{2\pi i}\int_C \frac{a}{z'-z}\,dz',$$

i.e., that if z is any point inside the simple closed curve C, then $\int_C \frac{1}{z'-z}\,dz' = 2\pi i$; this is a result worth remembering by itself. Now since f is constant, we must have $f'(z) = 0$, and hence $f^{(n)}(z) = 0$ for all $n \geq 1$; the above formula then gives

$$0 = f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{a}{(z'-z)^{n+1}}\,dz',$$

which gives

$$\int_C \frac{1}{(z'-z)^{n+1}}\,dz' = 0$$

whenever z' is inside the simple closed curve C and $n \geq 1$. Note that this does *not* follow from the Cauchy integral theorem since the integrand here is *not* analytic within the curve C. Thus we have an extension of the Cauchy integral theorem in this case. Again, this result is worth remembering all by itself.

25. Taylor series. Now that we know that any analytic function must have arbitrary many derivatives, we know that we can formally write out its Taylor expansion

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n, \tag{4}$$

where a is any point in the region on which f is analytic. The existence of the derivatives of f, though, does not prove that this series actually converges to f anywhere except at $z = a$ (where it does trivially since by convention the series above is simply $f(a)$ when $z = a$). Here we shall derive the Taylor expansion by a different method, namely as an application of the Cauchy integral formula. Our exposition closely follows that of Goursat, §35.

Since the series in (4), if it converges anywhere except at $z = a$, must converge on a disk centred at a, let us take our curve C to be a circle of radius R centred at a. Now for any z inside C we have the Cauchy integral formula for f:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z} \, dz'.$$

We shall show how to expand $\frac{1}{z'-z}$ in a power series. We have

$$\frac{1}{z' - z} = \frac{1}{(z' - a) - (z - a)} = \frac{1}{z' - a} \frac{1}{1 - \frac{z-a}{z'-a}}; \tag{5}$$

factoring out $z' - a$ like this is legitimate since here we are only concerned with the expression $1/(z' - z)$ when z' is a point *on* the curve C, and the point a is inside the curve. In fact, in this case, since the curve C is a circle of radius R centred at a, we actually have $|z' - a| = R$. Suppose that $|z - a| = r$; since z also lies inside C, we must have $r < R$. Now we would like to expand the second term in (5) above in a series. We shall augment our treatment in the lecture by providing a careful proof. (Our treatment in the lecture corresponded to taking $N \to \infty$ immediately and dropping the remainder terms, namely those terms coming from w^{N+1} below.) Recall the geometric series

$$\sum_{n=0}^{N} w^n = \frac{1 - w^{N+1}}{1 - w},$$

which is valid for any complex number $w \neq 1$;[2] from this we have

$$\frac{1}{1 - w} = \sum_{n=0}^{N} w^n + \frac{w^{N+1}}{1 - w}.$$

In our case, this gives from (5)

$$\begin{aligned}
\frac{1}{z' - z} &= \frac{1}{z' - a} \left[\sum_{n=0}^{N} \left(\frac{z - a}{z' - a} \right)^n + \frac{1}{1 - \frac{z-a}{z'-a}} \left(\frac{z - a}{z' - a} \right)^{N+1} \right] \\
&= \sum_{n=0}^{N} \frac{(z - a)^n}{(z' - a)^{n+1}} + \frac{1}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1}.
\end{aligned}$$

Substituting this back in to (4), we see that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_C \sum_{n=0}^{N} (z - a)^n \frac{f(z')}{(z' - a)^{n+1}} + \frac{f(z')}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1} \, dz' \\
&= \frac{1}{2\pi i} \left[\sum_{n=0}^{N} (z - a)^n \int_C \frac{f(z')}{(z' - a)^{n+1}} \, dz' \right] + \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1} \, dz'.
\end{aligned} \tag{6}$$

Let us consider the last term above. Since f is continuous on C, it must be bounded on C; let $M > 0$ be such that $|f(z')| < M$ when z' is on the curve C. Now since $|z' - a| = R$ and $|z - a| = r < R$, we see that $|z' - z| \geq R - r$ (this is just the triangle inequality $|z' - a| \leq |z' - z| + |z - a|$); thus

$$\left| \frac{1}{z' - z} \right| = \frac{1}{|z' - z|} \leq \frac{1}{R - r}.$$

[2] In fact, this formula is valid in any *ring* as long as $1 - w$ is invertible in that ring; i.e., it is a purely algebraic result.

Further,

$$\left|\frac{z-a}{z'-a}\right|^{N+1} = \left(\frac{r}{R}\right)^{N+1}.$$

Thus the absolute value of the second term can be bounded as follows:

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz'\right| \leq \frac{1}{2\pi}\cdot 2\pi R\cdot M\cdot \frac{1}{R-r}\cdot\left(\frac{r}{R}\right)^{N+1} = \frac{MR}{1-R}\left(\frac{r}{R}\right)^{N+1}.$$

Since $r < R$, this quantity must go to zero in the limit as $N \to \infty$; substituting this into (6) gives

$$\frac{1}{2\pi i}\left[\sum_{n=0}^{\infty}(z-a)^n \int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz'\right] = f(z) - \lim_{N\to\infty}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz' = f(z),$$

or to write it out more clearly,

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz'.$$

But by the Cauchy integral formula for $f^(n)$, the integral here is simply $\frac{1}{n!}f^{(n)}(a)$, and we have thus proven the Taylor series expansion for f,

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n,$$

which will be valid on any disk centred at a on which f is analytic. Note that the above argument shows quite rigorously both that the above series converges and that it converges to $f(z)$, given only the general Cauchy integral formula. So if you had never seen a proof that a Taylor series converges to the function it comes from, now you have!

26. Laurent series. It turns out that for many applications it is important to be able to treat functions which have varies kinds of *singularities*, i.e., which fail to be analytic at various points or regions of the plane. While such functions will still clearly have Taylor series expansions on any disk not containing any of these singularities, it turns out to be useful to consider a more general type of expansion which will represent the function on a region surrounding the singularities. These are called *Laurent series.*

Thus suppose that we have a function f which is analytic on an *annulus*; specifically, suppose that C and C' are two circles, centred at a point a, with radii R and R' respectively, where $R > R'$ (so that C' is the inner circle), and both oriented counterclockwise, and that f is analytic on the region between C and C'. We shall extract a series expansion for f from the general Cauchy integral theorem in the same way we found the Cauchy integral formula and then used it to extract the Taylor expansion for f in the previous two sections. Our first step is thus to produce a generalisation of the Cauchy integral formula to the present case. The generalisation is not at all hard. Let z be any point in the annulus between C and C', and let γ be a small circle centred at z and with radius r, oriented counterclockwise and entirely contained in the region between C and C'. Then by the general Cauchy integral theorem in §23, we have

$$\int_C \frac{f(z')}{z'-z}\,dz' = \int_{C'}\frac{f(z')}{z'-z}\,dz' + \int_\gamma \frac{f(z')}{z'-z}\,dz'.$$

Now since γ is entirely contained in the region between C and C', f must be analytic everywhere on and inside γ, which means that by the usual Cauchy integral formula the second integral above is just

$$\int_\gamma \frac{f(z')}{z'-z}\,dz' = 2\pi i f(z),$$

and the above formula gives

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz';$$

in other words, we can generalise the Cauchy integral formula to the case of a function analytic *between* two curves if we integrate over both of them with the correct orientation (equivalently, including the correct minus sign). Evidently we could also extend the formula to a situation where a function was analytic on a region with *multiple* holes, but we do not need that here.

Now the first integral above can be treated exactly as before, giving ultimately

$$\sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',$$

but note that in this case we *cannot* replace the integral with $f^{(n)}(a)/n!$, since f is not known to be analytic at a (f might not even be defined at a, for that matter!). The second integral can be treated by slightly adapting this method. Since in the second integral the point z' lies on C', letting $|z-a| = r$ we have $|z'-a| = R' < r$; thus we may write

$$-\frac{1}{z'-z} = \frac{1}{z-z'} = \frac{1}{(z-a)-(z'-a)} = \frac{1}{z-a}\frac{1}{1-\frac{z'-a}{z-a}};$$

thus we have an analogue to formula (6) but with z' and z interchanged except inside f:

$$\begin{aligned}-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}(z'-a)^n\int_{C'}\frac{f(z')}{(z-a)^{n+1}}\,dz\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}dz\\ &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}\frac{1}{(z-a)^{n+1}}\int_{C'}f(z')(z'-a)^n\,dz'\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}dz'.\end{aligned}$$

Since we now have, as just noted, $|z'-a| = R' < r = |z-a|$, the argument given above shows that the second integral vanishes in the limit as $N \to \infty$, and we obtain the series expansion

$$-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz = \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

Thus, finally, we find that $f(z)$ can be expressed as the sum of two series:

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

To simplify this a bit, let us make the definitions

$$a_n = \frac{1}{2\pi i}\int_C\frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \geq 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1}f(z')\,dz', \quad (n \geq 1)$$

where in b_1 we have $(z'-a)^0 = 1$ since $z' \neq a$, as z' is on C' and a is inside C'. Then we can write

$$f(z) = \sum_{n=0}^{\infty}a_n(z-a)^n + \sum_{n=1}^{\infty}b_n\frac{1}{(z-a)^n};$$

in other words, whereas in the previous section, when f was analytic everywhere inside the circle C and we could write it as a sum of powers of $z-a$, in the case when f is analytic only on an annular region, we must write f as an infinite series of powers of $z-a$ *and* $1/(z-a)$. This is reasonable since $1/(z-a)$ will not be analytic at $z = a$; but note that f may be singular at other points inside C' than just a.

Before ending with an example, it is probably worthwhile to step back a bit and consider what be the importance of the results we have derived in the last three sections. As a concise summary, and for comparison, these are

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz',$$
$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',$$
$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_{C'} \frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz',$$

where f is assumed to be analytic within the arbitrary simple closed curve C in the first line, within the circle C in the second, and between the circles C' and C in the third. All three of these are *representation formulæ*; i.e., they give $f(z)$ as a special type of expression (an integral in the first case, series in the latter two). One of the uses of formulæof this sort is that they give us concrete ways of writing out f, which allow us to perform certain manipulations which would be much harder without them. Another, slightly more abstract, perspective is that these formulægive us a way of breaking f down into other data, which may encode the information we need for a specific problem in a more convenient way than the map $z \mapsto f(z)$ all by itself. For example, if we are only interested in knowing $f(1)$, then the simpler the formula for f the better; but if we are interested in knowing $\int_{C'} f(z)\,dz$, then the simpler the expression for b_1 the better.

On the other hand, these formulæare so general that it will require a fair bit more work before we get to the concrete applications in which they are so powerful. Thus unfortunately we shall have to stop at the vague indications in the previous paragraph for the time being, with a promise to say more about it later.

Let us do an example.

EXAMPLE. Let p be a positive integer, let $a \in \mathbf{C}$, and define the function f on $C\backslash\{a\}$ by

$$f(z) = \frac{1}{(z-a)^p}.$$

Then f is analytic everywhere on the plane except at $z = a$. (This kind of singularity, incidentally, is called a *pole* of order p; we shall study these systematically later.) Thus we expect to be able to expand f as a Laurent series. Actually it is quite obvious that $f(z)$ as given *is* a (single-term) Laurent series, so actually we already know this without any calculation; but let us work out the integrals anyway to see what happens. In this case we may take C and C' to be any circles centred at a, say with radii R and R', where the only condition on these radii is that $R > R'$. We have first of all

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' = \frac{1}{2\pi i}\int_C \frac{1}{(z'-a)^{n+p+1}};$$

now $n \geq 0$, while $p \geq 1$, so $n + p \geq 1$ and by the example we did at the end of §24 above we must have $a_n = 0$. Similarly,

$$b_n = \frac{1}{2\pi i}\int_{C'} (z'-a)^{n-1} f(z')\,dz' = \frac{1}{2\pi i}\int_{C'} (z'-a)^{n-1-p}\,dz';$$

if $1 \leq n < p$ (note that if $p = 1$ there will not be any such n, but that doesn't matter) then we must have $n-1-p < -1$, so this integral is zero for the same reason. Now if instead we have $n > p$, then $n-1-p \geq 0$, so the integrand is actually analytic, and by the Cauchy integral theorem we have again $b_n = 0$. The only case left is $n = p$; in this case we have

$$b_p = \frac{1}{2\pi i}\int_{C'} \frac{1}{z'-a}\,dz' = 1,$$

by the first example at the end of §24 above. Thus we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $a_n = 0$ for all n and $b_n = 0$ except for $n = p$, where $b_p = 1$. The series on the left thus trivially give $\frac{1}{(z'-a)^p}$, as they should.

For those of you who have seen orthogonal bases in vector spaces with an inner product, it is worth noting the formal similarity between the above procedure and that of determining components along the basis vectors in an orthonormal basis. We are not going to make this formal similarity precise, but it is worth noting anyway.

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JUNE 8 – 12

Due Tuesday, June 16, at 3:30 PM EDT.

1. [9 marks] Let

$$f(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

By choosing and parameterising an appropriate (potentially only piecewise-smooth) curve, determine the function

$$F(z) = \int_0^z f(z')\, dz'.$$

Explain why the result does not depend on your choice of curve.

While there are many different choices of curve we could make, it is probably simplest to choose a piecewise-smooth curve each part of which is parallel to one of the coordinate axes, since in this way on each portion of the curve only one of the coordinates x and y will change at a time and we will only have to integrate single trigonometric or hyperbolic trigonometric functions. Specifically, then, let us suppose that the complex number z can be written as $z = x + iy$ and define our curve γ to be

$$\gamma(t) = \begin{cases} t, & t \in [0, x] \\ x + it, & t \in [0, y] \end{cases};$$

thus γ gives first a line along the real axis from 0 to the real part of z and then a line parallel to the imaginary axis from there to z. Since the integral along a piecewise-smooth curve is equal to the sum of the integrals along the different pieces, we may then write

$$\begin{aligned}
\int_\gamma f(z')\, dz' &= \int_0^x f(t) \frac{d}{dt} t\, dt + \int_0^y f(x+it) \frac{d}{dt}(x+it)\, dt \\
&= \int_0^x \cos t \cosh 0 - i \sin t \sinh 0\, dt + \int_0^y [\cos x \cosh t - i \sin x \sinh t]\,(i)\, dt \\
&= [\sin t]|_0^x + i\,[\cos x \sinh t - i \sin x \cosh t]|_0^y \\
&= \sin x + i\,[\cos x \sinh y - i \sin x \cosh y - (-i \sin x)] \\
&= \sin x - \sin x + \sin x \cosh y + i \cos x \sinh y \\
&= \sin x \cosh y + i \cos x \sinh y.
\end{aligned}$$

This integral will be independent of the path chosen by the Cauchy integral theorem, since the function $f(z)$ is analytic everywhere in the plane – this follows from the Cauchy-Riemann equations, or from noting that $f(z) = \cos z$.

Having noted that $f(z) = \cos z$, we note also that $F(z) = \sin z$, so that F is indeed an antiderivative of f, as it should be.

Marking: 2 marks for choosing a piecewise-smooth path from 0 to z; 5 marks for evaluating the integral (roughly as follows: 1 mark each for calculating γ' and substituting in γ correctly, 3 marks for actually evaluating the t integrals); 1 mark for invoking the Cauchy integral theorem (or similar argument), 1 mark for noting that this holds since f is analytic (some justification for f (some justification is required).

2. [9 marks] Using the Cauchy integral formula, evaluate the following integrals:

$$\int_\gamma \frac{\cos z}{z}\, dz, \qquad \gamma \text{ the square with sidelength 2 centred at the origin, oriented counterclockwise.}$$

$$\int_\gamma \frac{1}{(z-z_0)^2}\, dz, \qquad \gamma \text{ any simple closed curve containing the point } z_0\text{, in any orientation.}$$

$$\int_\gamma \frac{e^z}{z}\, dz, \qquad \gamma \text{ the unit circle, oriented } \textit{clockwise}.$$

We recall the Cauchy integral formula. If f is a function which is analytic on and within a simple closed curve C, and z_0 is some point within this curve, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'.$$

This formula applies to smooth and piecewise-smooth curves.

Thus for the first integral we may take $n = 0$, $C = \gamma$, $z_0 = 0$, and $f(z) = \cos z$ to obtain

$$\int_\gamma \frac{\cos z}{z}\,dz = \int_\gamma \frac{f(z')}{z'-z_0}\,dz' = 2\pi i f(z_0) = 2\pi i.$$

For the second integral, we may take $n = 1$, $C = \gamma$, $z_0 = z_0$, and $f(z) = 1$ to obtain

$$\int_\gamma \frac{1}{(z-z_0)^2}\,dz = \int_\gamma \frac{f(z')}{(z'-z_0)^2}\,dz' = \frac{2\pi i}{1!}f'(z_0) = 0,$$

since $f'(z) = 0$ for all z as f is constant.

Finally, for the third integral we may take $n = 0$, $z_0 = 0$, $f(z) = e^z$; taking C to be γ in the *opposite* orientation, hence oriented counterclockwise, and noting that changing the notation only changes the integral by introducing an extra minus sign, we have

$$\int_\gamma \frac{e^z}{z}\,dz = -\int_C \frac{f(z')}{z'-z_0}\,dz' = -2\pi i f(z_0) = -2\pi i e^0 = -2\pi i.$$

Marking: For the first integral, 1 mark for identifying f, 1 mark for identifying $z_0 = 0$, 1 mark for the final answer. For the second integral, 1 mark for identifying $n = 1$, 1 mark for identifying f, 1 mark for the final answer. For the third integral, 1 mark for identifying f, 1 mark for the minus sign, 1 mark for the final answer.

3. [6 marks] Let γ denote the unit circle, oriented counterclockwise, and let $z^{1/2}$ denote any branch of the square root function (be sure to clearly indicate which one you are using!). By direct computation, evaluate the integral

$$\int_\gamma \frac{z^{1/2}}{z}\,dz,$$

where we can evaluate the integral since the function is defined and bounded everywhere except at a single point on the curve (alternatively, you can view the above integral as a limit of an open segment of the circle as the two endpoints come towards the branch cut). Does your result contradict the Cauchy integral theorem or formula? Why or why not?

Just to make things interesting, let us take the branch obtained by cutting along the line $\theta = 5\pi/4$ and requiring the angle to lie in $(-3\pi/4, 5\pi/4)$. Note that we may parameterise the unit circle by

$$\gamma(t) = \cos t + i\sin t, \quad t \in [\theta_0, \theta_0 + 2\pi],$$

where θ_0 is any real number. Since by our choice of branch the angle – and hence the parameter value t – must lie in the interval $(-3\pi/4, 5\pi/4)$, we choose $\theta_0 = -3\pi/4$. We still have to figure out how to perform the integral when the function is not defined on the branch cut itself. There are a couple ways of looking at this. Probably the most rigorous one is that given in the parenthesis in the problem statement: essentially, consider the resulting t integral as an improper integral and evaluate it by taking limits towards the endpoints. Since along γ we may write

$$z^{1/2} = \cos\frac{t}{2} + i\sin\frac{t}{2},$$

this gives

$$\lim_{L_1\to -3\pi/4^-}\ \lim_{L_2\to 5\pi/4^+}\int_{L_1}^{L_2} \frac{\cos\frac{t}{2} + i\sin\frac{t}{2}}{\cos t + i\sin t}(-\sin t + i\cos t)\,dt;$$

i.e., instead of integrating from $-3\pi/4$ to $5\pi/4$, we integrate from some value L_1 to some other value L_2, both of which are inside the interval $(-3\pi/4, 5\pi/4)$, and then take the limit as they approach the two endpoints. The integral above may be evaluated as follows. Note that $\cos t + i\sin t = e^{it}$, so $1/(\cos t + i\sin t) = e^{-it} = \cos t - i\sin t$ (this can also be determined directly, using division of complex numbers, of course); thus the above integral equals

$$\begin{aligned}\int_{L_1}^{L_2}\left[\cos\frac{t}{2} + i\sin\frac{t}{2}\right](\cos t - i\sin t)(-\sin t + i\cos t)\,dt &= \int_{L_1}^{L_2}\left[\cos\frac{t}{2} + i\sin\frac{t}{2}\right] i\,dt \\ &= 2i\left[\sin\frac{t}{2} - i\cos\frac{t}{2}\right]\Bigg|_{L_1}^{L_2}.\end{aligned}$$

Now note that the result here is a continuous function of L_1 and L_2, so we may evaluate the limits above by substituting in the limiting values $L_1 = -3\pi/4$ and $L_2 = 5\pi/4$ (this is basically what the remark in the problem statement that 'we can evaluate the integral since the function is defined and bounded everywhere except at a single point on the curve' was getting at!); recalling that $\sin(x - \pi) = -\sin x$ and $\cos(x - \pi) = -\cos x$, we obtain

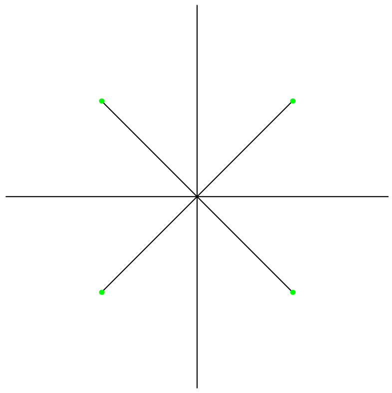
$$\begin{aligned}\int_\gamma \frac{z^{1/2}}{z}\,dz &= 2i\left[\left(\sin\frac{5\pi}{8} - i\cos\frac{5\pi}{8}\right) - \left(\sin -\frac{3\pi}{8} - i\cos -\frac{3\pi}{8}\right)\right] \\ &= 2i\left[2\sin\frac{5\pi}{8} - 2i\cos\frac{5\pi}{8}\right] = 4(\cos 5\pi/8 + i\sin 5\pi/8).\end{aligned}$$

Had we instead made a branch cut at $\theta = \alpha$, and required our angle to lie in $(\alpha - 2\pi, \alpha)$, we would evidently have obtained $4(\cos\alpha/2 + i\sin\alpha/2)$ instead. Note though that this number will never be 0 since it always lies on the circle of radius 4 centred at the origin. This does not contradict the Cauchy integral formula – which, if applied naively to the current integral, would have given $2\pi i 0^{1/2} = 0$ – since the function $z^{1/2}$ is not analytic at the origin, which lies inside the curve γ. (It is interesting to note that, had we integrated along a closed curve which wrapped around the origin *twice*, and used the full root function rather than a branch, the integral would have been zero. This is related to the fact that $z^{1/2}$ is double-valued. In this context, we should note that a closed curve which wraps twice around the origin cannot be continuously deformed into a curve wrapping once around the origin without passing through the origin.)

Marking: picking a branch; parameterising the curve with the correct interval; correct integrand; integrating; final answer; explanation, 1 mark each. Explicitly evaluating the integral as an improper one – as done above – was not required for full marks.

1. (a) [16 marks] Find all complex numbers z which satisfy the equation $1 + z^4 = 0$, and plot them on the complex plane.

We are given $z^4 + 1 = 0$, so $z^4 = -1$. Suppose that $z = r(\cos\theta + i\sin\theta)$; then $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$, so we must have $r^4(\cos 4\theta + i\sin 4\theta) = \cos\pi + i\sin\pi$, hence $r = 1$, $4\theta = \pi + 2n\pi$ for some $n \in \mathbf{Z}$, or $\theta = \pi/4 + n\pi/2$, $n \in \mathbf{Z}$. Since values of θ differing by 2π represent the same point z, we have four distinct values for z, with $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. These can be drawn as follows:



[2 marks for each correct expression for a root, 2 marks for each correct plotted root.]

(b) [16 marks] Write $z = x + iy$, expand out z^4, and use the Cauchy-Riemann equations to show that $1 + z^4$ is analytic at all points in the complex plane. [This part corresponds to Quiz 1.]

We have

$$\begin{aligned}(x+iy)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^{4-k}(iy)^k = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \\ &= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3). \qquad \text{[4 marks]}\end{aligned}$$

Letting P denote the real and Q the imaginary part of $1 + z^4$, we have [2 marks/derivative, 8 marks total]

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial P}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial Q}{\partial x} = 12x^2y - 4y^3, \quad \frac{\partial Q}{\partial y} = 4x^3 12xy^2,$$

so [1 mark/equation, 2 marks total]

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

and since the derivatives are continuous everywhere [2 marks], we see that $1 + z^4$ is analytic on the whole complex plane.

[Marking: as indicated.]

(c) [4 marks] Use the results of (a) and (b) to find the region in the complex plane where the function

$$\frac{1}{1+z^4}$$

is analytic (in the sense of having a complex derivative).

By the quotient rule and (b)[1 mark], $1/(1 + z^4)$ will be analytic as long as the denominator is nonzero [1 mark]. By (a), this will happen when $z \neq \cos(\pi/4 + n\pi/2) + i\sin(\pi/4 + n\pi/2)$ for any $n \in \mathbf{Z}$ [1 mark]. Thus $1/(1 + z^4)$ is analytic on $\mathbf{C}\backslash\{e^{i(\pi/4+n\pi/2)} \,|\, n \in \mathbf{Z}\}$.[1 mark]

[Marking: as indicated.]

The following is for Question 2 and Question 3:

Now let us define a (potentially multi-valued) function f of the complex variable z by the rule

$$f(z) = \int_0^z \frac{1}{1+z'^4}\, dz',$$

where the value of f may depend on the curve chosen from 0 to z (if so, then f will be multi-valued). If x is a real number, let

$$g(x) = \int_0^x \frac{1}{1+u^4}\, du,$$

where the integral is the usual real-variable integral; in other words, $g(x)$ is $f(x)$ where the contour defining f is required to lie along the real axis.

2. (a) [12 marks] Consider the function $f(iy)$, where the contour is taken along the imaginary axis. By parameterising this contour, show how to express $f(iy)$ in terms of $g(y)$.

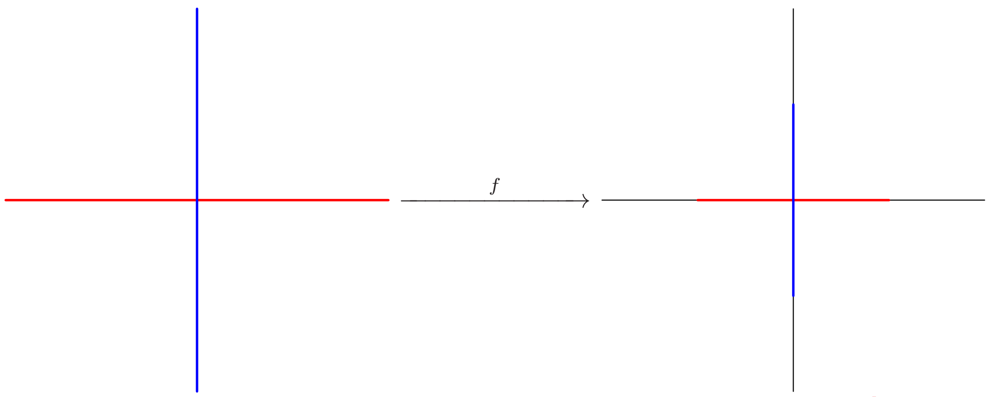
Let $y \in \mathbf{R}$, and define the curve $\gamma : [0,1] \to \mathbf{C}$ by $\gamma(t) = ity$ [2 marks]; then γ will be a curve along the imaginary axis from 0 to iy, and along this curve we have, since $\gamma'(t) = iy$[1 mark],

$$\int_\gamma \frac{1}{1+z^4}\, dz = \int_0^1 \frac{1}{1+(ity)^4} iy\, dt[2\text{ marks}] = \int_0^1 \frac{1}{1+(ty)^4} iy\, dt[2\text{ marks}]$$
$$= i\int_0^y \frac{1}{1+u^4}\, du[3\text{ marks}] = ig(y)[2\text{ marks}].$$

[Marking: as indicated.]

(b) [12 marks] Use your result from (a) to draw the image of the real and imaginary axes under the function $z \mapsto f(z)$, where in both cases we require the contours used to lie along the respective axes. (You may assume that the function g maps the real line onto some open interval around 0.)

Evidently, along the real axis, $f(x) = g(x)$ [2 marks], while from (a), $f(iy) = ig(y)$ [2 marks]. Thus f will take the real axis into some open interval around 0 on the real axis [2 marks], and the imaginary axis into some open interval around 0 on the imaginary axis [2 marks]. If we indicate this pictorially, we have



[2 marks each for correctly indicating the mapping of the real and imaginary axis]

[Marking: as indicated. Some explanation in words of the image sets should be given, but it can be less formal than what is here.]

(c) [8 marks] What is $f'(0)$? Can we conclude that f is conformal at $z = 0$? Explain how this relates to your picture in (b).

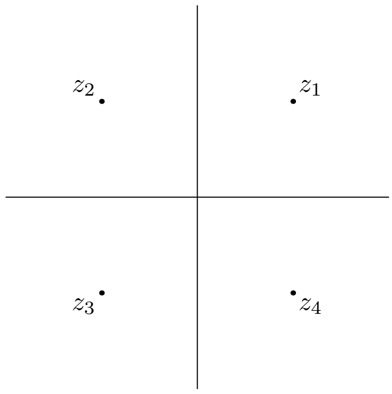
Since $1/(1+z^4)$ is analytic near $z = 0$ [2 marks], we will have $f'(z) = 1/(1+z^4)$ for z near 0, and in particular $f'(0) = 1/(1+0^4) = 1/1 = 1$ [2 marks]. Since f' exists and is nonzero, we see that f is conformal at $z = 0$ [2 marks]. This is exemplified by the fact that the image 'curves' (in this case, line segments) in (b) make the same angle with each other as the original ones do [2 marks].

[Marking: as indicated.]

3. This is a continuation of Question 2.

(a) [8 marks] Find all points z at which f does not possess a complex derivative. (You should give a reason for your answer, but you do not need to give a full proof.) Plot these points on the complex plane. Label them z_1, z_2, z_3, z_4, in any order you wish. [Hint: it will be useful to have z_1 and z_2 lie on the same side of the real axis.]

By the fundamental theorem of calculus [2 marks] [Goursat, §31], f will have a derivative at every point at which $1/(1+z^4)$ does [2 marks]; hence it must have a derivative at every point except the points $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, as we discussed in 1(c) above [2 marks]. We plot these as follows: [2 marks]



[Marking: as indicated. A citation to Goursat or anywhere else is of course not required.]

(b) [16 marks] Use your solution to 1(a) to factor $1+z^4$, and then apply the Cauchy integral formula to determine the value of the integral

$$\int_\gamma \frac{1}{1+z'^4}\,dz',$$

where γ is any curve enclosing z_1 but none of the other points you found in (a). Simplify your answer as much as possible.

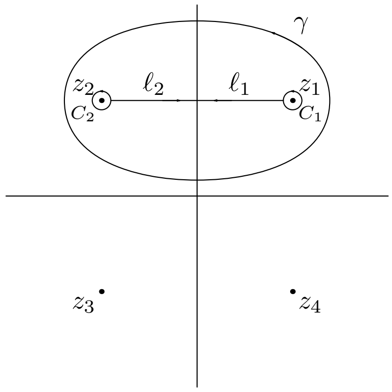
We see that we may write $1+z^4=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ [4 marks]. Since γ does not enclose z_2, z_3, or z_4, the function $1/(z-z_2)(z-z_3)(z-z_4)$ will be analytic on and within γ, and hence we may apply the Cauchy integral formula to write[2 marks]

$$\begin{aligned}\int_\gamma \frac{1}{1+z^4}\,dz &= \int_\gamma \frac{\frac{1}{(z-z_2)(z-z_3)(z-z_4)}}{z-z_1}\,dz && \text{[3 marks]}\\ &= \frac{2\pi i}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} && \text{[2 marks]}\\ &= \frac{2\pi i}{\sqrt{2}(\sqrt{2}+i\sqrt{2})(\sqrt{2}i)} = \frac{2\pi i}{2^{3/2}(-1+i)}\\ &= \pi\frac{1-i}{2^{3/2}}. && \text{[5 marks]}\end{aligned}$$

[Marking: as indicated. The answer must be simplified to a numeric form for full marks.]

(c) [32 marks] Repeat (b), but now let γ enclose only z_1 and z_2. [Hint: can you see how to use the Cauchy integral theorem to replace γ with two small circles around z_1 and z_2?]

Consider the following picture: [2 marks for each of the curves C_1, C_2]



By the Cauchy integral theorem, we have

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{C_1}\frac{1}{z+z^4}\,dz + \int_{C_2}\frac{1}{1+z^4}\,dz + \int_{\ell_1}\frac{1}{1+z^4}\,dz + \int_{\ell_2}\frac{1}{1+z^4}\,dz;$$

now the last two integrals cancel, since $1/(1+z^4)$ is continuous along the line ℓ_1 and ℓ_2, and so we have

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{C_1} \frac{1}{1+z^4}\,dz + \int_{C_2} \frac{1}{1+z^4}\,dz. \qquad \text{[4 marks]}$$

But now $\int_{C_1} \frac{1}{1+z^4}\,dz = \pi\frac{1-i}{2^{3/2}}$[10 marks] by (b); and $\int_{C_2} \frac{1}{1+z^4}\,dz$ can be computed in the same way:

$$\begin{aligned}
\int_{C_2} \frac{1}{1+z^4}\,dz &= \int_{C_2} \frac{\frac{1}{(z-z_1)(z-z_3)(z-z_4)}}{z-z_2}\,dz \qquad \text{[3 marks]}\\
&= \frac{2\pi i}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\text{[2 marks]} = \frac{2\pi i}{-\sqrt{2}(\sqrt{2}i)(-\sqrt{2}+\sqrt{2}i)}\\
&= -\frac{2\pi}{2^{3/2}(-1+i)} = \pi\frac{1+i}{2^{3/2}}, \qquad \text{[5 marks]}
\end{aligned}$$

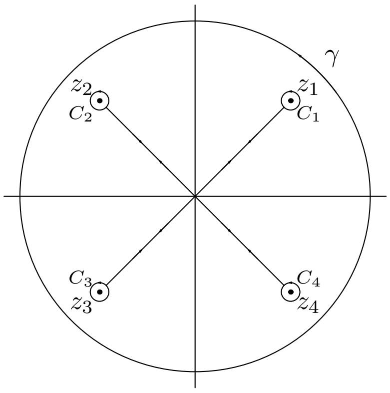
so

$$\int_\gamma \frac{1}{1+z^4}\,dz = -\pi i\frac{(1+i)^2}{2^{3/2}} = \frac{\pi}{2^{1/2}}. \qquad \text{[4 marks]}$$

[Marking: as indicated. The lines ℓ_1 and ℓ_2 do not need to be used for full marks.]

(d) [32 marks] Repeat (b), but now let γ enclose all four points.

Consider the following picture: [1 mark for each curve C_1]



By the same logic as in (c), we see that

$$\int_\gamma \frac{1}{1+z^4}\,dz = \int_{C_1} \frac{1}{1+z^4}\,dz + \int_{C_2} \frac{1}{1+z^4}\,dz + \int_{C_3} \frac{1}{1+z^4}\,dz + \int_{C_4} \frac{1}{1+z^4}\,dz. \qquad \text{[4 marks]}$$

We have already calculated the first two integrals; the remaining two may be calculated similarly:

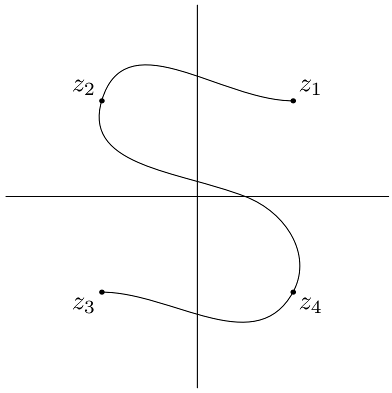
$$\begin{aligned}
\int_{C_3} \frac{1}{1+z^4}\,dz &= \int_{C_3} \frac{\frac{1}{(z-z_1)(z-z_2)(z-z_4)}}{z-z_3}\,dz \qquad \text{[3 marks]}\\
&= \frac{2\pi i}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\text{[2 marks]} = \frac{2\pi i}{-(\sqrt{2}+\sqrt{2}i)(-\sqrt{2}i)(-\sqrt{2})}\\
&= \frac{2\pi i}{2^{3/2}(1-i)} = \pi\frac{-1+i}{2^{3/2}} \qquad \text{[5 marks]}\\
\int_{C_4} \frac{1}{1+z^4}\,dz &= \frac{2\pi i}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}\text{[5 marks]} = \frac{2\pi i}{(-\sqrt{2}i)(\sqrt{2}-\sqrt{2}i)(\sqrt{2})}\\
&= -\frac{2\pi i}{2^{3/2}(1+i)} = -\pi\frac{1+i}{2^{3/2}}, \qquad \text{[5 marks]}
\end{aligned}$$

and we see that $\int_\gamma \frac{1}{1+z^4}\,dz = 0$.[4 marks]

[Marking: as indicated. Again, the lines between the curves do not need to be given for full marks.]

(e) [24 marks] Now let C be any simple (non-selfintersecting) piecewise-smooth curve whose endpoints are any two of the points z_1, z_2, z_3, z_4, which also passes through the remaining two points [Hint: it will be useful to have it start at z_1 and go to z_2 next], and which crosses the real axis exactly once, at some point $x_0 > 0$. Draw this curve on the plane you drew in (a). Let $D = \mathbf{C}\backslash C$ denote the complex plane with the curve C removed. Use your result from (d), together with the Cauchy integral theorem if necessary, to show that if we require the contour in the definition of f to be strictly within D, then f becomes a single-valued function.

We choose the following curve: [2 marks for each of the following: (i) initial point is one of the z_i; (ii) curve passes through all four z_i; (iii) end point is one of the z_i; (iv) curve only crosses the real axis once.]



Now let z be any point in the plane not lying on C, and let γ_1, γ_2 be two curves in $\mathbf{C}\backslash C$ from 0 to z [2 marks]. Then if any of the points z_1, z_2, z_3, z_4 lies in between γ_1 and γ_2, the whole curve C must also lie between them [6 marks]; by (d), then, the integral

$$\int_{\gamma_1-\gamma_2} \frac{1}{1+z^4}\,dz = \int_{\gamma_1} \frac{1}{1+z^4}\,dz - \int_{\gamma_2} \frac{1}{1+z^4}\,dz$$

must vanish[3 marks]. If none of the points lie between γ_1 and γ_2, then this integral will vanish by the Cauchy integral theorem[3 marks]. Thus in either case,

$$\int_{\gamma_1} \frac{1}{1+z^4}\,dz = \int_{\gamma_2} \frac{1}{1+z^4}\,dz,$$

and f will be single-valued.[2 marks]

[Marking: as indicated.]

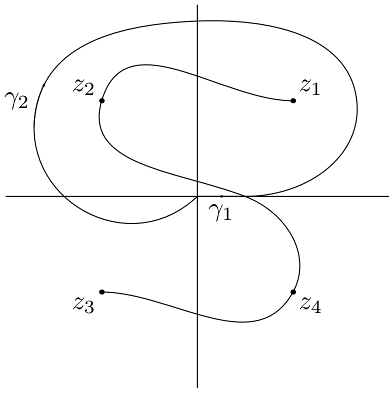
(f) [16 marks] Using the single-valued version of f described in (e), calculate

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z),$$

where $z \to x_0^{\pm}$ means that z approaches from the right (+) or left (−) of the curve C. [Hint: can you see how to apply your result from (c)?] Does this difference of limits depend on your choice of C? (You do not need to give a justification.)

[2(a) and 3(f) correspond to Quiz 2.]

Since f is single-valued, we may use any two curves γ_1, γ_2[3 marks]. We use the following curves: [2 marks for each curve; curves should form a closed loop containing only two of the z_i – it doesn't matter which two – and not intersecting the curve C except at its intersection with the real axis]



Now $1/(1+z^4)$ is continuous at x_0, so in the limit

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z) = \int_{\gamma_2} \frac{1}{1+z^4}\,dz - \int_{\gamma_1} \frac{1}{1+z^4}\,dz$$

$$\text{[2 marks]} = -\int_{\gamma_1-\gamma_2} \frac{1}{1+z^4}\,dz \text{[3 marks]} = -\frac{\pi}{2^{1/2}} \text{[2 marks]}$$

by (c). It is quite clear that this does not depend on the choice of C, since regardless of the choice of C the curve $\gamma_1 - \gamma_2$ must enclose exactly the two points z_1 and z_2, which entirely determines the integral of $1/(1+z^4)$ over $\gamma_1 - \gamma_2$.[2 marks]

[Marking: as indicated.]

1. (a) [16 marks] Find all complex numbers z which satisfy the equation $z^4 - z^2 + 1 = 0$, and plot them on the complex plane. [Hint: remember the quadratic formula: $az^2 + bz + c = 0$ has solutions

$$z = \frac{1}{2a}\left(-b + (b^2 - 4ac)^{1/2}\right),$$

where the square root denotes the full (multi-valued) complex square root.]

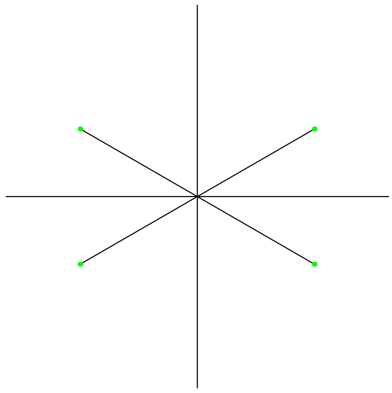
We may apply the quadratic formula to this equation by making the substitution $u = z^2$; then $u^2 - u + 1 = 0$, so

$$u = \frac{1}{2}\left(1 + (1-4)^{1/2}\right) = \frac{1}{2}(1 \pm i\sqrt{3}) = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{\pm i\pi/3}.$$

Now $u = z^2$, so we obtain the four solutions

$$z = e^{\pm i\pi/6},\ e^{\pm i(\pi/6+\pi)} = e^{i\pi/6},\ e^{5i\pi/6},\ e^{7i\pi/6},\ e^{11i\pi/6}.$$

These can be drawn as follows:



[2 marks for each correct expression for a root, 2 marks for each correct plotted root.]

(b) [16 marks] Write $z = x + iy$, expand out z^4, and use the Cauchy-Riemann equations to show that it is analytic at all points in the complex plane. [This part corresponds to Quiz 1.]

We have

$$\begin{aligned}(x+iy)^4 &= \sum_{k=0}^{4}\binom{4}{k} x^{4-k}(iy)^k = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \\ &= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3). \qquad \text{[4 marks]}\end{aligned}$$

Letting P denote the real and Q the imaginary part of z^4, we have [2 marks/derivative, 8 marks total]

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial P}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial Q}{\partial x} = 12x^2y - 4y^3, \quad \frac{\partial Q}{\partial y} = 4x^3 12xy^2,$$

so [1 mark/equation, 2 marks total]

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

and since the derivatives are continuous everywhere [2 marks], we see that z^4 is analytic on the whole complex plane.

[Marking: as indicated.]

(c) [4 marks] Use the results of (a) and (b) to find the region in the complex plane where the function

$$\frac{1}{1 - z^2 + z^4}$$

is analytic (in the sense of having a complex derivative). [You may assume without proof that z^2 is analytic.]

By the quotient rule and (b)[1 mark], as well as the fact that z^2 is analytic, $1/(1 - z^2 + z^4)$ will be analytic as long as the denominator is nonzero [1 mark]. By (a), this will happen when $z \neq e^{i\pi/6}$, $e^{5i\pi/6}$, $e^{7i\pi/6}$, $e^{11i\pi/6}$ [1 mark]. Thus $1/(1 - z^2 + z^4)$ is analytic on $\mathbf{C}\backslash\{e^{i\pi/6}, e^{5i\pi/6}, e^{7i\pi/6}, e^{11i\pi/6}\}$.[1 mark]

[Marking: as indicated.]

The following is for Question 2 and Question 3:

Now let us define a (potentially multi-valued) function f of the complex variable z by the rule

$$f(z) = \int_0^z \frac{1}{1 - z'^2 + z'^4}\, dz',$$

where the value of f may depend on the curve chosen from 0 to z (if so, then f will be multi-valued). If x is a real number, let

$$g(x) = \int_0^x \frac{1}{1 + u^2 + u^4}\, du,$$

where the integral is the usual real-variable integral; in other words, $g(x)$ is $f(x)$ where the curve defining f is required to lie along the real axis.

2. (a) [12 marks] Consider the function $f(iy)$, where the curve is taken along the imaginary axis. By parameterising this curve, show how to express $f(iy)$ in terms of $g(y)$. [Here y is an arbitrary real number.]

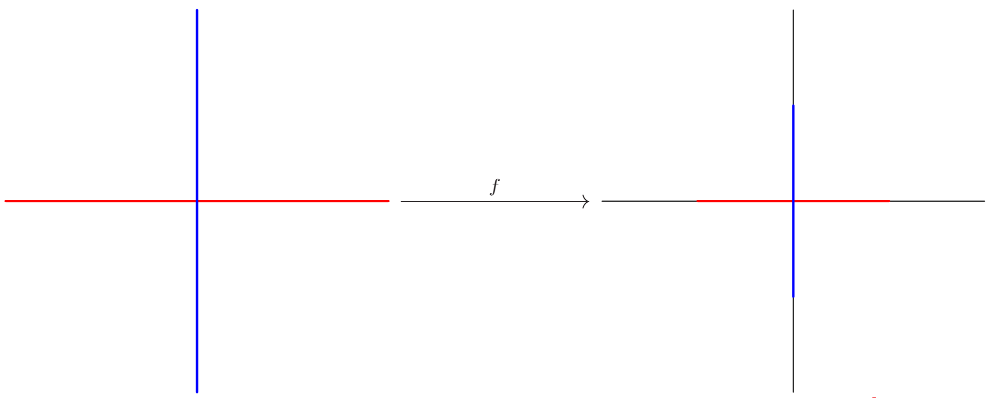
Let $y \in \mathbf{R}$, and define the curve $\gamma : [0, 1] \to \mathbf{C}$ by $\gamma(t) = ity$ [2 marks]; then γ will be a curve along the imaginary axis from 0 to iy, and along this curve we have, since $\gamma'(t) = iy$[1 mark],

$$\int_\gamma \frac{1}{1 - z^2 + z^4}\, dz = \int_0^1 \frac{1}{1 - (ity)^2 + (ity)^4} iy\, dt \text{[2 marks]} = \int_0^1 \frac{1}{1 + (ty)^2 + (ty)^4} iy\, dt \text{[2 marks]}$$
$$= i \int_0^y \frac{1}{1 + u^2 + u^4}\, du \text{[3 marks]} = ig(y) \text{[2 marks]}.$$

[Marking: as indicated.]

(b) [12 marks] Use your result from (a) to draw the image of the real and imaginary axes under the function $z \mapsto f(z)$, where in both cases we require the curves used to lie along the respective axes. (You may assume that the function g maps the real line onto some open interval around 0.)

Evidently, along the real axis, $f(x) = \int_0^x \frac{1}{1-x^2+x^4}\, dx$ [2 marks], while from (a), $f(iy) = ig(y)$ [2 marks]. Thus f will take the real axis into some open interval around 0 on the real axis [2 marks], and the imaginary axis into some open interval around 0 on the imaginary axis [2 marks]. If we indicate this pictorially, we have



[2 marks each for correctly indicating the mapping of the real and imaginary axis]

[Marking: as indicated. Some explanation in words of the image sets should be given, but it can be less formal than what is here.]

(c) [8 marks] What is $f'(0)$? Can we conclude that f is conformal at $z = 0$? Explain how this relates to your picture in (b).

Since $1/(1 - z^2 + z^4)$ is analytic near $z = 0$ [2 marks], we will have $f'(z) = 1/(1 - z^2 + z^4)$ for z near 0, and in particular $f'(0) = 1/(1 - 0^2 + 0^4) = 1/1 = 1$ [2 marks]. Since f' exists and is nonzero, we see that

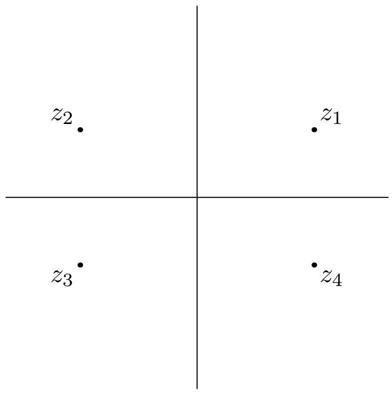
f is conformal at $z = 0$ [2 marks]. This is exemplified by the fact that the image 'curves' (in this case, line segments) in (b) make the same angle with each other as the original ones do [2 marks].

[Marking: as indicated.]

3. This is a continuation of Question 2.

(a) [8 marks] Find all points z at which f does not possess a complex derivative. (You should give a reason for your answer, but you do not need to give a full proof.) Plot these points on the complex plane. Label them z_1, z_2, z_3, z_4, in any order you wish. [Hint: it will be useful to have z_1 and z_2 lie on the same side of the real axis.]

By the fundamental theorem of calculus [2 marks] [Goursat, §31], f will have a derivative at every point at which $1/(1 - z^2 + z^4)$ does [2 marks]; hence it must have a derivative at every point except the points $e^{i\pi/6}$, $e^{5i\pi/6}$, $e^{7i\pi/6}$, $e^{11i\pi/6}$, as we discussed in 1(c) above [2 marks]. We plot these as follows: [2 marks]



[Marking: as indicated. A citation to Goursat or anywhere else is of course not required.]

In the following three parts, you are free to choose the orientation of the curve.

(b) [16 marks] Use your solution to 1(a) and your notation in (a) to factor $1 - z^2 + z^4$, and then apply the Cauchy integral formula to determine the value of the integral

$$\int_\gamma \frac{1}{1 - z^2 + z'^4}\, dz',$$

where γ is any curve enclosing z_1 but none of the other points you found in (a). Simplify your answer as much as possible.

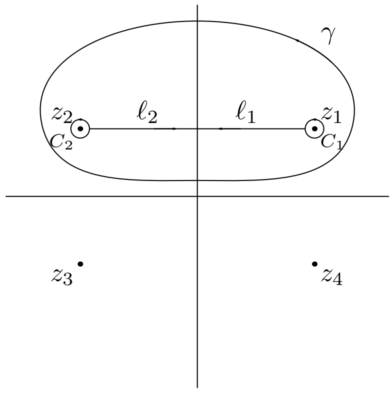
We see that we may write $1 - z^2 + z^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$ [4 marks]. Since γ does not enclose z_2, z_3, or z_4, the function $1/(z - z_2)(z - z_3)(z - z_4)$ will be analytic on and within γ, and hence we may apply the Cauchy integral formula to write[2 marks]

$$\begin{aligned}
\int_\gamma \frac{1}{1 - z^2 + z^4}\, dz &= \int_\gamma \frac{\frac{1}{(z-z_2)(z-z_3)(z-z_4)}}{z - z_1}\, dz && \text{[3 marks]}\\
&= \frac{2\pi i}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} && \text{[2 marks]}\\
&= \frac{2\pi i}{\sqrt{3}(\sqrt{3} + i)(i)} = \frac{2\pi i}{3 + i\sqrt{3}}\\
&= \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right). && \text{[5 marks]}
\end{aligned}$$

[Marking: as indicated. The answer must be simplified to a numeric form for full marks.]

(c) [32 marks] Repeat (b), but now let γ enclose only z_1 and z_2. [Hint: can you see how to use the Cauchy integral theorem to replace γ with two small circles around z_1 and z_2?]

Consider the following picture: [2 marks for each of the curves C_1, C_2]



By the Cauchy integral theorem, we have

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz + \int_{\ell_1} \frac{1}{1-z^2+z^4}\,dz + \int_{\ell_2} \frac{1}{1-z^2+z^4}\,dz;$$

now the last two integrals cancel, since $1/(1-z^2+z^4)$ is continuous along the line ℓ_1 and ℓ_2, and so we have

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz. \qquad \text{[4 marks]}$$

But now $\int_{C_1} \frac{1}{1-z^2+z^4}\,dz = \pi\frac{1-i}{2^{3/2}}$[10 marks] by (b); and $\int_{C_2} \frac{1}{1-z^2+z^4}\,dz$ can be computed in the same way:

$$\begin{aligned}\int_{C_2} \frac{1}{1-z^2+z^4}\,dz &= \int_{C_2} \frac{\frac{1}{(z-z_1)(z-z_3)(z-z_4)}}{z-z_2}\,dz \qquad \text{[3 marks]}\\ &= \frac{2\pi i}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\text{[2 marks]} = \frac{2\pi i}{-\sqrt{3}(i)(-\sqrt{3}+i)}\\ &= -\frac{2\pi i}{3-i\sqrt{3}} = \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right), \qquad \text{[5 marks]}\end{aligned}$$

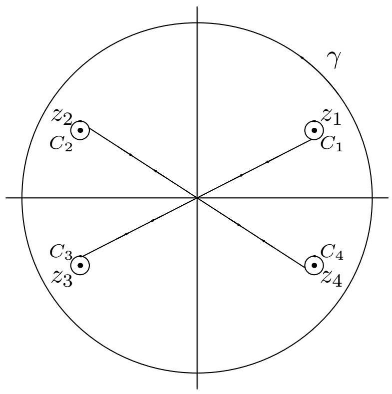
so

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \frac{\pi\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}+\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = \pi. \qquad \text{[4 marks]}$$

[Marking: as indicated. The lines ℓ_1 and ℓ_2 do not need to be used for full marks.]

(d) [32 marks] Repeat (b), but now let γ enclose all four points.

Consider the following picture: [1 mark for each curve C_1]



By the same logic as in (c), we see that

$$\int_\gamma \frac{1}{1-z^2+z^4}\,dz = \int_{C_1} \frac{1}{1-z^2+z^4}\,dz + \int_{C_2} \frac{1}{1-z^2+z^4}\,dz + \int_{C_3} \frac{1}{1-z^2+z^4}\,dz + \int_{C_4} \frac{1}{1-z^2+z^4}\,dz.$$

[4 marks]

We have already calculated the first two integrals; the remaining two may be calculated similarly:

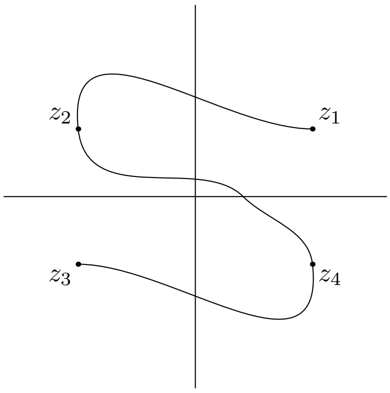
$$\begin{aligned}
\int_{C_3} \frac{1}{1-z^2+z^4}\,dz &= \int_{C_3} \frac{\frac{1}{(z-z_1)(z-z_2)(z-z_4)}}{z-z_3}\,dz \qquad \text{[3 marks]}\\
&= \frac{2\pi i}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\text{[2 marks]} = \frac{2\pi i}{-(\sqrt{3}+i)(-i)(-\sqrt{3})}\\
&= \frac{2\pi i}{\sqrt{3}-3i} = \frac{\pi\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) \qquad \text{[5 marks]}\\
\int_{C_4} \frac{1}{1-z^2+z^4}\,dz &= \frac{2\pi i}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}\text{[5 marks]} = \frac{2\pi i}{(-i)(\sqrt{3}-i)\sqrt{3}}\\
&= \frac{2\pi i}{-\sqrt{3}-3i} = \frac{\pi\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right), \qquad \text{[5 marks]}
\end{aligned}$$

and we see that $\int_\gamma \frac{1}{1+z^4}\,dz = 0$.[4 marks]

[Marking: as indicated. Again, the lines between the curves do not need to be given for full marks.]

(e) [24 marks] Now let C be any simple (non-selfintersecting) piecewise-smooth curve whose endpoints are any two of the points z_1, z_2, z_3, z_4, which also passes through the remaining two points [Hint: it will be useful to have it start at z_1 and go to z_2 next], and which crosses the real axis exactly once, at some point $x_0 > 0$. Draw this curve on the plane you drew in (a). Let $D = \mathbf{C}\backslash C$ denote the complex plane with the curve C removed. Use your result from (d), together with the Cauchy integral theorem if necessary, to show that if we require the contour in the definition of f to be strictly within D, then f becomes a single-valued function.

We choose the following curve: [2 marks for each of the following: (i) initial point is one of the z_i; (ii) curve passes through all four z_i; (iii) end point is one of the z_i; (iv) curve only crosses the real axis once.]



Now let z be any point in the plane not lying on C, and let γ_1, γ_2 be two curves in $\mathbf{C}\backslash C$ from 0 to z [2 marks]. Then if any of the points z_1, z_2, z_3, z_4 lies in between γ_1 and γ_2, the whole curve C must also lie between them [6 marks]; by (d), then, the integral

$$\int_{\gamma_1-\gamma_2} \frac{1}{1-z^2+z^4}\,dz = \int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz - \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz$$

must vanish[3 marks]. If none of the points lie between γ_1 and γ_2, then this integral will vanish by the Cauchy integral theorem[3 marks]. Thus in either case,

$$\int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz = \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz,$$

and f will be single-valued.[2 marks]

[Marking: as indicated.]

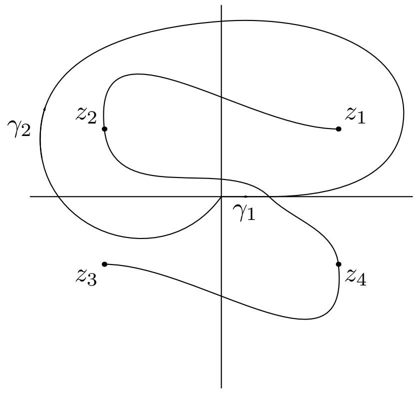
(f) [16 marks] Using the single-valued version of f described in (e), calculate

$$\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z),$$

where $z \to x_0^\pm$ means that z approaches from the right $(+)$ or left $(-)$ of the curve C. [Hint: can you see how to apply your result from (c)?] Does this difference of limits depend on your choice of C? (You do not need to give a justification.)

[2(a) and 3(f) correspond to Quiz 2.]

Since f is single-valued, we may use any two curves γ_1, γ_2[3 marks]. We use the following curves: [2 marks for each curve; curves should form a closed loop containing only two of the z_i – it doesn't matter which two – and not intersecting the curve C except at its intersection with the real axis]



Now $1/(1-z^2+z^4)$ is continuous at x_0, so in the limit

$$\begin{aligned}\lim_{z\to x_0^+} f(z) - \lim_{z\to x_0^-} f(z) &= \int_{\gamma_2} \frac{1}{1-z^2+z^4}\,dz - \int_{\gamma_1} \frac{1}{1-z^2+z^4}\,dz \qquad \text{[2 marks]}\\ &= -\int_{\gamma_1-\gamma_2} \frac{1}{1-z^2+z^4}\,dz\text{[3 marks]} = -\pi\text{[2 marks]}\end{aligned}$$

by (c). It is quite clear that this does not depend on the choice of C, since regardless of the choice of C the curve $\gamma_1 - \gamma_2$ must enclose exactly the two points z_1 and z_2, which entirely determines the integral of $1/(1-z^2+z^4)$ over $\gamma_1 - \gamma_2$.[2 marks]

[Marking: as indicated.]

Summary:
- We give an additional formula for the coefficients in a Laurent series expansion, and discuss how to determine the region of convergence of the series.
- We then discuss poles, essential singularities, and zeros, based on Taylor and Laurent series expansions.
- We define residues, and state the residue theorem.

(Goursat, §§40 – 43.)

24. Laurent series, revisited. Recall from before the term test that if we have a function f, analytic between two circles both centred at a, say C and C', with C' contained inside C, then if we define

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \geq 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'} (z'-a)^{n-1} f(z')\,dz', \quad (n \geq 1) \tag{1}$$

we have the following series expansion for f:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}, \tag{2}$$

and the series on the right-hand side converge on the annular region between C and C'. Now since f is analytic between C and C', as are the quantities $(z'-a)^{-(n+1)}$ and $(z'-a)^{n-1}$ in the definitions of a_n and b_n, the Cauchy integral theorem allows us to evaluate a_n and b_n over *any* simple closed curve γ which lies entirely in the annular region between C and C'. Suppose that we therefore replace C and C' in (1) by γ. Then, noting that we may write

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{-n+1}}\,dz',$$

we see that if we define

$$J_n = \begin{cases} a_n, & n \geq 0, \\ b_{-n}, & n < 0, \end{cases}$$

we see that we have for all $n \in \mathbf{Z}$, positive and negative,

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{n+1}}\,dz', \tag{3}$$

and moreover that we may write the series expansion for f given above as

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=1}^{\infty} J_{-n}(z-a)^{-n} \\ &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=-1}^{-\infty} J_n (z-a)^n = \sum_{n=-\infty}^{\infty} J_n (z-a)^n, \end{aligned} \tag{4}$$

where this last sum is essentially defined by the expression on the right-hand side of the first line.[0]

There is no essential difference between the expansions (2) and (4); they are just different ways of expressing the same information. On the other hand, the expression (3) is certainly more symmetric and probably easier to remember than the expression (1). We pay for this in some sense, though, by the fact that the expansion (4) in some sense hides the singular terms, and we must keep in mind that if n is negative the terms $(z-a)^n$ are singular at $z = a$.

25. Region of convergence. There are various ways of determining the region of convergence of a power or Laurent series. Perhaps the most elementary way is to apply the root test we learned in elementary

[0] Note that this is how we define improper integrals of the form $\int_{-\infty}^{+\infty} f(x)\,dx$ in elementary calculus: we pick some point $a \in \mathbf{R}$ and define the integral to be the sum $\int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$, much as we may define this last series as $\sum_{n=-\infty}^{-1} J_n(z-a)^n + \sum_{n=0}^{\infty} J_n(z-a)^n$.

calculus. Let us first recall the root test for a series of positive real numbers: if $c_n \geq 0$ for all n, then the series

$$\sum_{n=0}^{\infty} c_n$$

will converge if the quantity $\lim_{n\to\infty} c_n^{1/n} < 1$ and diverge if $\lim_{n\to\infty} c_n^{1/n} > 1$; if $\lim_{n\to\infty} c_n^{1/n} = 1$ then the test is indeterminate and we must do something else. Now suppose that we have a Taylor series

$$\sum_{n=0}^{\infty} a_n(z-a)^n.$$

This series will converge if it is *absolutely convergent*, i.e., if the series of absolute values

$$\sum_{n=0}^{\infty} |a_n(z-a)^n|$$

converges. If we apply the root test to this series, we see that the series is convergent if

$$\lim_{n\to\infty} |a_n(z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} < 1$$

and divergent if

$$\lim_{n\to\infty} |a_n(z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} > 1.$$

Let us now define the quantity R by

$$R^{-1} = \lim_{n\to\infty} |a_n|^{1/n};$$

if this limit is zero we set $R = \infty$, while if the limit is infinite we set $R = 0$. Then from the foregoing we see that the series will converge if

$$|z-a| < R$$

and diverge if

$$|z-a| > R.$$

(If $|z-a| = R$, the root test fails and the series may either converge or diverge.)

Now suppose that we consider instead the Laurent series

$$\sum_{n=1}^{\infty} b_n(z-a)^{-n};$$

by exactly the same logic, if in this case we define R' by

$$R' = \lim_{n\to\infty} |b_n|^{1/n},$$

then we see that the series will converge if

$$|z-a|^{-1} < \frac{1}{R'}$$

and diverge if

$$|z-a|^{-1} > \frac{1}{R'};$$

in other words, it will converge if

$$|z-a| > R'$$

and diverge if

$$|z-a| < R'.$$

(As usual, the test fails when $|z - a| = R'$, meaning that it tells us nothing about the convergence or divergence of the series.) This means that, while Taylor series converge on disks, the singular part of a Laurent series converges instead on the *exterior* of a disk. The full Laurent series, being the sum of a Taylor series and a singular part, will converge on the intersection of one disk with the exterior of another disk, i.e., on an annulus (exactly as we might expect!).[1]

The preceding method will allow us to find the region where any given Taylor or Laurent series converges, assuming that we can calculate the two limits involved. Thus it is useful when the only thing we know is the series itself. On the other hand, if we know the function f to which the series converges, then there is a much simpler method, as follows. Let us consider Laurent series; the same logic applies to Taylor series (which are, after all, just Laurent series without no singular part, i.e., with singular part equal to 0). Suppose that f is a function which is analytic between the circles C' and C, centred at a (with C' inside of C as usual). Then our results with Laurent series show that the Laurent series of f converges to f everywhere on this annulus. Now if f were analytic on a larger annulus, say between circles C_1' and C_1, also centred at a, then its Laurent series on the annulus between C_1' and C_1 would converge between C_1' and C_1; but by the Cauchy integral theorem (for example, recall that we can define the coefficients using any curve γ lying in the annular region) the coefficients of this Laurent series would be the same as those of the original Laurent series. Thus the original Laurent series must also converge on this larger annulus.

If we take this reasoning to its logical conclusion, we see that the Laurent series will converge on the largest annular region on which f is analytic. To be a bit more careful, recall that we need f to be continuous on the boundary curves C and C'; thus the Laurent series will converge on the largest annular region such that f is analytic in the interior and continuous on the boundary. This means that, if we are attempting to find where the Laurent series for a *given* function f is convergent, we need only determine the points where f is not analytic; the Laurent series will then converge on the largest annular region, centred at a, which does not contain any of these points.

(It is worth pointing out that this method does *not* apply to the case where we are simply given a Laurent series and do not know the function to which it converges. This is because it is in general not possible to determine the points where a function is not analytic simply by examining its Laurent series.)

Let us give an example.

EXAMPLE. Let us define a function f by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}.$$

We will find where these series converge, and then discuss how to use the formulas above to determine its Laurent series. (In a certain sense this second part is quite pointless since a Laurent series is unique, so the series expansion on the right-hand side *is* the Laurent expansion of f around 0. But on the other hand it will give us some practice in the use of the general formulas.) First of all, it can be shown that

$$\lim_{n\to\infty} (n!)^{1/n} = \infty;$$

thus also

$$\lim_{n\to\infty} [(n!)^2]^{1/n} = \infty,$$

and so the first series must converge on the entire complex plane.

Now it can be shown, using L'Hôpital's rule, that

$$\lim_{n\to\infty} n^{1/n} = 1,$$

[1] Note that it is, in principle, quite possible that this intersection might be empty, i.e., that the disk *outside* of which the singular part converges is larger than the disk *inside* of which the analytic part converges. In this case, the series simply does not converge at any point in the complex plane. If we obtained this result by starting from some specific function, it would indicate that the function was not analytic on any annular region centred at a. (This does not, incidentally, mean that the function is never analytic, just that the region on which it is analytic does not contain any annulus around a.)

whence

$$\lim_{n\to\infty}[n^2]^{1/n} = 1,$$

and the quantity R' above will be 1, meaning that the second series converges when

$$|z| > 1.$$

We pause here for a moment to discuss the relation of this last result to our second method for determining the region of convergence of a Laurent series. Let us define

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}$$

wherever this series converges. Since the series diverges for $|z| < 1$, the function g must have some singularity on the unit circle. Now the series is clearly absolutely convergent on the unit circle itself (since if $|z| = 1$ the absolute value of the terms of the series is simply $1/n^2$, giving a convergent series); thus g does not diverge at any point on the unit circle. Let us look at its derivative. Differentiating term-by-term, we obtain

$$g'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n-1}}{n};$$

thus

$$zg'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n}}{n},$$

so

$$\frac{d}{dz}[zg'(z)] = \sum_{n=1}^{\infty} z^{-n-1} = z^{-2}\sum_{n=0}^{\infty} z^{-n} = z^{-2}\frac{1}{1-z^{-1}} = \frac{1}{z^2-z} = \frac{1}{z(z-1)}.$$

Now this function clearly has a singularity at $z = 1$, which is on the unit circle. While it is not entirely clear how to determine the full function g given this rather peculiar differential operator on g, we can say that if g were analytic at $z = 1$, then so would be $d/dz[zg'(z)]$; since this latter quantity, as just noted, is *not* analytic at $z = 1$, g cannot be analytic there either. (Some further study suggests that g in fact has a branch point at $z = 1$, but we shall not show that here.) This explains why the series cannot converge on a larger annulus.

Proceeding, let us see how to calculate the coefficients in the Laurent series expansion for f using the formula above. (This is very similar to, though more complicated than, the example we did right before the break.) We shall use the formula (3) with $a = 0$, and γ any curve contained in the region $|z| > 1$:

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{z'^{\,n+1}}\,dz'.$$

Substituting in the series definition of f, we have

$$J_n = \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\frac{z'^{\,k}}{z'^{\,n+1}} + \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{z'^{,k+n+1}}\,dz',$$

so that if we assume we can interchange sum and integral, then we only need to evaluate the integrals

$$\int_\gamma z'^{\,k-n-1}\,dz', \qquad \int_\gamma z'^{-k-n-1}\,dz'.$$

It is sufficient to determine

$$\int_\gamma z'^{\,m}\,dz'$$

where m is any integer (positive or negative). Now if $m \geq 0$, then the integrand will be analytic everywhere on the plane (as usual, we set $z^0 = 1$ for all z by convention). Now suppose that $m \leq -1$, and write $n = -m - 1 \geq 0$; then by the general Cauchy integral formula

$$\begin{aligned}\int_\gamma z'^{\,m}\,dz' = \int_\gamma \frac{1}{z'^{\,-m}}\,dz' = \int_\gamma \frac{1}{z'^{\,n+1}}\,dz' \\ = 2\pi i \frac{d^n}{dz^n}[1]|_{z=0},\end{aligned}$$

which will be $2\pi i$ if $n = 0$ and 0 if $n > 0$, since the derivative of a constant is 0. Pulling all of this together, then, we see that

$$\int_\gamma z'^{\,m}\,dz' = \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1. \end{cases}$$

(This is a very useful formula to keep in mind, in general.) Applying this to the above formula for J_n, we see that for $n \geq 0$

$$\begin{aligned}J_n &= \frac{1}{2\pi i}\sum_{k=0}^\infty \frac{1}{(k!)^2}\int_\gamma z'^{\,k-n-1}\,dz' + \frac{1}{2\pi i}\sum_{k=1}^\infty \frac{1}{k^2}\int_\gamma z'^{\,-k-n-1}\,dz' \\ &= \frac{1}{(n!)^2},\end{aligned}$$

since the first integral will be nonzero only when $k - n - 1 = -1$, i.e., $k = n$, in the which case it equals $2\pi i$, while the second will be nonzero only when $-k - n - 1 = -1$, i.e., when $k = -n$; since in the second series $k \geq 1$, while $n \geq 0$, this is impossible. Similarly, when $n \leq -1$

$$\begin{aligned}J_n &= \frac{1}{2\pi i}\sum_{k=0}^\infty \frac{1}{(k!)^2}\int_\gamma z'^{\,k-n-1}\,dz' + \frac{1}{2\pi i}\sum_{k=1}^\infty \frac{1}{k^2}\int_\gamma z'^{\,-k-n-1}\,dz' \\ &= \frac{1}{(-n)^2},\end{aligned}$$

by the foregoing. Thus we may write

$$J_n = \begin{cases} \frac{1}{(n!)^2}, & n \geq 0, \\ \frac{1}{n^2}, & n \leq -1, \end{cases}$$

which is exactly the coefficients in the original series (as advertised).

26. Isolated singularities. We will now apply Laurent series to study the ways in which functions can fail to be analytic at a single point. More precisely, suppose that $a \in \mathbf{C}$, and let f be a function whose domain contains a. If f is analytic at a, then a is called a *regular point* of f; if f is not analytic at a, then a is called a *singular point* of f. Now suppose that a is a singular point, but that f is analytic everywhere else near a; i.e., that there is some $r > 0$ such that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\},$$

i.e., f is analytic everywhere on the disk of radius r centred at a, except of course at a itself. In this case a is called an *isolated singular point* of f. (Note that branch points are *not* isolated singular points.)

Let us see what we can learn about isolated singular points from Laurent series. Suppose that a is an isolated singular point of a function f, and that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\}.$$

Then on any annular region contained in this punctured disk we will have the Laurent series expansion

$$f = \sum_{n=0}^\infty a_n(z-a)^n + \sum_{n=1}^\infty b_n(z-a)^{-n},$$

where a_n and b_n are given by the formulas (1) above, and moreover do not depend on the particular choice of annular region; thus we may consider this to be an expansion for f on the entire punctured disk (which is, strictly speaking, *not* an annular region in the sense in which we have been using that term). Let us consider the singular part of this expansion, namely

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

There are three possibilities: (i) $b_n = 0$ for all n; (ii) $b_n \neq 0$ for only finitely many n; (iii) $b_n \neq 0$ for infinitely many n. In the first case, it can be shown that by defining $f(a) = a_0$, the function f can be made analytic on the whole disk $\{z \,|\, |z-a| < r\}$, so that the 'singularity' is not really a singularity at all; this is called a 'removable singularity'. In the second case we say that the function has a *pole* at a, while in the third case we say that it has an *essential singularity* at a.

Let us give a couple examples.

EXAMPLES. 1. Let $f = 1/\sin z$; then clearly f is analytic everywhere except where $\sin z = 0$, i.e., everywhere except for $z = n\pi$, $n \in \mathbf{Z}$. Each of these points must therefore be an isolated singularity of f. Let us consider the isolated singularity at $z = 0$, and see if we can determine the Laurent series for f there. Now we have

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Now the final series above must converge on the entire complex plane; essentially, this is because it converges at $z = 0$, and other than at that point it is equal to a function analytic on the punctured plane $\{z \,|\, z \neq 0\}$. (If you want to use this fact in your homework solutions, you need to write in a bit more detail!) Let us denote this function by $\phi(z)$, i.e.,

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Then since $\phi(0) = 1 \neq 0$, there must be some disk D around 0 on which $\phi \neq 0$. On this disk, then, $1/\phi$ must be analytic, and hence can be expanded as

$$\frac{1}{\phi(z)} = \sum_{k=0}^{\infty} a_k z^k$$

for some set of coefficients a_k, which we could determine by division of series but won't. Thus f can be written as

$$f(z) = \frac{1}{z\phi(z)} = \frac{1}{z} \sum_{k=0}^{\infty} a_k z^k = \frac{a_0}{z} + \sum_{k=0}^{\infty} a_{k+1} z^k.$$

Thus $z = 0$ is a pole for f. We say that it is a *pole of order 1*; we shall define this in general momentarily.

2. Now let us consider the function $f(z) = e^{1/z}$. Clearly, f is analytic everywhere except at $z = 0$, so that 0 is an isolated singular point of f. Now for any complex number z we have

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k;$$

thus for any nonzero complex number z we must have

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k},$$

and this must therefore be the Laurent expansion for f around 0. Since f has infinitely many nonzero coefficients in its singular part, 0 must be an essential singularity for f.

To return to our general treatment, suppose that a is a pole of a function f; this means that a is an isolated singularity of f, and that the singular part of the Laurent expansion of f around a has finitely many nonzero coefficients. Suppose that m is the largest integer for which $b_m \neq 0$, i.e., that $b_m \neq 0$ while $b_k = 0$ for all $k > m$; then we say that a is a pole *of order m* of f. This explains our terminology in the first example above.

In other words, a function f has a pole of order m at a if near a

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \cdots + \frac{b_m}{(z-a)^m},$$

and moreover $b_m \neq 0$.

There is a nice relationship between poles and zeros. Recall that a polynomial $p = a_0 + \cdots + a_n z^n$ is said to have a *zero of order m* at a point a if it is divisible by $(z-a)^m$, i.e., if there is another polynomial q such that $q(a) \neq 0$ and

$$p(z) = (z-a)^m q(z).$$

Now evidently this same definition can be applied to Taylor series. Specifically, suppose that a function f is analytic near a point a; then we may expand it in its Taylor series about a as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Now f is said to have a *zero of order m* at a if there is an analytic function ϕ near a such that $\phi(a) \neq 0$ and

$$f(z) = (z-a)^m \phi(z).$$

In this case, it is easy to see that the first m terms of the Taylor series for f must all vanish, i.e., that we must have

$$f(z) = a_m (z-a)^m + a_{m+1}(z-a)^{m+1} + \cdots.$$

Now consider $1/f(z)$ near $z = a$; by the foregoing, we have

$$\frac{1}{f(z)} = \frac{1}{(z-a)^m \phi(z)} = \frac{1}{\phi(z)}(z-a)^{-m};$$

now since $\phi(a) \neq 0$, as we saw in Example 1 above, $\phi(z)$ must be nonzero on some disk centred at a; thus $1/\phi(z)$ must be analytic on this disk, say with power series

$$\frac{1}{\phi(z)} = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

and moreover $c_0 = \frac{1}{\phi(a)} \neq 0$. Thus

$$\begin{aligned}\frac{1}{f(z)} &= \frac{1}{(z-a)^m} \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+m} (z-a)^n + \frac{c_{m-1}}{z-a} + \cdots + \frac{c_m}{(z-a)^m},\end{aligned}$$

which shows that $1/f$ must have a pole of order m at $z = a$. This logic works to show the reverse implication also, namely that if a function g has a pole of order m at $z = a$, then $1/g$ must have a zero of order m at $z = a$. Thus, in some sense, poles and zeroes are complementary to each other.

27. Residues. Suppose that a is an isolated singular point of the function f, and let γ be a simple closed curve in the punctured disk about a on which f is analytic, and which moreover encloses the point a. Then we may write the Laurent series for f about a,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

Thus, assuming that we can interchange sum and integral, we have

$$\int_\gamma f(z')\,dz' = \sum_{n=0}^{\infty} a_n \int_\gamma (z'-a)^n\,dz' + \sum_{n=1}^{\infty} b_n \int_\gamma (z'-a)^{-n}\,dz';$$

by the same logic we used in the examples in the previous section, all of these integrals will vanish except for the one with the power $(z'-a)^{-1}$, and that one will give $2\pi i$. Thus we have

$$\int_\gamma f(z')\,dz' = 2\pi i b_1.$$

The quantity b_1, which is the coefficient of the $(z'-a)^{-1}$ term in the Laurent expansion of f about $z=a$, is called the *residue* of f at the point a.

At present, this logic might seem slightly circular, since the equation above is actually equivalent to the definition of b_1 given above. If we had no other way to find Laurent series than through the definitions of a_n and b_n in terms of integrals, then this would indeed be circular. However, as our examples above have hopefully suggested, there are ways of computing Laurent series that do not involve calculating integrals at all. This means that there are other methods of computing residues. These methods allow us to apply the formula above in meaningful ways.

We may generalise this result as follows to obtain an even more useful one. Suppose that we begin with a curve γ, and that f is analytic everywhere inside γ except at certain isolated singular points $z_1, \cdots, z_n$. Let β_j denote the residue of f around z_j, for $j = 1, \cdots, n$. Then it follows from the generalised Cauchy theorem that we may write

$$\int_\gamma f(z')\,dz' = 2\pi i \sum_{j=1}^{n} \beta_j.$$

This result, known as the *residue theorem*, is extremely useful in the applications we shall make of contour integrals to evaluating definite integrals on the real line.

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 6 – 10

Due Tuesday, July 14, at 3:30 PM EDT.

1. Using the formulas in Goursat, §§35, 37, find the Laurent series for

$$f(z) = \frac{\sin z}{z}$$

around $z = 0$. How does this series compare to the Taylor series for $\sin z$ around $z = 0$? On what set does it converge? Justify your answer.

Let us write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

and let γ be a circle around $z = 0$; then we have the formulas

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z')}{(z'-0)^{n+1}}\, dz'$$
$$b_n = \frac{1}{2\pi i} \int_\gamma f(z')(z'-0)^{n-1}\, dz'.$$

Substituting in $f(z) = (\sin z)/z$, the first of these gives

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{\sin z'}{z'^{n+2}}\, dz' \text{[1 mark]}$$
$$= \frac{1}{2\pi i} \cdot \frac{2\pi i}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \sin z \bigg|_{z=0} = \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \sin z \bigg|_{z=0}; \text{[2 marks]}$$

thus $a_0 = 1$, $a_1 = 0$, $a_2 = -1/6$, $a_3 = 0$, etc., and in general $a_{2k+1} = 0$ while $a_{2k} = (-1)^k/(2k+1)!$[1 mark]. Now

$$b_1 = \frac{1}{2\pi i} \int_\gamma \frac{\sin z'}{z'}\, dz' \text{[1 mark]} = \frac{1}{2\pi i} \sin 0 = 0, \text{[1 mark]}$$

while if $n > 1$

$$b_n = \frac{1}{2\pi i} \int_\gamma \sin z'(z'-0)^{n-2}\, dz' = 0 \text{[1 mark]}$$

by the Cauchy integral theorem, since the integrand is analytic everywhere. Thus we have

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}. \text{[1 mark]}$$

Now recall that the Taylor series for $\sin z$ is

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1};$$

in other words, the Laurent series for $(\sin z)/z$ is just the Taylor series for $\sin z$, divided term-by-term by z, as we might expect[1 mark]. Since the Taylor series for $\sin z$ converges everywhere on the complex plane, the Laurent series above will converge to an analytic function everywhere except possibly at $z = 0$; but at $z = 0$ the series also clearly converges to 1, so the series converges on the entire complex plane.[1 mark]

2. Again using the formulas in Goursat, §§35, 37, find the Laurent series for

$$f(z) = \frac{e^z}{(z-2)^2}$$

around $z = 2$. How does this compare to the Taylor series for e^z around $z = 2$? [Hint: recall that $e^{a+b} = e^a e^b$; can you use this to find the Taylor series?] On what set does this Laurent series converge? Again, justify your answer.

We use the same formulas as in question 1. Thus we have, first of all, letting now γ denote a circle around $z = 2$,

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-2)^{n+1}}\,dz'$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{(z'-2)^{n+3}}\,dz'\text{[1 mark]} = \frac{1}{(n+3)!}\frac{d^{n+2}}{dz^{n+2}}e^z\Big|_{z=2}\text{[2 marks]} = \frac{e^2}{(n+2)!},\text{[1 mark]}$$

by the Cauchy integral formula for derivatives, while

$$b_1 = \frac{1}{2\pi i}\int_\gamma f(z')\,dz'\text{[1 mark]} = \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{(z'-2)^2}\,dz' = e^2,\text{[1 mark]}$$
$$b_2 = \frac{1}{2\pi i}\int_\gamma f(z')(z'-2)\,dz' = \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{z'-2}\,dz' = e^2,\text{[1 mark]}$$

while for $n > 2$

$$b_n = \frac{1}{2\pi i}\int_\gamma f(z')(z'-2)^{n-1}\,dz' = \frac{1}{2\pi i}\int_\gamma e^{z'}(z'-2)^{n-3}\,dz' = 0\text{[1 mark]}$$

by the Cauchy integral theorem. Thus we have the expansion

$$f(z) = \frac{e^2}{(z-2)^2} + \frac{e^2}{z-2} + \sum_{n=0}^\infty \frac{e^2}{(n+2)!}(z-2)^n.\text{[2 marks]}$$

Now the Taylor series for e^z around $z = 2$ can be found as follows:

$$e^z = e^2 e^{z-2} = e^2\sum_{n=0}^\infty \frac{1}{n!}(z-2)^n = \sum_{n=0}^\infty \frac{e^2}{n!}(z-2)^n;\text{[2 marks]}$$

from this we see that the above series is simply this series, divided by $(z-2)^2$ term-by-term, as we would expect.[1 mark]

Now the Taylor series for e^z around $z = 2$ converges on the entire complex plane; thus the series above for f will converge everywhere where it is defined, i.e., on the punctured plane $\{z \in \mathbf{C} \mid z \neq 2\}$.[2 marks]

Summary:
- We give examples of applications of the residue theorem to the evaluation of definite integrals on the real axis.
- We then give theorems for showing that integrals over semicircles go to zero, and provide additional methods for computing residues.

(Goursat, §§44 – 46.)

28. Evaluation of definite integrals. Recall the residue theorem from last time: if f is a function which is continuous on a simple closed curve γ, and analytic inside γ except potentially at a finite number of isolated singularities $z_1, \cdots, z_n$, at which it has residues $\beta_1, \cdots, \beta_n$, respectively. Then we have

$$\int_\gamma f(z')\,dz' = 2\pi i \sum_{j=1}^n \beta_j.$$

Let us see by way of an example – which could also be done by elementary methods – how this result can be applied to evaluate definite integrals.

EXAMPLE. Consider the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx.$$

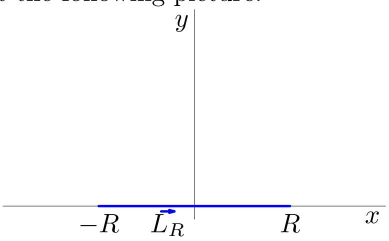
Since this integral converges, it is equal to the limit

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx;$$

since $\arctan x$ is an antiderivative of $1/(1+x^2)$, by the fundamental theorem of calculus this integral is equal to

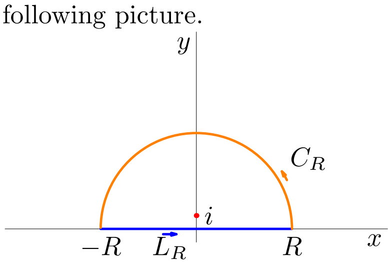
$$\lim_{R\to+\infty} (\arctan R - \arctan(-R)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Now suppose that we consider $\int_{-R}^{R} \frac{1}{1+x^2}\,dx$ as a contour integral in the complex plane, with the contour taken along the real axis; then we get the following picture.



As it stands this does not seem to have gotten us anywhere. Note though that the integrand here is analytic on the entire plane except for (simple) poles at $\pm i$. Thus if it were possible to somehow *augment* the contour L_R in order to obtain a closed curve (we speak of *closing the contour*), we would be able to apply either the Cauchy integral theorem – if the closed curve did not contain either of the poles – or the residue theorem – otherwise – in order to evaluate the integral over the full closed contour. If, additionally, it were possible somehow to compute the integral over the additional contour, at least in the limit of large R, we would then be able to compute our original integral.

In general there are multiple ways of closing the contour; i.e., there are multiple different possible choices for the additional curve to be used to produce a closed contour from the original one. Consider the semicircle C_R in the upper half-plane as in the following picture.



Remember that our technique will only be useful if we have a way of computing $\int_{C_R} 1/(1+z^2)\,dz$; we claim that in fact

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2}\,dz = 0.$$

In class we showed this by parameterising the curve C_R and considering the resulting integrand; here we use a slightly simpler method. Recall that, if f is continuous on a simple closed curve γ, with maximum M on γ, then

$$\left|\int_\gamma f(z)\,dz\right| \leq M|\gamma|,$$

where $|\gamma|$ is the length of the curve γ. Now clearly $|C_R| = \pi R$ (since C_R is a *semi*circle of radius R); further, if z is any point on C_R then we may write $z = Re^{it}$ for some $t \in [0, \pi]$, and thus

$$\left|\frac{1}{1+z^2}\right| = \left|\frac{1}{1+R^2e^{2it}}\right| = \left|\frac{R^{-2}}{e^{2it}+R^{-2}}\right| = R^{-2}\left|e^{2it}+R^{-2}\right|^{-1}.$$

Now by the triangle inequality we may write

$$\left|e^{2it}+R^{-2}\right| \geq \left|e^{2it}\right| - \left|R^{-2}\right| = 1 - R^{-2},$$

so that when we take the limit $R \to \infty$ we have

$$R^{-2}\left|e^{2it}+R^{-2}\right|^{-1} \leq R^{-2}(1-R^{-2})^{-1} \leq \frac{1}{R^2-1},$$

which goes to zero. Thus

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2}\,dz = 0$$

as claimed.

Now for any $R > 1$, the closed curve $L_R + C_R$ will enclose the single pole at i. Let us calculate the residue of $1/(1+z^2)$ at $z = i$. We have

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{1/(z+i)}{z-i};$$

if we think of expanding $1/(z+i)$ as

$$\frac{1}{z+i} = \sum_{n=0}^{\infty} c_n(z-i)^n$$

(which we can do since $1/(z+i)$ is analytic near i), then

$$\frac{1}{1+z^2} = \frac{c_0 + c_1(z-i) + c_2(z-i)^2 + \cdots}{z-i} = \frac{c_0}{z-i} + c_1 + c_2(z-i) + \cdots,$$

and the residue of $1/(1+z^2)$ is clearly c_0. But $c_0 = 1/(z+i)|_{z=i} = 1/(2i)$. Thus at the end of the day we have for all $R > 1$

$$\int_{L_R} \frac{1}{1+z^2}\,dz + \int_{C_R} \frac{1}{1+z^2}\,dz = 2\pi i \cdot \frac{1}{2i} = \pi;$$

and in the limit $R \to \infty$, the integral over L_R goes to $\int_{-\infty}^{+\infty} 1/(1+x^2)\,dx$ while the integral over C_R goes to zero. Thus at the end of the day

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi,$$

exactly as we found above.

There are two key points in the above procedure: (i) we have to find a curve C_R (which need not be a semicircle, in general, or any segment of a circular path) which will close the contour L_R, and over which we can integrate f; (ii) we have to evaluate the residues of f at its singularities inside the closed contour $L_R + C_R$. We shall now give methods for addressing these two points: first, by providing general conditions under which integrals along circular arcs like C_R go to zero as $R \to \infty$; second, by providing additional methods for calculating residues.

29. When $\int_{C_R} f(z)\,dz \to 0$. First we have the following fairly straightforward generalisation of the example from the previous section. Suppose that f is a function analytic on the exterior of a circle of radius R for a suitably large R (in other words, if f has any singularities they are not too far from the origin). Suppose that for suitably large R there is a number M_R such that for all $z \in C_R$ we have $|f(z)| < M_R$, and that $\lim_{R\to\infty} RM_R = 0$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

To see this, note that the length of C_R is πR; thus

$$\left|\int_{C_R} f(z)\,dz\right| \leq \pi R M_R,$$

and as this latter quantity goes to zero by assumption, the integral must also, by the squeeze theorem.

We may apply this to the example in the previous section as follows. Suppose that $z \in C_R$. Then we have

$$|f(z)| = \left|\frac{1}{1+z^2}\right| \geq \frac{1}{|z^2|-1} = \frac{1}{R^2-1},$$

where we have used the triangle inequality as before; since $R/(R^2-1) \to 0$ as $R \to \infty$, the above result shows that $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$ as well.

We note in passing that it is actually sufficient to show that if $z \to \infty$ along circles C_R, then we must have $\lim_{z\to\infty} zf(z) = 0$. More precisely, what this means is that $zf(z)$ can be made arbitrarily small by taking $z \in C_R$ with $|z| = R$ arbitrarily large. In cases like the foregoing this is easier to apply, since we have

$$\lim_{z\to\infty} \frac{z}{1+z^2} = \lim_{z\to\infty} \frac{z^{-1}}{1+z^{-2}},$$

and since the numerator goes to 0 while the denominator goes to 1, the fraction must go to zero. To be fully rigorous, though, we would have to explain how this kind of a limit – restricting z to lie on a particular family of curves – relates to usual limits, but we shall not do that here. This method can also be used to show easily what we saw in class: suppose that

$$f(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials, and $\deg Q \geq \deg P + 2$. Then we can write

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z(a_0 + \cdots + a_n z^n)}{b_0 + \cdots + b_{n+2} z^{n+2}},$$

where $b_{n+2} \neq 0$ but we may have $a_n = 0$. By dividing numerator and denominator by z^{n+2}, this becomes

$$\lim_{z\to\infty} \frac{a_n z^{-1} + a_{n-1} z^{-2} + \cdots + a_0 z^{-n-1}}{b_{n+2} + b_{n+1} z^{-1} + \cdots + b_0 z^{-n-2}} = 0,$$

since the numerator goes to 0 while the denominator goes to $b_{n+2} \neq 0$.

Let us do an example.

EXAMPLE. Compute

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx.$$

We note first that the integrand has poles in the complex plane at $\pm i$, just like the example above. Let us now consider what kind of contour we can use to close the line L_R. Now we have

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Now suppose that $z = a + ib$; then

$$\cos z = \frac{1}{2}\left(e^{ia-b} + e^{-ia+b}\right),$$

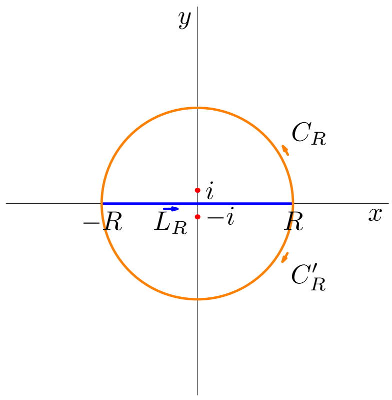
and we see that the first term goes to zero on the upper half-plane ($b > 0$) while the second term goes to infinity exponentially there, and vice versa on the lower half-plane. (We ignore for the moment what happens on the real axis when $b = 0$.) Thus it does not seem that there is any way of closing the contour so as to have $\int_{C_R} f(z)\,dz = 0$ as regardless of whether C_R is in the upper or lower half-plane the integrand will have one term going to infinity.

There are two ways of dealing with this. The more general one is to split the original integral up into two pieces,

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{2(1+x^2)}\,dx, \qquad \int_{-\infty}^{+\infty} \frac{e^{-ix}}{2(1+x^2)}\,dx,$$

and then closing these two integrals in the upper and lower half-plane, respectively; for example, using in turn the curves C_R and C_R' in the following figure. We shall see similar cases to this in the future. For now we use a simpler method. Note that we have also

$$\cos z = \operatorname{Re} e^{iz},$$



so since we are integrating along the real axis, we may write

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\operatorname{Re} e^{ix}}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{1+x^2}\,dx.$$

We try closing this integral in the upper half-plane as described above; thus let C_R denote a semicircle from R to $-R$ in the upper half-plane, as indicated in the above figure. If $z = a + ib \in C_R$, then $b \geq 0$, so we have

$$\left|e^{iz}\right| = \left|e^{ia}e^{-b}\right| = e^{-b} \leq 1,$$

and

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|1+z^2|} \leq \frac{1}{R^2-1}$$

as before, and since $\lim_{R\to\infty} R/(R^2-1) = 0$ we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{1+z^2}\,dz = 0.$$

Thus we need only calculate the residues of $e^{iz}/(1+z^2)$ in the upper half-plane. Now in the upper half-plane this function is singular only at $z = i$; if we proceed in the same way we did in the previous example (we shall give a general method for this right after this example), we see that the residue is

$$\frac{e^{i \cdot i}}{2i} = \frac{1}{2ei},$$

and finally by the residue theorem and the fact that $\int_{C_R} \frac{e^{iz}}{1+z^2}\, dz \to 0$ as $R \to \infty$, that our original integral is

$$\int_{-\infty}^{+\infty} \frac{e^{iz}}{1+z^2}\, dz = 2\pi i \cdot \frac{1}{2ei} = \frac{\pi}{e}.$$

This is already real, so that we have

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}.$$

30. Methods for computing residues. In the previous two examples we have tacitly applied the following result:

Suppose that $f(z)$ has a simple pole at $z = a$. Then

$$\mathrm{Res}_a f(z) = \lim_{z \to a} (z-a) f(z).$$

This is quite simple to see. Since f has a simple pole, there must be a function $\phi(z)$ which is analytic and nonzero at a such that

$$f(z) = \frac{\phi(z)}{z-a}.$$

Then, proceeding as in the two examples above, it is easy to see that $\mathrm{Res}_a f(z) = \phi(a)$; alternatively, we may use the Cauchy integral formula (here γ is a small circle around a such that f is analytic on and within γ):

$$\mathrm{Res}_a f(z) = \frac{1}{2\pi i} \int_\gamma f(z)\, dz = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{z-a}\, dz = \phi(a).$$

But

$$\phi(a) = \lim_{z \to a} \phi(z) = \lim_{z \to a} (z-a) f(z),$$

which establishes our result.

We may extend the above result to poles of higher order. Suppose that f has instead a pole of order m at $z = a$. Then we may write

$$f(z) = \frac{\phi(z)}{(z-a)^m},$$

where as before ϕ is analytic and nonzero at a; thus (letting as before γ denote a small circle around a within and on which f is analytic)

$$\begin{aligned}
\mathrm{Res}_a f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{(z-a)^m}\, dz = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \phi(z) \bigg|_{z=a} \\
&= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z).
\end{aligned}$$

In other words, to calculate the residue of a function f at a pole of order m, we first multiply f by $(z-a)^m$, differentiate $m-1$ times, evaluate at a, and divide by $(m-1)!$. (Note that $m \geq 1$, so that $m - 1 \geq 0$ and the foregoing makes sense.) Note that this formula reduces to the previous one in the case $m = 1$. Note also that to apply it we must first determine the order of the pole a.

In the case that $f(z) = P(z)/Q(z)$, as before, where P and Q have no common factors and Q has no repeated roots, we see that every pole of f will be simple, and the residue at a pole z_0 will be

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{P(z)}{Q(z)/(z-z_0)} = \frac{P(z_0)}{Q'(z_0)},$$

since $Q(z_0) = 0$ (as z_0 is a pole of f and therefore must be a zero of Q) and this allows us to write

$$\lim_{z\to z_0} \frac{Q(z)}{z - z_0} = \lim_{z\to 0} \frac{Q(z) - Q(z_0)}{z - z_0} = Q'(z_0).$$

$Q'(z_0) \neq 0$ since by assumption the roots of Q are not repeated.

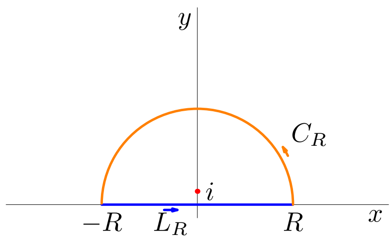
Let us give an example.

EXAMPLES. 1. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + x^2)^2}\, dx.$$

We note that the integrand, extended to the complex plane, has poles at $\pm i$, each of order 2. We expect that we can close the contour using a half-circle C_R from R to $-R$ in the upper half-plane, as we have done in the other examples above (see the picture). To see that we can in fact do this, we apply the result from the previous section: for $z \in C_R$, we have

$$|1 + z^2| \geq R^2 - 1, \qquad |1 + z^2|^2 \geq (R^2 - 1)^2, \qquad \left|\frac{1}{(1 + z^2)^2}\right| \leq \frac{1}{(R^2 - 1)^2},$$



and since $R/(R^2 - 1)^2$ clearly goes to zero as $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{(1 + z^2)^2}\, dz = 0.$$

Thus we need only calculate the residue of $1/(1 + z^2)^2$ at i. Since i is a pole of order 2 of $1/(1 + z^2)^2$, this will be, since $(1 + z^2)^2 = (z - i)^2(z + i)^2$,

$$\frac{1}{(2 - 1)!} \lim_{z\to i} \frac{d^{2-1}}{dz^{2-1}} (z - i)^2 \cdot \frac{1}{(1 + z^2)^2} = \frac{d}{dz} \frac{1}{(z + i)^2}\bigg|_{z=i} = -\frac{2}{(z + i)^3}\bigg|_{z=i} = -\frac{2}{-8i} = -\frac{1}{4}i,$$

and the integral will be

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + x^2)^2}\, dx = 2\pi i \cdot \left(-\frac{1}{4}i\right) = \frac{\pi}{2}.$$

(It is worth pointing out here that it is *always* a good idea to make sure that our final answer makes sense: for example, here we are integrating a real-valued function, so we expect to get a real number as the result; and it is in fact a *positive* real-valued function, so we expect to get a positive real number as the result, as we have. Had we gotten a negative real number, or a complex number with a nonzero imaginary part, it would mean we had made a mistake somewhere earlier which we would need to go back and fix.)

2. Let us evaluate the integral

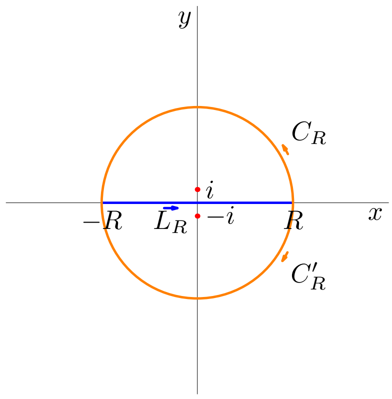
$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1 + x^2)^2}\, dx.$$

Here we clearly have the same kind of issue that we had in our example in section 29 above; namely, whether we close the contour in the upper or the lower half-plane will depend on the sign of k. Let us first suppose that $k \geq 0$. Then if $z = a + ib$ is in the upper half-plane, so that $b \geq 0$, then as in the example just cited we have

$$e^{ikz} = e^{-kb} e^{ika},$$

which is bounded in absolute value by $e^{-kb} \leq 1$; in other words, the integrand here on the semicircle C_R in the next figure will be bounded by the same quantity as we had for $1/(1+z^2)^2$ in the previous example, and the integral over C_R will go to zero as $R \to \infty$ as there. More carefully, recall that we just showed that on C_R

$$\left|\frac{1}{(1+z^2)^2}\right| \leq \frac{1}{(R^2-1)^2};$$



thus when $k \geq 0$ and $z \in C_R$ (so that z is in the upper half-plane)

$$\left|\frac{e^{ikz}}{(1+z^2)^2}\right| \leq \frac{1}{(R^2-1)^2}$$

as well, and since $R/(R^2-1)^2 \to 0$ as $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0$$

by our general results above. Thus it suffices to calculate the residue of $e^{ikz}/(1+z^2)^2$ at the pole i, and since this is still a pole of order 2, its residue is

$$\begin{aligned}\frac{1}{(2-1)!} \lim_{z\to i} \frac{d^{2-1}}{dz^{2-1}} (z-i)^2 \cdot \frac{e^{ikz}}{(1+z^2)^2} &= \frac{d}{dz} \frac{e^{ikz}}{(z+i)^2}\bigg|_{z=i} \\ &= \frac{ike^{ikz}}{(z+i)^2}\bigg|_{z=i} - 2\,\frac{e^{ikz}}{(z+i)^3}\bigg|_{z=i} = \frac{ike^{-k}}{-4} - 2\frac{e^{-k}}{-8i} = -ie^{-k}\left(\frac{1}{4} + \frac{1}{4}k\right),\end{aligned}$$

so that the integral for $k \geq 0$ is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\, dx = 2\pi i \cdot (-i) e^{-k} \left(\frac{1}{4} + \frac{1}{4}k\right) = \frac{\pi}{2} e^{-k} (k+1).$$

Now suppose that $k \leq 0$, and consider the curve C'_R in the above figure. If $z = a + bi \in C'_R$, then $b \leq 0$, so

$$e^{ikz} = e^{-kb} e^{ika}$$

will still be bounded by 1 in absolute value since $k \leq 0$ and $b \leq 0$ implies that $kb \geq 0$, i.e., $-kb \leq 0$. The exact same logic used above now shows that

$$\lim_{R\to\infty} \int_{C'_R} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0,$$

and we are left with calculating the residue at $-i$. This is very similar to calculating the residue at i; it is equal to

$$\frac{d}{dz} \frac{e^{ikz}}{(z-i)^2}\bigg|_{z=-i} = \frac{ike^{ikz}}{(z-i)^2}\bigg|_{z=-i} - 2\,\frac{e^{ikz}}{(z-i)^3}\bigg|_{z=-i} = \frac{ike^{k}}{-4} - 2\frac{e^{k}}{8i} = ie^{k}\left(\frac{1}{4} - \frac{1}{4}k\right).$$

Before we can determine the value of the integral, though, there is one additional wrinkle we have not yet mentioned: note that the curve $L_R + C_R$ was oriented counterclockwise, as required by the Cauchy integral formula; but $L_R + C'_R$ is oriented *clockwise*, which means that we must put in an extra minus sign when applying the Cauchy integral formula. More carefully, we have

$$\int_{L_R} \frac{e^{ikz}}{(1+z^2)^2}\,dz + \int_{C'_R} \frac{e^{ikz}}{(1+z^2)^2}\,dz = -2\pi i \mathrm{Res}_{-i} \frac{e^{ikz}}{(1+z^2)^2};$$

thus, finally, our integral is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = -2\pi i \cdot \left[ie^k\left(\frac{1}{4} - \frac{1}{4}k\right)\right] = \frac{\pi}{2}e^k\,(1-k)\,.$$

Pulling all of this together, then, we have finally that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = \frac{\pi}{2}e^{-|k|}(1+|k|),$$

since $|k| = k$ when $k \geq 0$ and $|k| = -k$ when $k \leq 0$.

For those who know something about Fourier transforms, it is interesting to note the following about the differentiability of this function. If we expand the exponential out in a Taylor series, we see that (dropping the $\pi/2$ coefficient for convenience)

$$\begin{aligned} e^{-|k|}(1+|k|) &= \left(1 - |k| + \frac{1}{2}|k|^2 - \frac{1}{6}|k|^3 + \cdots\right)(1+|k|) \\ &= 1 - |k|^2 + \frac{1}{2}|k|^2 + \frac{1}{2}|k|^3 - \frac{1}{6}|k|^3 + \cdots. \end{aligned}$$

Now a little thought should convince you that $|k|^n$ has n derivatives everywhere and $n+1$ derivatives everywhere except 0, where the $n+1$th derivative is discontinuous. This shows that $e^{-|k|}(1+|k|)$ has 3 continuous derivatives, which agrees nicely with the fact that the function $1/(1+x^2)^2$ has 3 moments in L^2 (i.e., that the integral of the square of $x^n/(1+x^2)^2$ over all of $\mathbf{R}^1$ will be finite for $n \leq 3$).

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 13 – 17

Due Wednesday, July 22, at 3:30 PM EDT.

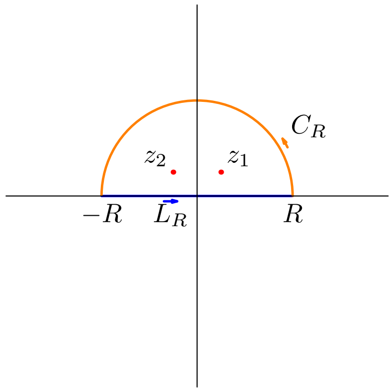
1. [20 marks] Evaluate the following integrals:

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx, \qquad \int_{-\infty}^{+\infty} \frac{1}{1-x^2+x^4}\,dx.$$

(You may cite the term test solutions on the course website in your solution, if you wish.)

To evaluate the first integral, we proceed as follows. Let $f(z) = 1/(1+z^4)$, let $R > 1$, let L_R denote the line segment from $-R$ to R along the real axis, and let C_R denote the semicircle of radius R centred at 0 in the upper half-plane. Then the only poles of f inside the closed contour $L_R + C_R$ are at (see the picture)

$$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$



so that we may write, by the residue theorem,

$$\int_{L_R} \frac{1}{1+z^4}\,dz + \int_{C_R} \frac{1}{1+z^4}\,dz = 2\pi i\left[\operatorname{Res}_{z_1}\frac{1}{1+z^4} + \operatorname{Res}_{z_2}\frac{1}{1+z^4}\right].$$

Moreover, if $z = Re^{it}$, $t \in [0, \pi]$ is some point in C_R, then we have

$$R|f(z)| = R\left|\frac{1}{1+R^4e^{4it}}\right| \le \frac{R}{R^4-1} = \frac{R^{-3}}{1-R^{-4}},$$

which clearly goes to zero as $R \to \infty$. Thus we have

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^4}\,dz = 0$$

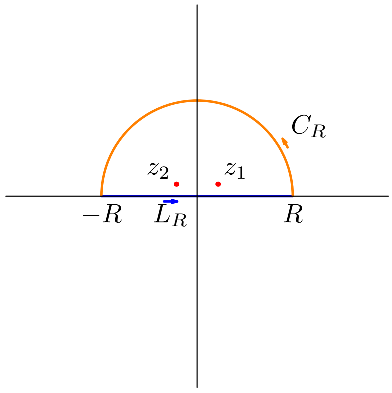
as well, and therefore our original integral is

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^4}\,dx = \lim_{R\to\infty}\int_{L_R}\frac{1}{1+z^4}\,dz = 2\pi i\left[\operatorname{Res}_{z_1}\frac{1}{1+z^4} + \operatorname{Res}_{z_2}\frac{1}{1+z^4}\right] = \frac{\pi}{\sqrt{2}},$$

where the final answer follows from question 3(c) on the main sitting of the term test.

To evaluate the second integral, we proceed in an analogous fashion. Let $f(z) = 1/(1-z^2+z^4)$, let $R > 1$, let L_R denote the line segment from $-R$ to R along the real axis, and let C_R denote the semicircle of radius R centred at 0 in the upper half-plane. Then the only poles of f inside the closed contour $L_R + C_R$ are at (see the picture)

$$z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad z_2 = -\frac{\sqrt{3}}{2} + i\frac{1}{2},$$



so that we may write, by the residue theorem,

$$\int_{L_R} \frac{1}{1-z^2+z^4}\,dz + \int_{C_R} \frac{1}{1-z^2+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{1}{1-z^2+z^4} + \mathrm{Res}_{z_2} \frac{1}{1-z^2+z^4}\right].$$

Moreover, if $z = Re^{it}$, $t \in [0, \pi]$ is some point in C_R, then we have

$$R|f(z)| = R\left|\frac{1}{1-R^2e^{2it}+R^4e^{4it}}\right| \leq \frac{R}{R^4-R^2-1} = \frac{R^{-3}}{1-R^{-2}-R^{-4}},$$

which clearly goes to zero as $R \to \infty$. Thus we have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1-z^2+z^4}\,dz = 0$$

as well, and therefore our original integral is

$$\int_{-\infty}^{+\infty} \frac{1}{1-x^2+x^4}\,dx = \lim_{R\to\infty} \int_{L_R} \frac{1}{1-z^2+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{1}{1-z^2+z^4} + \mathrm{Res}_{z_2} \frac{1}{1-z^2+z^4}\right] = \pi$$

where the final answer follows from question 3(c) on the makeup sitting of the term test.

[Marking: for each integral, 2 marks for the setup (description or picture of L_R, C_R, and the poles); 2 marks for the residue theorem (or the Cauchy integral theorem, to relate it to the integral on the term test); 3 marks for showing that the integral over C_R goes to zero; 3 marks for deducing the final value.]

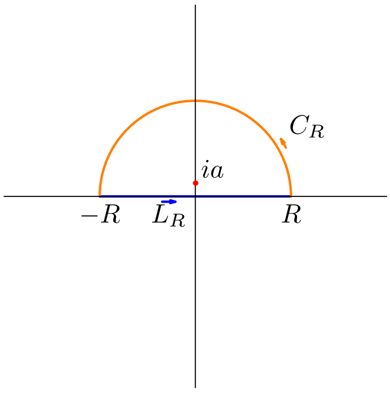
2. [30 marks] Evaluate the following integrals:

$$\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx, \qquad m,\ a \text{ real},\ a \neq 0.$$

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\,dx, \qquad k \text{ any real number. [Hint: Apply your work from problem 1.]}$$

As mentioned in the announcement of July 21, for the first integral all that is required is to set things up and then evaluate the residue, ignoring the singularity at $z = 0$. We thus let L_R be the line segment from $-R$ to R, and C_R the semicircle centred at 0 with radius R in the upper half-plane; then we see that the only singularity the integrand has inside the closed curve $L_R + C_R$ is at $z = ia$ (where we assume that $a > 0$ without loss of generality). (See the picture.) Now by the residue theorem we have

$$\int_{L_R} \frac{\sin mz}{z(z^2+a^2)}\,dz + \int_{C_R} \frac{\sin mz}{z(z^2+a^2)}\,dz = 2\pi i \mathrm{Res}_{ia} \frac{\sin mz}{z(z^2+a^2)}.$$



We leave the issue of how to deal with the integral over C_R for later and simply show how to calculate the residue. Since $z^2 + a^2 = (z - ia)(z + ia)$, we see that $z = ia$ is a simple pole of $(\sin mz)/[z(z^2 + a^2)]$, and thus the residue may be calculated as follows:

$$\begin{aligned}\mathrm{Res}_{ia} \frac{\sin mz}{z(z^2 + a^2)} &= \lim_{z \to ia} (z - ia) \frac{\sin mz}{z(z^2 + a^2)} = \lim_{z \to ia} \frac{\sin mz}{z(z + ia)} \\ &= -\frac{\sin mia}{2a^2} = -\frac{i \sinh ma}{2a^2},\end{aligned}$$

whence we see that

$$2\pi i \mathrm{Res}_{ia} \frac{\sin mz}{z(z^2 + a^2)} = \frac{\pi \sinh ma}{a^2}.$$

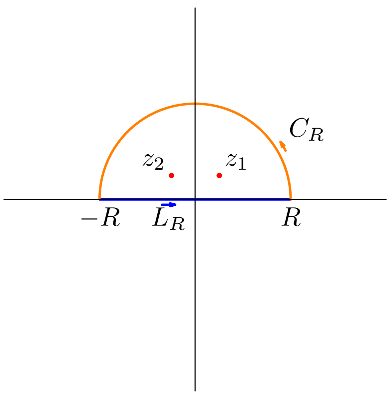
[Marking: again, 2 marks for the setup, 2 marks for the residue theorem; then 3 marks for computing the residue.]

Finally, for the last integral we must consider two separate cases depending on whether $k > 0$ or $k < 0$. (For $k = 0$ this is simply the integral from question 1.) Suppose that $k > 0$, and let L_R denote the line segment from $-R$ to R and C_R the semicircle centred at 0 with radius R in the *upper* half-plane, as usual; then for $R > 1$ the only poles will be those at

$$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$

as found in question 1 (see the picture), and by the residue theorem we have as usual

$$\int_{L_R} \frac{e^{ikz}}{1 + z^4}\, dz + \int_{C_R} \frac{e^{ikz}}{1 + z^4}\, dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{e^{ikz}}{1 + z^4} + \mathrm{Res}_{z_2} \frac{e^{ikz}}{1 + z^4} \right].$$



Now note that for $R > 1$ we have, for any $z = Re^{it}$ on C_R,

$$\left| \frac{1}{1 + z^4} \right| = \left| \frac{1}{1 + R^4 e^{4it}} \right| \leq \frac{1}{R^4 - 1},$$

which clearly goes to zero as $R \to \infty$; thus by the Jordan lemma we must have $\int_{C_R} \frac{e^{ikz}}{1+z^4}\, dz \to 0$ as $R \to \infty$. Thus we are left with computing the residues. Now the poles are all of order 1, and thus we have

$$\mathrm{Res}_{z_i} \frac{e^{ikz}}{1+z^4} = \lim_{z \to z_i} (z - z_i) \frac{e^{ikz}}{1+z^4} = e^{ikz_i} \lim_{z \to z_i} (z - z_i) \frac{1}{1+z^4}, \tag{1}$$

where we have used the product rule for limits and the continuity of the exponential function. Now from the term test we have

$$\begin{aligned} \mathrm{Res}_{z_1} \frac{1}{1+z^4} &= \lim_{z \to z_1} (z - z_1) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(-1+i)} = -\frac{1+i}{2^{5/2}}, \\ \mathrm{Res}_{z_2} \frac{1}{1+z^4} &= \lim_{z \to z_2} (z - z_2) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(1+i)} = \frac{1-i}{2^{5/2}}; \end{aligned}$$

thus we have finally

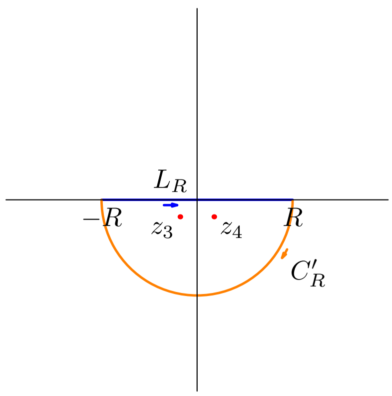
$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\, dx &= 2\pi i \left[\mathrm{Res}_{z_1} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_2} \frac{e^{ikz}}{1+z^4} \right] \\ &= 2\pi i \left[-e^{k\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)} \frac{1+i}{2^{5/2}} + e^{k\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)} \frac{1-i}{2^{5/2}} \right] \\ &= -4\pi \mathrm{Im}\, e^{-\frac{k}{\sqrt{2}} - i\frac{k}{\sqrt{2}}} \frac{1-i}{2^{5/2}} = \frac{\pi}{\sqrt{2}} e^{-k/\sqrt{2}} \left(\cos\frac{k}{\sqrt{2}} + \sin\frac{k}{\sqrt{2}} \right). \end{aligned}$$

The integral for $k < 0$ is analogous except that now we close using the semicircle C_R' in the *lower* half-plane, as shown in the figure below; this means that we pick up the poles at

$$z_3 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \qquad z_4 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

As before we have by the residue theorem that

$$\int_{L_R} \frac{e^{ikz}}{1+z^4}\, dz + \int_{C_R'} \frac{e^{ikz}}{1+z^4}\, dz = -2\pi i \left[\mathrm{Res}_{z_3} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_4} \frac{e^{ikz}}{1+z^4} \right],$$



where the minus sign is required since the curve $L_R + C_R'$ is now oriented *clockwise*. Now by a straightforward modification of Jordan's lemma to the case where $k < 0$ and we close in the *lower* half-plane, since on C_R' we have ($z = Re^{it}$)

$$\left| \frac{1}{1+z^4} \right| = \left| \frac{1}{1+R^4 e^{4it}} \right| \le \frac{1}{R^4 - 1}$$

as before, we have that $\int_{C_R'} \frac{e^{ikz}}{1+z^4}\, dz \to 0$ as $R \to \infty$. Now formula (1) holds just as well for $i = 3, 4$ as for $i = 1, 2$; since from the term test we have

$$\begin{aligned}
\mathrm{Res}_{z_3} \frac{1}{1+z^4} &= \lim_{z \to z_3} (z - z_3) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(1-i)} = \frac{1+i}{2^{5/2}}, \\
\mathrm{Res}_{z_4} \frac{1}{1+z^4} &= \lim_{z \to z_4} (z - z_4) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(-1-i)} = \frac{-1+i}{2^{5/2}},
\end{aligned}$$

we have finally in this case

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\, dx &= -2\pi i \left[\mathrm{Res}_{z_3} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_4} \frac{e^{ikz}}{1+z^4} \right] \\
&= -2\pi i \left[e^{k\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)} \frac{1+i}{2^{5/2}} - e^{k\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)} \frac{1-i}{2^{5/2}} \right] \\
&= 4\pi \mathrm{Im}\, e^{\frac{k}{\sqrt{2}} - i\frac{k}{\sqrt{2}}} \frac{1+i}{2^{5/2}} = \frac{\pi}{\sqrt{2}} e^{k/\sqrt{2}} \left(\cos \frac{k}{\sqrt{2}} - \sin \frac{k}{\sqrt{2}} \right).
\end{aligned}$$

We see that we may combine these two expressions into one as follows:

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}} e^{-|k|/\sqrt{2}} \left(\cos \frac{k}{\sqrt{2}} + \sin \frac{|k|}{\sqrt{2}} \right).$$

[Marking: 11 marks for each, plus 1 mark for getting the extra minus sign in the second integral. For each integral, 1 mark for the setup, 1 mark for the residue theorem, 1 mark for the application of Jordan's lemma, 2 marks for each residue, and 4 marks for the final computations.]

Summary:

- We tie down a few loose ends from previous lectures.
- We prove Jordan's lemma and give examples of its application to the evaluation of definite integrals.
- We then give additional examples of finding contours in the complex plane for the evaluation of definite integrals on the real line.

(Goursat, §§41, 44 – 46.)

31. On zeroes and poles. Recall that we have defined poles of a definite order and zeroes of a definite order, as follows:

Pole of order m at a:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}, \quad b_m \neq 0$$
$$= \frac{\phi(z)}{(z-a)^m}, \quad \phi \text{ analytic and nonzero at } a$$

Zero of order m at a:

$$f(z) = \sum_{n=m}^{\infty} a_n(z_a)^n, \quad a_m \neq 0$$
$$= (z-a)^m\phi(z), \quad \phi \text{ analytic at } a.$$

We now claim that function which is analytic except for isolated singularities can have only finitely many poles, and that a nonzero analytic function can have only finitely many zeroes, on any finite region. Thus, let C be a simple closed curve on and within which the function f is analytic, and suppose first that f has infinitely many zeroes at a_1, a_2, $\cdots$ within the curve C. We shall only give the main idea (the details will be given later when we talk about analytic continuation). We need the celebrated *Bolzano-Weierstrass Theorem*:

Let $\{a_1, a_2, \cdots\}$ be an infinite set within a simple closed curve C. Then there must be a point a within or on C such that every disk around a contains infinitely many points of this set.

This is proved in courses on analysis, but it is also quite reasonable intuitively since if there infinitely many points in a finite region, surely they cannot all be staying a finite distance away from each other: they must be 'clustering' somewhere.[0] By this theorem, there must be a point a within or on C such that every disk around a contains infinitely many zeroes of f; thus any disk around a must contain some point at which f is zero, which means that $f(a)$ must be zero: in other words, a is a zero of f. Let us write out the Taylor series of f at a:

$$f(z) = \sum_{n=1}^{\infty} \alpha_n(z-a)^n.$$

We claim that all of the coefficients must be zero. Suppose that $\alpha_k \neq 0$ for some k. Then we would be able to write

$$f(z) = (z-a)^k\phi(z),$$

where ϕ is analytic at a and – crucially – *nonzero* at a. Now this would imply that there would be a disk around a on which ϕ is still nonzero; but since $(z-a)^k$ is zero only when $z = a$, f would not be zero anywhere on this disk either, contradicting our choice of a. Thus all of the coefficients in the Taylor series of f must be zero, which means that f must be identically zero on every disk around a at which it is analytic. Note though that this does not automatically allow us to conclude that it must be identically zero on C. The idea to complete the proof – which we shall go over more carefully when we talk about analytic continuation later – is as follows. f must be analytic on some disk around a. Now let us take a point near the boundary of this disk; then since f is identically zero near that point, its Taylor series around that point must still be identically zero. Thus f must be identically zero on all disks around this new point on which it is still analytic. We can then continue extending the region until we show that f must actually be identically zero on all of C. (Specifically, as we shall see when we talk about analytic continuation, we actually proceed by extending f along a curve to any other point in C, which allows us to conclude that f must still be zero at that point, and hence at every point in C.)

[0] In other words, there is no way for an infinite group of people to practice social distancing within a grocery store!

Now suppose that the points a_1, a_2, $\cdots$ were in fact poles. Then as before there would be a point a, any disk around which would contain infinitely many of the poles a_i. Then clearly a cannot be an isolated singularity of f, since any disk around it contains additional singularities of f; but a cannot be a regular point either, since f is not analytic on any disk around a. Thus f could not be analytic except for isolated singularities, completing the proof in this case.

32. Cauchy principal value. Recall that in elementary calculus we give the following definition:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{a} f(x)\,dx + \lim_{L_2\to\infty}\int_{a}^{L_2} f(x)\,dx, \tag{1}$$

where the integral on the left exists if and only if the two limits on the right-hand side both exist as finite numbers. Here a is any real number; it is easy to show that the definition does not depend on the choice of a, so for convenience we shall take $a = 0$.

Now the careful student may have noted that the integrals we have calculated so far are *not* in the form of a sum of two different limits, but rather of a single limit,

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx.$$

This limit is called the *Cauchy principal value* of the integral, and we denote it by PV $\int_{-\infty}^{\infty} f(x)\,dx$.[1] Now it is easy to see that if the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists as defined above, then the Cauchy principal value also exists and is equal to it; for in this case

$$\begin{aligned}\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\\ &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{0} f(x)\,dx + \lim_{L_2\to\infty}\int_{0}^{L_2} f(x)\,dx\\ &= \int_{-\infty}^{\infty} f(x)\,dx.\end{aligned}$$

The logic, however, does not work in reverse, and for a very simple reason. Note that in going from the first to the second line above we used the fact that if the limit of two quantities exist, then the limit of their sum exists and equals the sum of the limits. It is, however, most definitely *not* true that if the limit of a sum exists, then the limit of the two terms in the sum both exist! (As a simple example, consider $f(x) = 1 - 1/x$ and $g(x) = 1/x$ as $x \to 0$: clearly, $f(x) + g(x) = 1$, and the limit of this exists as $x \to 0$, while neither f nor g has a limit which exists.) Thus the logic cannot be run backwards. To sum up, then: if $\int_{-\infty}^{\infty} f(x)\,dx$ exists, so does PV $\int_{-\infty}^{\infty} f(x)\,dx$, and the two must be equal; but the converse is not necessarily true.

There is one case where the converse is true, though: when f is even. In this case, we see that

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx = 2\lim_{L\to\infty}\int_{0}^{L} f(x)\,dx,\ \lim_{L\to-\infty}\int_{L}^{\infty} f(x)\,dx = \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx,$$

so that if the Cauchy principal value exists, then so do both of the limits in (1) above, and hence so does the integral $\int_{-\infty}^{\infty} f(x)\,dx$. To summarise, then, we have

$$\text{if the integral } \int_{-\infty}^{\infty} f(x)\,dx \text{ exists, then so does PV} \int_{-\infty}^{\infty} f(x)\,dx, \text{ and the two are equal;}$$
$$\text{if PV} \int_{-\infty}^{\infty} f(x)\,dx \text{ exists and } f \text{ is even, then so does } \int_{-\infty}^{\infty} f(x)\,dx \text{ and the two are equal.}$$

[1] The terminology may be slightly misleading. The Cauchy principal value is something which can be computed independently of the actual integral as defined above: it requires only evaluating the single limit just given. In particular, we do *not* determine the Cauchy principal value by first evaluating the full integral and then doing something to that number!

Another way of looking at the difference between (1) and the principal value is to note that in (1) we have a two-dimensional limit, while the principal value is effectively the directional limit along the line $L_1 = -L_2$. As we learned in multivariable calculus, if the two-dimensional limit of a quantity exists, then so does the limit along any curve – but if all we know is that the limit along one particular line exists, we really do not know anything at all about the full two-dimensional limit, in general. Thus knowing that the Cauchy principal value exists does not, in general, tell us anything about the integral in (1).

It is worth noting that the techniques we have studied so far all amount to calculating the Cauchy principal value rather than the integral as defined in (1). Hence, if we are asked to compute the integral $\int_{-\infty}^{\infty} f(x)\,dx$, in order to show that it equals the Cauchy principal value we must first show that it exists (as it will, as just shown, when f is even, for example).

33. Jordan's lemma. Recall that there are two main steps to computing integrals using contours: one, finding a way of 'closing the contour' in such a way that we can calculate the integral of our function over the additional part of the contour (for example, using a semicircle the integral over which goes to zero as its radius goes to infinity); two, evaluating residues. In the previous lecture we saw additional methods for the second step; now we shall prove a result helping us to deal with the first step. First of all, we note that for $x \in [0, \pi/2)$ we have $0 \le \cos x \le 1$, so

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2}\cos x \le 0,$$

since $\tan x \ge x$ for $x \in [0, \pi/2)$. This means that the function $\sin x/x$ is decreasing on $[0, \pi/2)$, so its minimum value on $[0, \pi/2]$ is achieved at $x = \pi/2$, and is therefore $(\sin \pi/2)/(\pi/2) = 2/\pi$. Thus for $x \in [0, \pi/2]$ we have $\sin x \ge \frac{2}{\pi}x$.

With this preliminary, we may now prove Jordan's lemma:

Let C_R denote the semicircle of radius R centred at the origin in the upper half-plane. Let f be a function which is analytic in the upper half-plane on the exterior of some semicircle of radius R_0, and such that for every $R > R_0$ there is a constant M_R such that $|f(z)| \le M_R$ on C_R, and $M_R \to 0$ as $R \to \infty$. Then $\int_{C_R} f(z)e^{iaz}\,dz \to 0$ as $R \to \infty$ for any positive number a.

To prove this, parameterise C_R by $z(t) = R(\cos t + i\sin t)$, $t \in [0, \pi]$; then

$$e^{iaz} = e^{iaR(\cos t + i\sin t)} = e^{iaR\cos t}e^{-aR\sin t}.$$

Now the first factor has modulus one, while for $t \in [0, \pi/2]$ we have

$$\sin t \ge \frac{2}{\pi}t, \qquad -\sin t \le -\frac{2}{\pi}t, \qquad e^{-aR\sin t} \le e^{-\frac{2aR}{\pi}t};$$

thus

$$\begin{aligned}\left|\int_{C_R} f(z)e^{iaz}\,dz\right| &\le \int_0^\pi \left|f(Re^{it})\right| e^{-aR\sin t}R\,dt \le RM_R\int_0^\pi e^{-aR\sin t}\,dt = 2RM_R\int_0^{\pi/2} e^{-aR\sin t}\,dt \\ &\le 2RM_R\int_0^{\pi/2} e^{-\frac{2aR}{\pi}t}\,dt = -\frac{M_R\pi}{a}e^{-\frac{2aR}{\pi}t}\Big|_0^{\pi/2} = \frac{M_R\pi}{a}\left(1 - e^{-aR}\right),\end{aligned}$$

which goes to zero as $R \to \infty$, since M_R does and the quantity in parentheses goes to 1. This completes the proof.

We note that the same is true if we replace 'upper' everywhere by 'lower' and require that a be negative: for now we may parameterise C_R by $z(t) = -R(\cos t + i\sin t)$, in the which case, proceeding as before, the integral over C_R of $f(z)e^{iaz}$ can be bounded by

$$2RM_R\int_0^{\pi/2} e^{aR\sin t}\,dt.$$

But now, as before, aR is negative since a is, and the proof proceeds as before with $-aR$ replaced by aR everywhere.

We now give some examples.

EXAMPLES. 1. Evaluate $\int_0^\infty \frac{x\sin x}{1+x^2}\,dx$.

We first note that the integrand is even so that we have

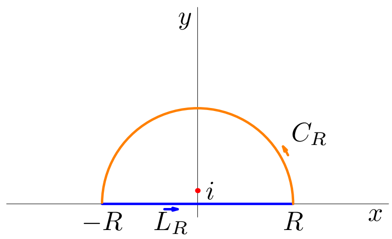
$$\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = 2\int_{-\infty}^\infty \frac{x\sin x}{1+x^2}\,dx,$$

where this last integral will exist exactly when the principal value exists, as shown above. Thus it suffices to compute the principal value of this last integral. We wish to apply Jordan's lemma. Now we have $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$; but if we were to use this formula, it would require us to compute two separate integrals which would need to be closed in different half-planes. That would be possible but would be more work than is necessary. Instead we write

$$\sin x = \operatorname{Im} e^{ix},$$

and note that this allows us to write

$$\int_{-\infty}^\infty \frac{x\sin x}{1+x^2}\,dx = \operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx.$$



This looks like the kind of function to which we should be able to apply Jordan's lemma. We only need a bound on $x/(1+x^2)$ on the upper semicircle C_R. Now on C_R we have

$$\left|\frac{z}{1+z^2}\right| = \frac{|z|}{|1+z^2|} \ge \frac{R}{R^2-1} = \frac{1/R}{1-R^{-2}},$$

which clearly goes to zero as $R \to \infty$. Thus by Jordan's lemma

$$\int_{C_R} \frac{ze^{iz}}{1+z^2}\,dz \to 0 \quad \text{as} \quad R \to \infty.$$

Since the integrand has only one pole in the upper half-plane, at $z = i$, we may write, by the residue theorem,

$$\begin{aligned}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx &= \lim_{R\to\infty}\left[-\int_{C_R}\frac{ze^{iz}}{1+z^2}\,dz + 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{z^2+1}\right] \\ &= 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{(z-i)(z+i)} = 2\pi i \frac{ie^{i^2}}{2i} = \frac{2\pi i}{e}. \end{aligned} \tag{2}$$

Thus we have finally

$$\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx = \frac{1}{2}\operatorname{Im}\frac{2\pi i}{e} = \frac{\pi}{e}.$$

It is worth noting that the integral in (2) is a pure imaginary number; this can be traced to the fact that $x\cos x$ is odd, which means that the principal value of its integral over the real line is zero.

2. Evaluate

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx. \tag{3}$$

This integral introduces some additional twists to our standard procedure. First of all, by $\sin x/x$ we mean actually the function

$$\begin{cases} \frac{\sin z}{z}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

which by what we have seen on a previous homework assignment is analytic everywhere on the complex plane, and in fact has the Taylor series expansion

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

If we now think of closing the integral in (3) in the upper half-plane, it would appear that it evaluates to zero since there are no singularities and hence no residue. This would be wrong (and looking at a graph of $\sin x/x$ suggests as much), since as we have noticed before the integral over a semicircle in the upper half-plane of something involving $\sin x$ will not, in general, go to zero since $\sin x$ includes a term e^{-ix}. Let us look at this a bit more carefully. Let L_R, as usual, denote the line segment from $-R$ to R along the real axis, and C_R denote the upper semicircle of radius R centred at the origin. Then we have by the Cauchy integral theorem

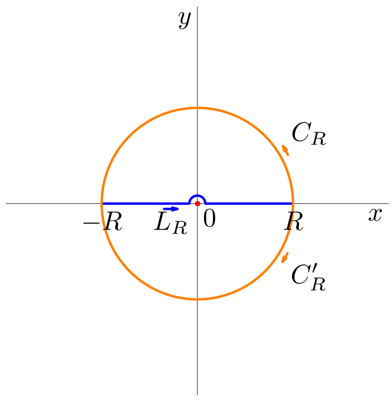
$$0 = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{\sin z}{z}\, dz = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{e^{iz} - e^{-iz}}{2iz}\, dz.$$

We would like to break this up in such a way that we can close the integral involving e^{-iz} in the lower half-plane. This suggests splitting $\sin x$ up in the first integrand:

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L_R} \frac{e^{ix} - e^{-ix}}{2ix}\, dx.$$

This is perfectly fine, but unfortunately we cannot break this integral up into two separate pieces as it stands since the individual pieces would have a singularity at the origin, which lies on the line L_R. (Note that we do need to break this integral up in order to obtain a closed curve with either C_R or $-C_R$ – the lower semicircle – and hence to apply the residue theorem.) But by the Cauchy integral theorem, since $\sin z/z$ is analytic everywhere on the plane, we may replace L_R with any other curve passing from $-R$ to R; let us use a contour which goes along the real axis from $-R$ to $-\epsilon$ and ϵ to R, and joins $-\epsilon$ to ϵ by a small semicircle of radius ϵ centred at the origin, in the upper half-plane. Denote this contour by L'_R. Then we have

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L'_R} \frac{\sin z}{z}\, dz = \int_{L'_R} \frac{e^{iz}}{2iz}\, dz - \int_{L'_R} \frac{e^{-iz}}{2iz}\, dz.$$



We may now evaluate these integrals by closing in the upper and lower half-planes, respectively. Let us look at the first integral. By the residue theorem,

$$\int_{L'_R} \frac{e^{iz}}{2iz}\, dz + \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

But now on C_R we clearly have

$$\left|\frac{1}{2iz}\right| = \frac{1}{2}R^{-1} \to 0 \quad \text{as} \quad R \to \infty,$$

so by Jordan's lemma we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

Thus

$$\lim_{R\to\infty} \int_{L'_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

The second integral is more interesting. We have by the residue theorem

$$\int_{L'_R} \frac{e^{-iz}}{2iz}\, dz + \int_{-C_R} \frac{e^{-iz}}{2iz}\, dz = 2\pi i \mathrm{Res}_0 \frac{e^{-iz}}{2iz} = \pi,$$

while $|1/(2iz)| = 1/(2R) \to 0$ as $R \to \infty$ shows that the second integral vanishes as $R \to \infty$, by Jordan's lemma. Thus we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \mathrm{PV} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \int_{L_R} \frac{\sin x}{x}\, dx = \pi.$$

It is worth noting that the method we used in the previous example – of replacing $\sin x$ by e^{ix} and then taking an imaginary part – does not work directly in this case since the curve L'_R we integrate over does not lie along the real axis, so we cannot simply recover the integral over it of $\sin x/x$ from that of e^{ix}/x by taking an imaginary part.

34. Another way of closing the contour. Let us consider, by way of example, another method for closing the contour.

Example. Evaluate the integrals

$$\int_0^\infty \sin x^2\, dx, \qquad \int_0^\infty \cos x^2\, dx.$$

We do this by considering the integral

$$\int_0^\infty e^{iz^2}\, dz.$$

Let L_R in this case denote the line segment from 0 to R along the real axis. We shall close the contour in two different ways. First, though, it is probably worthwhile to consider why the methods we have been using so far do not work in this case. Clearly, if the above integral exists then it will equal half the Cauchy principal value of

$$\int_{-\infty}^\infty e^{iz^2}\, dz.$$

Now if we consider the integral from $-R$ to R of e^{iz^2} and then close it with the semicircle C_R of radius R centred at the origin in the upper half-plane, then it would appear initially – as in the previous example – that we would get zero. However, as before, the integral along C_R of e^{iz^2} does not vanish as $R \to \infty$. While not a proof, we may see that this is reasonable by the following computation:

$$\int_{C_R} e^{iz^2}\, dz = \int_0^\pi e^{iR^2 e^{2it}} iRe^{it}\, dt = \int_0^\pi e^{iR^2\cos 2t} e^{-R^2 \sin 2t} iRe^{it}\, dt;$$

but $\sin 2t < 0$ when $t \in (\pi/2, \pi)$, so that the exponential above goes to infinity with R on that range of t. Thus evidently we need to do something else.

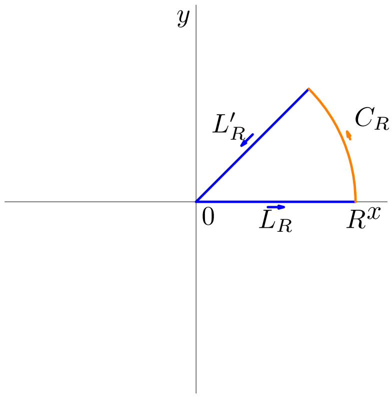
This calculation actually suggests something worth noting: if we were able to restrict $t \in (0, \pi/2)$, then the exponential above would go to zero as $R \to \infty$, and it is possible that the whole integral will also go to zero. Thus let us consider closing the contour with a piece C'_R of the full semicircle C_R together with a line segment back to the origin, i.e., with a pie-wedge shaped contour as in the following figure. The problem now will be how to calculate the integral over the additional line segment L'_R. Now the line segment L'_R may be parameterised as $\omega(R-t)$, $t \in [0, R]$ (the $R-t$ is because the line starts on C_R and ends at the origin), where ω is some complex number of unit modulus. This allows us to write the integral over L'_R as

$$\int_0^R e^{i\omega^2(R-t)^2}\,\omega(-dt) = -\omega \int_0^R e^{i\omega^2 t^2}\,dt.$$

Now if $\omega^2 = i$, then the integrand would become e^{-t^2}, and we can compute the integral of e^{-t^2} over the positive real axis by other methods, so it appears that we might be able to use the line L'_R in that case. We shall do this in detail below. Alternatively, if $\omega^2 = -1$, then the integral over L'_R will be $-\omega$ times the conjugate of that over L_R, so we may be able to find the integral over L_R in this case also by isolating and solving. We shall not use this method here, but it is very useful for this week's homework assignment. (Hint, hint!)

We shall thus take ω to satisfy $\omega^2 = i$. This means that we have two choices: $\omega = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and $\omega = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. We take the first because to close to the second would require us to use a circle along which we do not have good bounds for e^{iz^2}. Thus we let L'_R be the line parameterised by $(R-t)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$, $t \in [0, R]$, and let C'_R denote the segment of the semicircle of radius R centred at the origin from R to $R\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$. Then by the Cauchy integral theorem we have

$$\int_{L_R} e^{iz^2}\,dz + \int_{C'_R} e^{iz^2}\,dz + \int_{L'_R} e^{iz^2}\,dz = 0.$$



We deal with the integral over C'_R first. We have

$$\left|\int_{C'_R} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2e^{2it}} Rie^{it}\,dt\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2t}\,dt;$$

since $t \in [0, \pi/4]$, we have $2t \in [0, \pi/2]$, so $\sin 2t \ge \frac{2}{\pi}(2t) = \frac{4}{\pi}t$ and the above integral is bounded by

$$R\int_0^{\pi/4} e^{-\frac{4R^2}{\pi}t}\,dt = -\frac{\pi}{4R}e^{-\frac{4R^2}{\pi}t}\Big|_0^{\pi/4} = \frac{\pi}{4R}\left(1 - e^{-R^2}\right),$$

which goes to zero as $R \to \infty$. Thus the integral over C'_R does not contribute anything to the final integral. Now the integral over L'_R is equal to

$$\int_0^R e^{-(R-t)^2}\left[-\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]dt = -e^{i\pi/4}\int_0^R e^{-t^2}\,dt,$$

which in the limit as $R \to \infty$ becomes

$$-e^{i\pi/4}\frac{1}{2}\sqrt{\pi} = -\frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Thus we have finally

$$\int_0^\infty e^{iz^2}\,dz = -\lim_{R\to\infty}\int_{L'_R} e^{iz^2}\,dz = \frac{1}{2}\sqrt{\frac{\pi}{2}} + i\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

so we see that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Summary:
- We demonstrate additional methods of computing definite integrals using contours.
- We then proceed to discuss more general properties of analytic functions.
- In particular, we prove Liouville's Theorem, the argument principle, and Rouché's Theorem, and use these to give two different proofs of the fundamental theorem of algebra.
- We then discuss the Poisson kernel for Laplace's equation on a disk.

(Goursat, §§36, 45, 48 – 49.)

35. Additional methods of closing the contour. We show two more methods of closing the contour on a definite integral, by way of example.

EXAMPLE. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx. \tag{1}$$

We note first of all that this integral converges (apply the usual power test). Now to extend this to a contour integral over a complex curve we must first choose a branch of the cube root function. We shall choose the branch with a branch cut along the positive real axis*, and take the angle θ to lie in the interval $(0, 2\pi)$. To evaluate integral (1), we consider a so-called *keyhole contour* composed of four separate curves (see the figure): a line L_R from $i\epsilon$ to $R + i\epsilon$; a circular arc C_R running counterclockwise from $R + i\epsilon$ to $R - i\epsilon$, centred at the origin; another line L'_R running from $R - i\epsilon$ to $-i\epsilon$; and finally a semicircle C'_ϵ running clockwise from $-i\epsilon$ to $i\epsilon$ in the third and second quadrants, again centred at the origin. (There are of course other slightly different contours which would perform the same task equally well.) Let us let $\gamma = L_R + C_R + L'_R + C'_\epsilon$ denote the full curve. Note that since the branch point and branch cut of $z^{1/3}$ lie entirely outside of this curve (this is our first indication that taking a branch cut along the line of integration was in fact the correct thing to do!) the integrand $z^{1/3}/(1+z^2)^2$ has only poles within the contour, at $z = \pm i$, we see from the residue theorem that

$$\int_\gamma \frac{z^{1/3}}{(1+z^2)^2}\,dz = 2\pi i\left[\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2}\right].$$

We will come back to the calculation of these residues later and consider first how the integral at left relates to the integral in (1). In particular, we claim that

$$\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad \text{and} \quad \int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad \text{as} \quad R\to\infty,\ \epsilon \to 0^+. \tag{2}$$

Note first that if z is on C_R or C'_ϵ, then $|z^{1/3}| = |z|^{1/3}$, where $|z|^{1/3}$ denotes the unique positive cube root of the positive real number z (positive since the curves C_R and C'_ϵ do not pass through the origin). Thus, parameterising C_R as Re^{it}, $t \in [\theta_0, 2\pi - \theta_0]$, we have (note that the equation $|z^{1/3}| = |z|^{1/3}$ does not depend on which branch of the cube root function is used to calculate $z^{1/3}$, and thus we do not need to worry here and in the next integral whether the point as parameterised has an angle lying within the interval $(0, 2\pi)$)

$$\begin{aligned}\left|\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{1/3}}{|1+R^2e^{2it}|^2} R\,dt \\ &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{4/3}}{(R^2-1)^2}\,dt = (2\pi - 2\theta_0)\frac{R^{4/3}}{(R^2-1)^2} \to 0\end{aligned}$$

as $R \to \infty$, since the degree of the denominator is greater than that of the numerator. Similarly, parameterising C'_ϵ as ϵe^{-it}, $t \in [\pi/2, 3\pi/2]$, we have

$$\begin{aligned}\left|\int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{1/3}}{|1+\epsilon^2e^{-2it}|^2} \epsilon\,dt \\ &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{4/3}}{(1+\epsilon^2)^2}\,dt = \pi\frac{\epsilon^{4/3}}{(1+\epsilon^2)^2} \to 0\end{aligned}$$

* It may seem like a very bad idea to take the branch cut along the positive real axis as, after all, this is exactly the line along which we wish to integrate! As we shall see shortly, though, this is exactly the right place to put the branch cut in this case.

as $\epsilon \to 0^+$, since the denominator goes to 1 and the numerator to 0. This completes the demonstration of (2).

We thus only need to consider how the integrals over L_R and L'_R relate to (1). We parameterise L_R by $t + i\epsilon$ and L'_R by $(R-t) - i\epsilon$, where in both cases $t \in [0, R]$. Since $\epsilon > 0$, we may express these numbers in polar notation as (here arctan gives the angle in $(-\pi/2, \pi/2)$ which has the given tangent)

$$t + i\epsilon = \sqrt{t^2 + \epsilon^2} e^{i \arctan \epsilon/t}, \tag{3}$$

$$\begin{aligned}(R-t) - i\epsilon &= \sqrt{(R-t)^2 + \epsilon^2} e^{i \arctan(-\epsilon/(R-t))} = \sqrt{(R-t)^2 + \epsilon^2} e^{-i \arctan \epsilon/(R-t)} \\ &= \sqrt{(R-t)^2 + \epsilon^2} e^{i(2\pi - \arctan \epsilon/(R-t))},\end{aligned} \tag{4}$$

where the 2π in (4) was added to make the angle lie in the interval $(0, 2\pi)$ corresponding to our chosen branch of $z^{1/3}$. Given this, then, the integrals over L_R and L'_R become

$$\int_{L_R} \frac{z^{1/3}}{(1+z^2)^2}\, dz = \int_0^R \frac{(t^2+\epsilon^2)^{1/6} e^{i\frac{1}{3}\arctan\frac{\epsilon}{t}}}{(1+(t+i\epsilon)^2)^2}\, dt,$$

$$\int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2}\, dz = -\int_0^R \frac{[(R-t)^2+\epsilon^2]^{1/6} e^{i\frac{1}{3}(2\pi - \arctan\frac{\epsilon}{R-t})}}{(1+(R-t-i\epsilon)^2)^2}\, dt.$$

If we take the limit of these expressions as $\epsilon \to 0^+$ and interchange it with the integrals, we obtain, since $\arctan 0 = 0$,

$$\lim_{\epsilon\to 0^+} \int_{L_R} \frac{z^{1/3}}{(1+z^2)^2}\, dz = \int_0^R \frac{t^{1/3}}{(1+t^2)^2}\, dt,$$

$$\lim_{\epsilon\to 0^+} \int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2}\, dz = -\int_0^R \frac{(R-t)^{1/3} e^{\frac{2\pi i}{3}}}{(1+(R-t)^2)^2}\, dt = -e^{\frac{2\pi i}{3}} \int_0^R \frac{t^{1/3}}{(1+t^2)^2}\, dt,$$

and we see finally that

$$\lim_{R\to\infty} \lim_{\epsilon\to 0^+} \int_\gamma \frac{z^{1/3}}{(1+z^2)^2}\, dz = \left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\, dx,$$

so that the integral (1) is equal to

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\, dx = \frac{2\pi i}{1 - e^{\frac{2\pi i}{3}}} \left[\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2}\right]. \tag{5}$$

We are thus left with the task of computing the residues. Note that both $\pm i$ are poles of order 2 of the function $z^{1/3}/(1+z^2)^2$; thus letting $\lambda = \pm i$, we may write

$$\mathrm{Res}_\lambda \frac{z^{1/3}}{(1+z^2)^2} = \lim_{z\to\lambda} \frac{d}{dz} (z-\lambda)^2 \frac{z^{1/3}}{(1+z^2)^2}.$$

Since $1 + z^2 = (z-i)(z+i)$, this allows us to write

$$\begin{aligned}\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z\to i} \frac{d}{dz} \frac{z^{1/3}}{(z+i)^2} = \lim_{z\to i} \frac{\frac{1}{3z^{2/3}}(z+i)^2 - 2(z+i)z^{1/3}}{(z+i)^4} \\ &= \frac{\frac{-4}{3e^{i\pi/3}} - 4ie^{i\pi/6}}{16} = -\frac{1}{4}\left[\frac{1}{3} e^{-i\pi/3} + ie^{i\pi/6}\right] \\ &= -\frac{1}{4}\left[\frac{1}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\right] = -\frac{1}{4}\left[\left(\frac{1}{3} - 1\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right] \\ &= \frac{1}{6}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right),\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z\to -i} \frac{d}{dz} \frac{z^{1/3}}{(z-i)^2} = \frac{1}{16}\left[\frac{1}{3e^{i\pi}}(-4) + 4ie^{i\pi/2}\right] \\ &= \frac{1}{4}\left(\frac{1}{3} - 1\right) = -\frac{1}{6},\end{aligned}$$

so

$$\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} = \frac{1}{6}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{6}e^{i\pi/3}.$$

At this point we may start to feel like something has gone wrong since our answer at the end of the day needs to be a real number, but it is not clear how we shall get a real number out of what we have so far. Everything does work out in the end, though: by (5) we have

$$\begin{aligned}\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx &= \frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\left[-\frac{1}{6}e^{\frac{i\pi}{3}}\right] \\ &= -\frac{\pi}{6}\frac{2i}{e^{-i\pi/3}-e^{i\pi/3}} = \frac{\pi}{6}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{12},\end{aligned}$$

which is thus our final answer.

[Deep breath!]

Note how taking the branch cut along the line of integration helped us: since branch points are singularities which are not poles, whatever closed contour we draw in the complex plane must exclude the branch point and the branch cut; and as we bring the edges of the keyhole contour (the lines L_R and L'_R) together, the discontinuity of $z^{1/3}$ across the branch cut will allow us to combine the two integrals without cancellation, to get a multiple of the integral along the branch cut. Thus it is precisely a multiple of the integral along the branch cut which is equal to the sum of the residues of the integrand at its poles within the contour.

Let us consider one more way of closing the contour.

Example. Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx. \tag{6}$$

This integral is much simpler than the previous one! It also demonstrates a slightly different application of the type of 'wedge contour' we saw when evaluating the Fresnel integrals previously. We start out with the line L_R which is just the interval $[0, R]$ in this case (since we have no branch cut!). We then consider closing the contour using a circular arc C_R of angle α followed by a line L'_R back to the origin (necessarily therefore making an angle α with the positive real axis). Now we wish the integral over L'_R to be related somehow to the original integral (6). Parameterising it as $(R-t)e^{i\alpha}$, we have

$$\int_{L'_R} \frac{1}{1+z^3}\,dz = -\int_0^R \frac{1}{1+(R-t)^3e^{3i\alpha}}e^{i\alpha}\,dt = -\int_0^R \frac{1}{1+t^3e^{3i\alpha}}e^{i\alpha}\,dt.$$

For this to be a multiple of (6), we must have $3\alpha = 2n\pi$ for some integer n. Since we wish also to include as few residues as possible in the closed contour, we want α to be as small as possible, and therefore take $\alpha = 2\pi/3$. Thus we will close using the contour shown in the figure. Now since $R/|1+z^3| \to 0$ as $R \to \infty$, when $z \in C_R$, we must have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^3}\,dz = 0.$$

Now the only poles of the integrand in (6) are at the complex cube roots of -1, which are $e^{i\pi/3}$, $e^{i\pi}$, and $e^{5\pi/3}$; only the first of these lies within the contour $L_R + C_R + L'_R$, and thus we may write

$$(1-e^{2\pi i/3})\int_0^\infty \frac{1}{1+x^3}\,dx = 2\pi i\mathrm{Res}_{\pi i/3}\frac{1}{1+z^3}. \tag{7}$$

Now this pole is simple, so the residue can be calculated as follows:

$$\mathrm{Res}_{\pi i/3}\frac{1}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{(1+z^3)-(1+[e^{i\pi/3}]^3)} = \left(\frac{d}{dz}(1+z^3))\bigg|_{z=e^{\pi i/3}}\right)^{-1};$$

compare our result on p. 6 of the lecture notes for July 14 – 16. This evaluates to

$$\frac{1}{3}e^{-2\pi i/3},$$

and so by (7) we have

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi i e^{-2\pi i/3}}{3(1-e^{2\pi i/3})} = \frac{\pi}{3}\frac{2i}{e^{-\pi i/3}-e^{\pi i/3}}e^{-\pi i} = \frac{\pi}{3}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{6}.$$

That this integral is exactly twice that in the previous example, is a complete coincidence. (As far as I know! – I picked both integrals out of a hat, pretty much.)

The idea in this method can be combined with that in the previous example – i.e., we can make a contour consisting of two lines at angles to each other, such that the integrals over these lines can be written in terms of each other, as well as a large circle and a small circle around the origin if that happens to be a branch point. This idea is useful on question 2 of the August 3 – 7 homework.

There is one more method for turning definite integrals into contour integrals which we shall have use for; see Goursat, §45. We, again, show this by way of an example.

Example. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx. \tag{8}$$

We begin by noting that the integrand is continuous on the real line, so that the integral is defined. Note that this integral is fundamentally different from the other integrals we have studied so far, since it is taken over a finite interval instead of an infinite one. Thus it does not seem that the method of considering it as a contour integral and then closing the contour, as we have done previously, will be of use here. We shall instead do something completely different: rewrite (8) as a contour integral by, effectively, *deparameterising* it: in other words, recognising integral (8) as the parameterised form of an integral over a closed contour in the complex plane.

To do this, note first of all that

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i};$$

now as x ranges from 0 to 2π, e^{ix} and e^{-ix} both trace out the unit circle. Thus it would seem that integral (8) may be the parameterisation of an integral over the unit circle. Now the only other part of the integrand which depends on x is dx; if we let $z = e^{ix}$, then $dz = ie^{ix}dx = izdx$, so $dx = dz/(iz)$ (note that since z is on the unit circle, $z \neq 0$ so $1/z$ is defined), and we have finally that (letting C denote the unit circle)

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = \int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\,\frac{dz}{iz},$$

since if $z = e^{ix}$ then $e^{-ix} = 1/z$. Now we may simplify this integral as follows:

$$\int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\,\frac{dz}{iz} = \int_C \frac{2}{4iz-z^2+1}\,dz = -2\int_C \frac{1}{z^2-4iz-1}\,dz. \tag{9}$$

This is exactly the kind of integral which can be evaluated using the residue theorem! We just need to find the poles of the integrand. Now $z^2-4iz-1=0$ can be solved using the quadratic formula:

$$z = 2i + (-4+1)^{1/2} = 2i + i\sqrt{3}, 2i - i\sqrt{3} = i(2\pm\sqrt{3}).$$

Now $\sqrt{3} \in (1,2)$, so that the root $i(2+\sqrt{3})$ lies outside the unit circle while the other root, $i(2-\sqrt{3})$, lies within it. Thus the integral (9) can be evaluated by computing the residue of the integrand at this root, which is

$$\begin{aligned}\mathrm{Res}_{i(2-\sqrt{3})}\frac{1}{z^2-4iz-1} &= \lim_{z\to i(2-\sqrt{3})}\frac{z-i(2-\sqrt{3})}{z^2-4iz-1} = \lim_{z\to i(2-\sqrt{3})}\frac{1}{z-i(2+\sqrt{3})}\\ &= \frac{1}{-2i\sqrt{3}} = \frac{i}{2\sqrt{3}},\end{aligned}$$

so that finally

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = -4\pi i \cdot \frac{i}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

Again, that this integral is *also!* an integral multiple of the previous too, is still a complete coincidence!

The method in this last example can be adapted to many other integrals with integrands which are rational functions of $\sin x$ and $\cos x$, defined everywhere on the interval of integration. See Goursat, §45, for a general discussion.

36. Liouville's Theorem. We prove the following result:

LIOUVILLE'S THEOREM. Let f be a function which is analytic and bounded on the entire complex plane. Then f must be constant.

This means that there must be a constant M such that $|f(z)| \le M$ for all $z \in \mathbf{C}$. The proof is an easy application of the Cauchy integral formula. Let $z \in \mathbf{C}$, let $R > 0$ be any positive real number, and let C_R denote the circle of radius R centred at z. Then we must have

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z')}{(z'-z)^2}\,dz',$$

so

$$|f'(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z')|}{R^2} R\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{R}\,dt = \frac{M}{R}.$$

But now $f'(z)$ cannot depend on R; since the right-hand side of the above expression goes to zero as $R \to \infty$, we must have $|f'(z)| = 0$, i.e., $f'(z) = 0$, for all $z \in \mathbf{C}$. This means that f must itself be constant, as claimed.

[I can't remember if we have ever proved that f' identically zero means that f must be constant when f is an analytic function. At any rate it is not hard to prove. Suppose that f is analytic on the interior of some simple closed curve, and pick any point z_0 inside that curve; then we can write, by the fundamental theorem of calculus,

$$f(z) = \int_{z_0}^{z} f'(z')\,dz' + f(z_0) = f(z_0),$$

since $f'(z') = 0$. This means that f must be constant.]

This can be used to prove that every (nonconstant) polynomial with complex coefficients has at least one complex root. To see this, let

$$P(z) = a_n z^n + \cdots + a_0,$$

where $a_n, \cdots, a_0 \in \mathbf{C}$ and we assume that $a_n \ne 0$. Suppose that P has no complex roots; we shall show that this implies that $n = 0$, so that P is constant. Since P has no complex roots, its reciprocal $1/P$ must be analytic everywhere on the complex plane. Now note that

$$\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \tag{10}$$

must go to zero as $|z|$ goes to infinity; thus there is an $R > 0$ such that the modulus of (10) is less than $\frac{1}{2}|a_n|$ when $|z| \ge R$. For such z, then, we have

$$\begin{aligned}|P(z)| = |z|^n \left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right| &\ge R^n \left[|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right|\right] \\ &\ge R^n \left[|a_n| - \frac{1}{2}|a_n|\right] = \frac{1}{2}R^n |a_n|,\end{aligned}$$

so since $a_n \ne 0$ we must have

$$\left|\frac{1}{P(z)}\right| \le \frac{2}{R^n |a_n|}$$

for $|z| \ge R$. Thus $1/P$ is bounded on the exterior of the disk of radius R centred at the origin. But since $|1/P|$ is a continuous function, and this disk is closed and bounded, $|1/P|$ must be bounded inside the disk as well; thus it must be bounded everywhere, and by Liouville's Theorem it must therefore be constant. Since it cannot be equal to zero, it must equal a nonzero constant, and hence $P = 1/(1/P)$ must be constant as well, as claimed.

37. The argument principle and Rouché's Theorem. We prove the following result:

THE ARGUMENT PRINCIPLE. Let C be a simple closed curve, and let f be a function analytic within and on C except possibly for poles within C, and which is nonzero on C. If Z denotes the number of zeroes and P the number of poles of f within C, counted with multiplicity, then

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i(Z - P).$$

To see this, we note the following result. If z_0 is a zero of f within C, say of multiplicity m, then there is a nonzero analytic function ϕ on a disk around z_0 such that near z_0

$$f(z) = (z - z_0)^m \phi(z);$$

while if z_0 is a pole of f within C, say of order n, then there is a nonzero analytic function ψ on a disk around z_0 such that near z_0

$$f(z) = (z - z_0)^{-n}\psi(z).$$

In the first case,

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1}\phi(z) + (z - z_0)^m \phi'(z)}{(z - z_0)^m \phi(z)} = \frac{m}{z - z_0} + \frac{\phi'(z)}{\phi(z)},$$

and we note that ϕ'/ϕ is analytic since ϕ is nonzero. Similarly, in the second case

$$\frac{f'(z)}{f(z)} = \frac{-n(z - z_0)^{-n-1}\psi(z) + (z - z_0)^{-n}\psi'(z)}{(z - z_0)^{-n}\psi(z)} = -\frac{n}{z - z_0} + \frac{\psi'(z)}{\psi(z)},$$

and again ψ'/ψ is analytic since ψ is nonzero. Now if z_0 is a point in C which is neither a pole nor a zero of f, then clearly f'/f is itself analytic near z_0. Thus the only poles of f'/f within C are the zeroes and the poles of f, and these are simple poles with residues equal to the multiplicitly of the zero or the negative of the order of the pole, respectively.

Now let $\{z_i\}$ and $\{p_j\}$ denote the zeroes and poles, respectively, of f within C. Then by the foregoing, and the residue theorem,

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\, dz &= 2\pi i\left[\sum_{z_i} \mathrm{Res}_{z_i} \frac{f'(z)}{f(z)} + \sum_{p_j} \mathrm{Res}_{p_j} \frac{f'(z)}{f(z)}\right]\\ &= 2\pi i\left[\sum m_i - \sum n_j\right] = 2\pi i(Z - P),\end{aligned}$$

where m_i denotes the multiplicity of the zero z_i and n_j denotes the order of the pole p_j, and the last equation follows by the definition of Z and P.

The following result can be derived from this after a further study of the geometrical meaning of Rouché's Theorem. We shall give more details on all of this later.

ROUCHÉ'S THEOREM. Suppose that f and g are analytic within and on a simple closed curve C, and that $|f(z)| > |g(z)|$ everywhere on C. Then $f + g$ and f have the same number of zeroes inside C.

We can proceed in two different ways: by contradiction, and directly. In the lecture we gave a proof by contradiction; here we show how to proceed directly (though the direct method requires the use of the Bolzano-Weierstrass Theorem enunciated in the previous set of lecture notes). Since the polynomial P is not constant, neither is $1/P$;

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 20 – 31

Due Saturday, August 1, at 10:00 PM EDT.

1. Consider the integral from question 2 of the previous homework assignment:

$$\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx,$$

and assume that both m and a are positive real numbers. By using an indented contour, evaluate this integral fully. [You are allowed to resubmit material submitted as part of the previous assignment if you wish.]

By $(\sin mx)/x$ in the integrand is meant the function

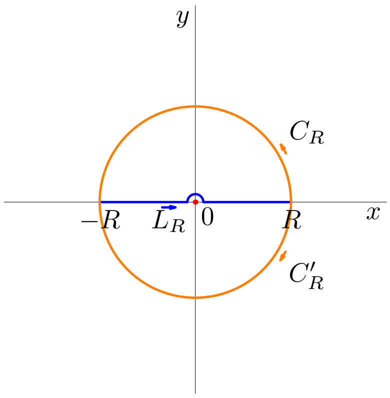
$$\begin{cases} \frac{\sin mx}{x}, & x \neq 0 \\ m, & x = 0, \end{cases}$$

which is just $mf(mx)$ if we set

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

which as we have seen already extends to an analytic function on the entire complex plane. By the Cauchy integral theorem, then, we can replace the integral over the real axis with an integral over an indented contour, as in the figure. We denote this by L_R, the upper semicircle by C_R, and the lower semicircle by C'_R, as shown in the figure. Now since the contour L_R does not pass through the origin, we may write

$$\begin{aligned}\int_{-R}^{R} \frac{\sin mx}{x(x^2+a^2)}\,dx &= \int_{L_R} \frac{\sin mz}{z(z^2+a^2)}\,dz \\ &= \int_{L_R} \frac{e^{imz}-e^{-imz}}{2iz(z^2+a^2)}\,dz = \int_{L_R} \frac{e^{imz}}{2iz(z^2+a^2)}\,dz - \int_{L_R} \frac{e^{-imz}}{2iz(z^2+m^2)}\,dz,\end{aligned}$$



and we may evaluate these integrals by closing in the upper and lower half-planes, respectively, and applying Jordan's lemma. Specifically, since $m > 0$ and on C_R and C'_R we have

$$\lim_{R\to\infty} \left|\frac{1}{2iz(z^2+a^2)}\right| \leq \lim_{R\to\infty} \frac{1}{2R(R^2-a^2)} = 0,$$

Jordan's lemma gives

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{imz}}{2iz(z^2+a^2)}\,dz = 0$$

and so, since ia is the only pole of the integrand in the contour $L_R + C_R$,

$$\lim_{R\to\infty}\int_{L_R}\frac{e^{imz}}{2iz(z^2+a^2)}\,dz = 2\pi i\mathrm{Res}_{ia}\frac{e^{imz}}{2iz(z^2+a^2)}$$
$$= 2\pi i\lim_{z\to ia}(z-ia)\frac{e^{imz}}{2iz(z-ia)(z+ia)}$$
$$= 2\pi i\frac{e^{-ma}}{-2\pi(2ia)} = -\frac{e^{-ma}}{2a^2}\pi.$$

Similarly, since $m > 0$, on the lower half-circle we have also by the modified form of Jordan's lemma we used once before

$$\lim_{R\to\infty}\int_{C'_R}\frac{e^{-imz}}{2iz(z^2+a^2)}\,dz = 0$$

and so, since the curve $L_R + C'_R$ is oriented *clockwise* and contains the two poles ia and 0,

$$\lim_{R\to\infty}\int_{L_R}\frac{e^{-imz}}{2iz(z^2+a^2)}\,dz = -2\pi i\left[\mathrm{Res}_{-ia}\frac{e^{-imz}}{2iz(z^2+a^2)} + \mathrm{Res}_0\frac{e^{-imz}}{2iz(z^2+a^2)}\right]$$
$$= -2\pi i\left[\lim_{z\to -ia}(z+ia)\frac{e^{-imz}}{2iz(z-ia)(z+ia)} + \lim_{z\to 0} z\frac{e^{-imz}}{2iz(z^2+a^2)}\right]$$
$$= -2\pi i\left[\frac{e^{-ma}}{2a(-2ia)} + \frac{1}{2ia^2}\right] = \pi\frac{e^{-ma}}{2a^2} - \frac{\pi}{a^2},$$

from which we obtain finally

$$\int_{-\infty}^{\infty}\frac{\sin mx}{x(x^2+a^2)} = -\frac{e^{-ma}}{2a^2}\pi - \left[\pi\frac{e^{-ma}}{2a^2} - \frac{\pi}{a^2}\right] = \frac{\pi}{a^2}\left(1 - e^{-ma}\right).$$

[Marking: 1 mark for using an indented contour; 1 mark for closing in both half-planes; in each half-plane, 1 mark for applying Jordan's lemma and 1 mark for the corresponding limit; in each half-plane, 1 mark for the application of the residue theorem; 2 marks for each of three residue calculations; 1 mark for the calculations giving the final answer. 15 marks total.]

2. Evaluate the following integral:

$$\int_0^{2\pi}\frac{dt}{(c-\cos 2t)^2}, \tag{1}$$

where c is a real number with absolute value greater than 1.

This integral becomes far simpler if we first make the substitution $x = 2t$ [1 mark]; the integral then becomes

$$\int_0^{4\pi}\frac{\frac{1}{2}dx}{(c-\cos x)^2} = \int_0^{2\pi}\frac{dx}{(c-\cos x)^2},$$

where the second equality follows since cos is periodic with period 2π. Now we use the usual procedure of viewing this integral as the integral over the unit circle of some function to be determined. We have

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right),$$

which will equal

$$\frac{1}{2}\left(z + z^{-1}\right)\text{ [2 marks]}$$

if $z = e^{it}$. Now given this z, we have $dz = ie^{it}dt = izdt$; since z is never zero, we may rewrite this as $dt = dz/(iz)$, and rewrite the original integral (1) as (letting C denote the unit circle)

$$\int_C\frac{dz/(iz)}{\left(c - \frac{1}{2}(z+z^{-1})\right)^2} = \frac{1}{i}\int_C\frac{4z\,dz}{(z^2-2cz+1)^2}.\text{[2 marks]}$$

Note that the integrand has two poles, both of second order;[1 mark]they are at the zeroes of $z^2 - 2cz + 1$. These may be found by using the quadratic formula; noting that the discriminant is $4c^2 - 4 = 4(c^2 - 1) > 0$, we may write these as

$$z = c \pm \sqrt{c^2 - 1}.\text{[2 marks]}$$

Now if $c > 1$, then clearly $c + \sqrt{c^2 - 1} > 1$, so only $c - \sqrt{c^2 - 1}$ lies inside the unit circle; while if $c < -1$, then clearly $c - \sqrt{c^2 - 1} < -1$, so only $c + \sqrt{c^2 - 1}$ lies inside the unit circle[2 marks]. Thus we consider these two cases separately. If $c > 1$, we have[4 marks, 1 for the residue theorem, 3 for the residue computation]

$$\begin{aligned}
\frac{1}{i}\int_C \frac{4z\,dz}{(z^2 - 2cz + 1)^2} &= 8\pi \mathrm{Res}_{c-\sqrt{c^2-1}} \frac{z}{(z^2 - 2z + 1)^2} \\
&= 8\pi \left.\frac{d}{dz} \frac{z}{(z - c - \sqrt{c^2 - 1})^2}\right|_{c-\sqrt{c^2-1}} = 8\pi \left[\frac{1}{c^2 - 1} + \frac{2(c - \sqrt{c^2 - 1})}{(c^2 - 1)^{3/2}}\right] \\
&= 8\pi \frac{2c - \sqrt{c^2 - 1}}{(c^2 - 1)^{3/2}},
\end{aligned}$$

while if $c < -1$, we have[4 marks, 1 for the residue theorem, 3 for the residue computation]

$$\begin{aligned}
\frac{1}{i}\int_C \frac{4z\,dz}{(z^2 - 2cz + 1)^2} &= 8\pi \mathrm{Res}_{c+\sqrt{c^2-1}} \frac{z}{(z^2 - 2cz + 1)^2} \\
&= 8\pi \left.\frac{d}{dz} \frac{z}{(z - c + \sqrt{c^2 - 1})^2}\right|_{c+\sqrt{c^2-1}} = 8\pi \left[\frac{1}{c^2 - 1} - \frac{2(c + \sqrt{c^2 - 1})}{(c^2 - 1)^{3/2}}\right] \\
&= 8\pi \cdot \frac{-2c - \sqrt{c^2 - 1}}{(c^2 - 1)^{3/2}},
\end{aligned}$$

and we see finally that

$$\int_0^{2\pi} \frac{dt}{(c - \cos 2t)^2} = \frac{16\pi|c|}{(c^2 - 1)^{3/2}} - \frac{8\pi}{c^2 - 1}.\text{[2 marks]}$$

[Marking: as indicated.]

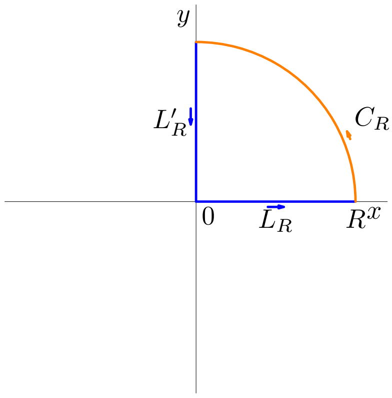
3. Choose *one* of the following integrals, and evaluate it:

$$\int_0^\infty \frac{\cos x^2 - \sin x^2}{x^8 + 1}\,dx, \qquad \int_0^\infty \frac{x^{1/2}}{x^2 + 1}.$$

You are strongly encouraged to also do the other integral for practice! [Hint for the first integral: try evaluating $\int_0^\infty \frac{e^{ix^2}}{x^8+1}\,dx$.]

For the first integral, we close using the quarter-circle contour shown in the figure. Note that, parameterising the circular arc as Re^{it}, $t \in [0, \pi/2]$, we have

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iz^2}}{z^8 + 1}\,dz\right| &\le \int_0^{\pi/2} R\left|\frac{e^{iR^2\cos 2t}e^{-R^2\sin 2t}}{z^8 + 1}\right| dt \\
&\le \frac{R}{R^8 - 1}\frac{1}{2}\int_0^\pi e^{-R^2\sin x}\,dx = \frac{R}{R^8 - 1}\int_0^{\pi/2} e^{-R^2\sin x}\,dx \\
&= \frac{R}{R^8 - 1}\frac{\pi}{2R^2}\left(1 - \frac{1}{e^2}\right),
\end{aligned}$$



where we have used the substitution $x = 2t$, the fact that sin is symmetric about $\pi/2$, and the Jordan inequality. Now clearly this last expression goes to zero as $R \to \infty$, so

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz^2}}{z^8+1}\,dz = 0.$$

Since the only singularities of the integrand inside the curve $L_R + C_R + L'_R$ are simple poles at $z = e^{i\pi/8}$ and $z = e^{3i\pi/8}$, we have

$$\int_{L_R} \frac{e^{iz^2}}{z^8+1}\,dz + \int_{L'_R} \frac{e^{iz^2}}{z^8+1}\,dz + \int_{C_R} \frac{e^{iz^2}}{z^8+1}\,dz = 2\pi i \left[\mathrm{Res}_{e^{i\pi/8}} \frac{e^{iz^2}}{z^8+1} + \mathrm{Res}_{e^{3i\pi/8}} \frac{e^{iz^2}}{z^8+1}\right]. \tag{2}$$

These residues may be calculated as follows. If z_0 denotes either of the poles, then

$$\mathrm{Res}_{z_0} \frac{e^{iz^2}}{z^8+1} = \lim_{z\to z_0} (z-z_0)\frac{e^{iz^2}}{z^8+1} = \lim_{z\to z_0} e^{iz^2} \frac{z-z_0}{(z^8+1)-(z_0^8+1)} = e^{iz_0^2} \frac{1}{\frac{d}{dz}z^8+1\big|_{z=z_0}} = \frac{e^{iz_0^2}}{8z_0^7},$$

so the residues in (2) above are

$$\begin{aligned} \frac{e^{ie^{i\pi/4}}}{8e^{7\pi i/8}} &= \frac{e^{-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}}{8e^{7\pi i/8}} = \frac{e^{-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}}{8}(-i)e^{-3\pi i/8}, \\ \frac{e^{ie^{3\pi i/4}}}{8e^{21\pi i/8}} &= -\frac{e^{-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}}}{8}e^{3\pi i/8}; \end{aligned}$$

if we let $\omega = a + ib$ denote the second of these, then the sum of residues becomes

$$\omega + i\overline{\omega} = a + ib + i(a - ib) = a(1+i) + ib(1-i),$$

so the sum of integrals in (2) will become

$$\begin{aligned} 2\pi i(a(1+i) + ib(1-i)) &= 2\pi[-a(1-i) - b(1-i)] = -2\pi(a+b)(1-i) \\ &= \frac{2\pi}{8}(1-i)e^{-\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right) + \sin\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right)\right]. \end{aligned} \tag{3}$$

We must now determine how to evaluate the integral over L'_R. If we parameterise $-L'_R$ as it, $t \in [0, R]$, then we may write

$$\int_{L'_R} \frac{e^{iz^2}}{z^8+1}\,dz = -i\int_0^R \frac{e^{-it^2}}{t^8+1}\,dt,$$

whence we see that

$$\int_{L_R} \frac{e^{iz^2}}{z^8+1}\,dz + \int_{L_R'} \frac{e^{iz^2}}{z^8+1}\,dz = \int_0^R \frac{e^{it^2} - ie^{-it^2}}{t^8+1}\,dt$$
$$= \int_0^R \frac{\cos t^2 - \sin t^2 - i(\cos t^2 - \sin t^2)}{t^8+1}\,dt = (1-i)\int_0^R \frac{\cos t^2 - \sin t^2}{t^8+1}\,dt,$$

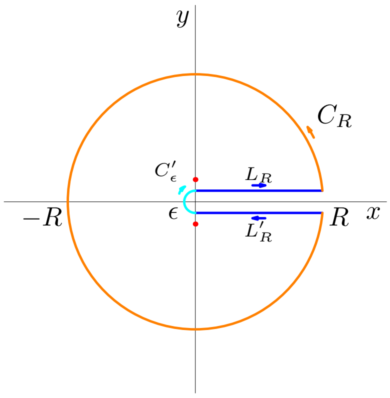
and we have finally from (3)

$$\int_{-\infty}^{\infty} \frac{\cos t^2 - \sin t^2}{t^8+1}\,dt = \frac{2\pi}{8}e^{-\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right) + \sin\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right)\right].$$

[Marking: 2 marks for the choice of curves; 1 mark for applying the residue theorem; 3 marks for showing the integral over C_R goes to 0 as $R \to \infty$; 4 marks for the residue calculations; 2 marks for the calculation of the sum of the residues; 3 marks for evaluating L_R' and obtaining the final result.]

The second integral is rather easier. Since we wish as usual to evaluate the integral by using a contour in the complex plane, we must pick a particular branch of the square root function appearing in the numerator and choose a contour which avoids the corresponding branch cut. As in the example in the notes, we shall take a branch cut along the positive real axis, and require the angle for the square root function to lie in the interval $(0, 2\pi)$. Having done this, we shall use the keyhole contour shown in the figure. Now for R large, we may write on C_R, noting that $|z^{1/2}| = \sqrt{|z|}$ no matter which branch of the square root function we use,

$$R\left|\frac{z^{1/2}}{1+z^2}\right| \le R\frac{R^{1/2}}{R^2-1}, \tag{4}$$



which clearly goes to zero as $R \to \infty$; thus we must have

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{1+z^2}\,dz = 0.$$

Similarly, for ϵ very small we have a result exactly analogous to (4):

$$\epsilon\left|\frac{z^{1/2}}{1+z^2}\right| \le \epsilon\frac{e^{1/2}}{1-\epsilon^2},$$

where we have $|1+z^2| \ge 1 - \epsilon^2$ since we are interested in ϵ small and may assume $\epsilon < 1$. This goes to zero as $\epsilon \to 0$, which implies that

$$\lim_{\epsilon\to 0^+} \int_{C_\epsilon'} \frac{z^{1/2}}{1+z^2}\,dz = 0.$$

Thus in the limits $R \to \infty$, $\epsilon \to 0$, we have, since the only poles of the integrand inside the contour are at $\pm i$, and the integrand has no other singularities inside the contour,

$$\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz + \int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz = 2\pi i\left[\mathrm{Res}_i \frac{z^{1/2}}{1+z^2} + \mathrm{Res}_{-i}\frac{z^{1/2}}{1+z^2}\right]. \tag{5}$$

Now

$$\begin{aligned}
\mathrm{Res}_i \frac{z^{1/2}}{1+z^2} &= \lim_{z\to i}(z-i)\frac{z^{1/2}}{(z-i)(z+i)}\\
&= \lim_{z\to i}\frac{z^{1/2}}{z+i} = \frac{e^{i\pi/4}}{2i} = \frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}},\\
\mathrm{Res}_{-i} \frac{z^{1/2}}{1+z^2} &= \lim_{z\to -i}(z+i)\frac{z^{1/2}}{(z-i)(z+i)}\\
&= \lim_{z\to -i}\frac{z^{1/2}}{z-i} = \frac{e^{3i\pi/4}}{-2i} = -\frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}},
\end{aligned}$$

so the sum of residues in (5) is simply $-i/(2\sqrt{2})$ and the sum of integrals in (5) is $\pi/\sqrt{2}$. Now we may parameterise the lines L_1 and $-L_2$ by $t + i\epsilon$ and $t - i\epsilon$, respectively, where $t \in [0, R]$ in both cases. Now in polar form

$$\begin{aligned}
t + i\epsilon &= \sqrt{t^2+\epsilon^2}\,e^{i\arctan\frac{\epsilon}{t}},\\
t - i\epsilon &= \sqrt{t^2+\epsilon^2}\,e^{i(2\pi - \arctan\frac{\epsilon}{t})},
\end{aligned}$$

where the range of arctan is $(-\pi/2, \pi/2)$ and the angle in the second line is chosen so as to lie in the interval $(0, 2\pi)$ corersponding to our chosen branch of $z^{1/2}$. Thus

$$\begin{aligned}
\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\left(t^2+\epsilon^2\right)^{1/4} e^{\frac{1}{2}i\arctan\frac{\epsilon}{t}}}{1+(t+i\epsilon)^2}\,dt,\\
-\int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\left(t^2+\epsilon^2\right)^{1/4} e^{\frac{1}{2}i(2\pi - \arctan\frac{\epsilon}{t})}}{1+(t-i\epsilon)^2}\,dt;
\end{aligned}$$

taking the limit as $\epsilon \to 0^+$ gives

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\sqrt{t}}{1+t^2}\,dt,\\
-\lim_{\epsilon\to 0^+}\int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{-\sqrt{t}}{1+t^2}\,dt,
\end{aligned}$$

so substituting in to equation (5) and taking the limit as $R \to \infty$ gives

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{1}{2}\cdot\frac{\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

[Marking: 1 mark for the contour; 2 marks each for showing the integrals over C_R and C'_ϵ vanish in the appropriate limits; 1 mark for applying the residue theorem; 2 marks for calculating the residues; 2 marks each for working out the L_R and L'_R integrals; 1 mark for adding up and solving.]

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR AUGUST 3 – 7

Due Monday, August 10, at 11:59 PM EDT.

1. [15 marks] Evaluate whichever of the integrals from problem 3 of the last assignment you did not do last week.

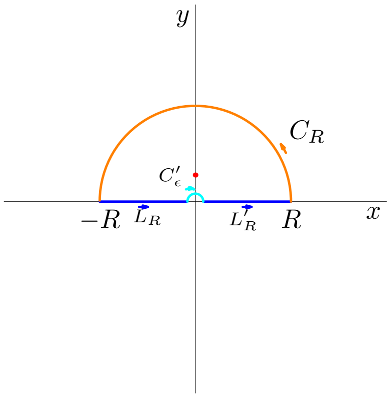
[See the previous set of solutions! – also for marking scheme!]

2. [20 marks] Evaluate the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx,$$

where $\log x$ denotes the usual real-valued logarithm of the positive real number x.

Our first inclination (well, Nathan's first inclination, anyway!) may be to use a keyhole contour like that used for the second integral in 1. However, this won't work, as the portions of the integrals on the line segments across the branch containing $\log x/(1+x^2)^2$ will cancel. Thus we instead use the following contour,[2 marks] which is almost the same as the one we used when evaluating integrals involving $\sin x/x$ except that we break it up into more pieces. As in the second integral in 1, in order to integrate over contours in the complex plane we must choose a particular branch of the logarithm.[1 mark]In this case it turns out to be convenient to take a branch cut which is far away from the contour; thus we shall take a branch cut along the negative imaginary axis, and require our angle to lie in the interval $(-\pi/2, 3\pi/2)$. [A branch cut along any line in the lower half-plane would work equally well, and in fact the calculations below would be unchanged. While one could also use a branch cut along part of the real axis, it would in this case only lead to additional complications.]



Now if z is any point on C_R, then we may write $z = Re^{i\theta}$, where $\theta \in [0,\pi]$; thus $\operatorname{Log} z = \log R + i\theta$, so $|\operatorname{Log} z| \le |\log R| + \pi$ and we may write

$$R\left|\frac{\operatorname{Log} z}{(1+z^2)^2}\right| \le R\frac{|\log R| + \pi}{(R^2-1)^2}, \text{[1 mark]}$$

which clearly goes to zero as $R \to \infty$ (remember that $\log R < R$ for all R), so

$$\lim_{R\to\infty}\int_{C_R}\frac{\operatorname{Log} z}{(1+z^2)^2}\,dz = 0.\text{[1 mark]}$$

Similarly, if z is any point in C'_ϵ, then we may write $z = \epsilon e^{i\theta}$, where again $\theta \in [0,\pi]$, so as before we have $|\operatorname{Log} z| \le |\log \epsilon| + \pi$ and

$$\epsilon\left|\frac{\operatorname{Log} z}{(1+z^2)^2}\right| \le \epsilon\frac{|\log \epsilon| + \pi}{(1-\epsilon^2)^2}.\text{[1 mark]}$$

Now clearly

$$\lim_{\epsilon\to 0^+}\epsilon\frac{\pi}{(1-\epsilon^2)^2} = 0; \text{[1 mark]}$$

also, by L'Hôpital's rule (for real functions!),

$$\lim_{\epsilon\to 0^+} \epsilon \log \epsilon = \lim_{\epsilon\to 0^+} \frac{\log \epsilon}{1/\epsilon} = \lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon\to 0^+} -\epsilon = 0, \text{[1 mark]}$$

from which we see that

$$\lim_{\epsilon\to 0^+} \int_{C'_\epsilon} \frac{\operatorname{Log} z}{(1+z^2)^2}\,dz = 0.$$

Thus, in the limits $\epsilon \to 0^+$ and $R \to \infty$, we have, since the only singularity of the integrand inside the contour is a pole at $z = i$,

$$\lim_{\epsilon\to 0^+} \lim_{R\to\infty} \int_{L_1} \frac{\operatorname{Log} z}{(1+z^2)^2}\,dz + \int_{L_2} \frac{\operatorname{Log} z}{(1+z^2)^2}\,dz = 2\pi i \operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2}.\text{[1 mark]}$$

Before computing the residue, we note that $-L_1$ may be parameterised as $te^{i\pi}$, $t \in [0, R]$, so that

$$\int_{L_1} \frac{\operatorname{Log} z}{(1+z^2)^2}\,dz = \int_0^R \frac{\log t + i\pi}{(1+t^2)^2}\,dt = \int_{L_2} \frac{\operatorname{Log} z}{(1+z^2)^2}\,dz + i\pi \int_0^R \frac{1}{(1+t^2)^2}\,dt.\text{[2 marks]}$$

Thus we have also to evaluate the integral

$$\int_0^\infty \frac{1}{(1+t^2)^2}\,dt = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+t^2)^2}\,dt;$$

this latter integral may be evaluated by closing the contour in the upper half-plane,[1 mark]noting that the integral over the extra semicircle will go to zero as its radius goes to infinity since

$$R\left|\frac{1}{(1+R^2e^{2it})^2}\right| \le R\frac{1}{(R^2-1)^2} \to 0 \quad \text{as} \quad R \to \infty, \text{[1 mark]}$$

and thus obtaining

$$\int_{-\infty}^\infty \frac{1}{(1+t^2)^2}\,dt = 2\pi i \operatorname{Res}_i \frac{1}{(1+z^2)^2}\text{[1 mark]} = 2\pi i \left.\frac{d}{dz}\frac{1}{(z+i)^2}\right|_{z=i} = 2\pi i\left(-\frac{2}{(2i)^3}\right) = \frac{\pi}{2}, \text{[2 marks]}$$

so finally

$$\int_0^\infty \frac{1}{(1+t^2)^2}\,dt = \frac{\pi}{4}$$

and

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{1}{2}\left[2\pi i \operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2} - i\frac{\pi^2}{4}\right].$$

Thus we need only compute this residue. We have

$$\operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2} = \left.\frac{d}{dz}\frac{\operatorname{Log} z}{(z+i)^2}\right|_{z=i} = \frac{\frac{1}{i}(2i)^2 - 2(2i)\operatorname{Log} i}{(2i)^4} = \frac{-\frac{4}{i} - 4i\frac{i\pi}{2}}{16} = \frac{i}{4} + \frac{\pi}{8}, \text{[3 marks]}$$

whence finally

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = \frac{1}{2}\left[-\frac{\pi}{2} + i\frac{\pi^2}{4} - i\frac{\pi^2}{4}\right] = -\frac{\pi}{4}.\text{[1 mark]}$$

We note that it is reasonable to obtain a negative number since $\log x \to -\infty$ as $x \to 0^+$.
[Marking: as above.]

3. [15 marks] Suppose that f is a function which is analytic on the entire complex plane. Let C_R denote the portion of the circle of radius R centred at the origin lying between the angles θ_1 and θ_2 with the positive real axis. Is it possible to have

$$\lim_{R\to\infty} \int_{C_R} |f(z)|\,ds = 0?$$

(The integral here is an arclength integral from multivariable calculus.) If not, prove it; otherwise, give an example. [Hint: Liouville's Theorem!]

Summary:
- We review the method of using analytic maps to perform a 'change of variables' in boundary-value problems involving Laplace's equation.
- We give an example, and then study the properties of a few special maps.

(Cf. Goursat, §§22, 24.)

38. Transformations of solutions to Laplace's equation. Last week we discussed the Poisson kernel and its relation to the Cauchy integral theorem: we saw that, just as the Cauchy integral formula gives the value of an analytic function everywhere inside a simple closed curve as an integral involving only the values of the function on the *boundary* of the region (i.e., the simple closed curve), so too the Poisson kernel allows us to write the values of a harmonic function inside a simple closed curve as an integral involving only the values of the function on the boundary (i.e., again, the curve).[0] Today we shall see another method of solving Laplace's equation given information about the function on the boundary, using the fact that analytic functions essentially take harmonic functions to harmonic functions and thus allow us to perform a 'change of variables' of sorts in Laplace's equation to replace a (potentially) hard problem with a (hopefully) easier one.

Before beginning this, we note one small point. When we speak of harmonic functions, we always mean real-valued functions of two real variables. On the other hand, when we speak of analytic functions on the plane we mean complex-valued functions of a complex variable. To avoid tiresome and unimportant notational issues, we shall agree that if u is any real-valued function of two real variables, and $z \in \mathbf{C}$ is any complex number, then the notation $u(z)$ (which is technically undefined) shall mean $u(\operatorname{Re} z, \operatorname{Im} z)$; in other words, we write as a shorthand

$$u(x + iy) = u(x, y).$$

With this out of the way, we have the following result, which we saw some version of back towards the start of the course: suppose that D, $E \subset \mathbf{C}$ are two regions (e.g., interiors of closed curves), that $u : D \to \mathbf{R}$ is a harmonic function on D, and that $f : E \to D$ is an analytic map with $f(E) \cap \partial D = \emptyset$.[1] Then the

[0] It is worth noting however that there is a very basic difference between these two formulas which we have so far glossed over: we have noted (though not proved) that the function u defined using the Poisson kernel approaches the function h in the limit as its argument (x, y) goes radially to a point on C. The same is *not* in general true for the Cauchy integral formula, for a very basic reason. As we noted earlier, the method by which we proved the Cauchy integral formula for derivatives allows us to prove also that any function defined as an integral of the form

$$F(z) = \int_C \frac{h(z')}{z' - z}\, dz', \tag{1}$$

where C is some simply closed curve, z is any point inside C, and h is a continuous function on C, must also have a derivative everywhere inside C, and hence must be analytic. But *this analytic function need not agree with h on the boundary curve C*, for the following reason: suppose that h were real-valued; then if $F(z) = h(z)$ for $z \in C$, the analytic function F would be real-valued on C, and hence its imaginary part would be zero everywhere on C. But its imaginary part must be harmonic, and any harmonic function which vanishes on a simple closed curve must vanish everywhere inside that curve; hence F must be real-valued inside C as well as on C, and must therefore be constant. But if h is not constant then this contradicts $F(z) = h(z)$ on C. Another way of looking at this is that the function F includes two nonzero harmonic functions, so somehow the integral (1) is actually giving us *two* harmonic functions, which clearly contain more information on the boundary than is included in the function h alone. (I should point out that in general there is no reason to believe that even the real part of F agrees with h on the boundary curve C; and here it is worth recalling that when we derived the Poisson kernel we had to add in an extra term by hand involving the point z^*.) This is a bit of a side comment, but worth keeping in mind to avoid making mistakes.

[1] In the applications we shall be mostly interested in cases where the boundary curves of D and E are simple (i.e., where D and E are simply connected), where $f(\partial E) = \partial D$, and where f^{-1} is also analytic. But nothing in this present result relies on any of these conditions, so we state it in greater generality.

function

$$v = u \circ f : E \to \mathbf{R}$$

is also harmonic on E.

To see this, let $a \in E$; then $f(a) \in D$, so that u must be harmonic at $f(a)$. Now we can find an $\epsilon > 0$ so that the disk of radius ϵ centred at $f(a)$ is still contained in D [we do not care whether it is contained in $f(E)$; that is actually totally irrelevant]; let us denote this disk by U. Thus u is actually harmonic on U, and there must then be a conjugate harmonic function on U, call it $\tilde{u}$, so that $u + i\tilde{u}$ is analytic on U. (See Goursat, §3, p. 10; or §9 in these lecture notes.) Let us denote this function by g. Then $g \circ f$ must be analytic at a, by the chain rule; but since the real and imaginary parts of all analytic functions are harmonic, its real part $u \circ f = v$ must be harmonic on E, as desired. (The restricting to a disk is only to make sure that the function $\tilde{u}$ is well-defined – as we have seen before (e.g., in the second quiz), on a non-simply connected region a conjugate harmonic function may become multiple-valued.)

For the applications, we need to restrict the regions E and D and the function f, and for this we need a bit more notation. We shall require E and D to be simply connected; equivalently, we require their boundary curves ∂D and ∂E to be simple closed curves. We shall also require E and D to represent only the interior of their boundary curves, i.e., that $E \cap \partial E = D \cap \partial D = \emptyset$; we set $\overline{E} = E \cup \partial E$, $\overline{D} = D \cup \partial D$.[2] We assume that $f^{-1} : D \to E$ (exists and) is analytic, and that f and f^{-1} extend to continuous functions mapping $\overline{E}$ to $\overline{D}$ and $\overline{D}$ to $\overline{E}$, respectively; at the risk of being extremely confusing, we shall denote these extensions by f and f^{-1} as well.

To put all of this more simply, we assume that f is invertible with analytic inverse, and that f and its inverse extend to continuous functions on the boundary of E and D, respectively.

Now suppose that u satisfies

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = h, \tag{2}$$

where h is some piecewise-continuous function on ∂D. We call any problem like (2) a *boundary-value problem,* since we have conditions on u on the boundary of the region. Now let E and $f : E \to D$ satisfy the conditions just described, and set $v = u \circ f$. Then v is harmonic on E, while if $a \in \partial E$ we must have $f(a) \in \partial D$, so that

$$v(a) = (u \circ f)(a) = u[f(a)] = h[f(a)]$$

by the boundary condition in (2). Thus v satisfies the boundary-value problem

$$\Delta v = 0 \text{ on } E, \qquad v|_{\partial E} = h \circ f. \tag{3}$$

Now it may happen that we can find E and f such that problem (3) is simpler than (2); in fact, in the examples we shall see the answer to (3) can be guessed at quite easily. Suppose thus that we can find some function v satisfying (3). Then running the above logic backwards, we see that the function $\tilde{u} = v \circ f^{-1}$ must satisfy the boundary-value problem

$$\Delta \tilde{u} = 0 \text{ on } D, \qquad \tilde{u}|_{\partial D} = h;$$

in other words, $\tilde{u}$ is a solution to our original boundary value problem (2).

Let us see an example.

Example. Solve the following boundary problem on the region D given in polar coordinates as $D = \{(r, \theta) \,|\, r \in (0, 1),\, \theta \in (-\pi/4, \pi/4)\}$:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases} \tag{4}$$

The factor of 2 in the sine, the square in $\pm r^2$, and the shape of the region suggest that perhaps f ought to have something to do with a square. Now $z \mapsto z^2$ increases the angle by a factor of 2; thus if we were to

[2] For those who know something of topology, we note that $\overline{E}$ and $\overline{D}$, as the notation suggests, are just the closure of E and D in the usual topology of $\mathbf{R}^2$ in this case.

take $f(z) = z^2$, the region E would be even narrower than the region D, and the problem would probably not be any simpler. If we take $f(z) = z^{1/2}$, though (assuming we can take an appropriate branch!), then we see that the region E would be a half-disk. Thus let us let $f(z) = z^{1/2}$, where we take a branch cut along the negative real axis and require $\theta \in (-\pi, \pi)$. Then if we set

$$E = \{(r, \theta) \,|\, r \in (0, 1),\, \theta \in (-\pi/2, \pi/2)\},$$

we see that $f : E \to D$ is one-to-one and onto, and its inverse is given by $z \mapsto z^2$, which is analytic on D. Further, we may extend f to a continuous function on $\overline{E}$ by setting $f(0) = 0$ (this will not extend to an analytic function, of course, but that does not matter). Now we must determine how the boundary condition changes; in other words, if we define $h : \partial D \to \mathbf{R}$ by

$$h = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases},$$

then we need to determine $h \circ f : \partial E \to \mathbf{R}$. Now ∂E also has three pieces, namely $r = 1$, $\theta \in [-\pi/2, \pi/2]$, $\theta = \pi/2$, $r \in [0, 1]$, and $\theta = -\pi/2$, $r \in [0, 1]$. Now if $z = re^{i\theta}$, where $\theta \in (-\pi, \pi)$, then $f(z) = r^{1/2}e^{\frac{1}{2}i\theta}$; thus these three boundary pieces are mapped to, respectively,

$$r = 1,\, \theta \in [-\pi/4, \pi/4]; \quad \theta = \pi/4,\, r \in [0, 1]; \quad \theta = -\pi/4,\, r \in [0, 1].$$

Now if $r = 1$ and $\theta \in [-\pi/2, \pi/2]$, then we have $r^{1/2} = 1$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2}e^{\frac{1}{2}i\theta}) = h(e^{\frac{1}{2}i\theta}) = \sin 2 \cdot \left(\frac{1}{2}\theta\right) = \sin\theta;$$

if $\theta = \pi/2$ and $r \in [0, 1]$, then we have $\frac{1}{2}\theta = \pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2}e^{\frac{1}{2}i\theta}) = \left(r^{1/2}\right)^2 = r;$$

finally, if $\theta = -\pi/2$ and $r \in [0, 1]$, then we have $\frac{1}{2}\theta = -\pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2}e^{\frac{1}{2}i\theta}) = -\left(r^{1/2}\right)^2 = -r.$$

Pulling this together, we have

$$h \circ f = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases},$$

so that we wish to solve the problem

$$\Delta v = 0 \text{ on } E, \qquad v|_{\partial E} = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases} \tag{5}$$

Now we note that if a, b, and c are real numbers, then the function

$$g(x, y) = a + bx + cy \tag{6}$$

will be harmonic at every point $(x, y) \in \mathbf{R}^2$, since all of its second order derivatives vanish. Let us see whether we can find a solution to (5) which is of this form. The idea is that, since all of these functions satisfy Laplace's equation, we only need to fit the boundary conditions. Now in polar coordinates, the function g may be written

$$g = a + br\cos\theta + cr\sin\theta.$$

If $r = 1$, this gives $a + b\cos\theta + c\sin\theta$, which will fit the boundary condition in (5) if $a = b = 0$, $c = 1$; if $\theta = \pm\pi/2$, then it gives

$$a \pm cr,$$

which will also fit the boundary condition in (5) if $a = 0$ and $c = 1$. In other words, we have found that the function

$$v(x, y) = y$$

is a solution to (5).

Working backwards, then, we know from our general work above that the function $u = v \circ f^{-1}$ will solve our original problem. Now $f^{-1}(z) = z^2$, so that we have finally the solution

$$u(x, y) = v[f(x + iy)] = v(x^2 - y^2, 2xy) = 2xy$$

to the boundary value problem (4).

If we step back and look at the broad sweep of the logic used in this example, we see that to make efficacious use of this technique in practice, we must have a catalogue of two things: one, transformations and regions; two, standard solutions to Laplace's equation on the domain regions of these transformations. For us, the main class of standard solutions will be the linear ones given in (6). We now describe a few elementary transformations which are useful.

EXAMPLES of transformations and regions useful for solving Laplace's equation.

1. Powers and roots on wedges. If $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^n$ takes a wedge with vertex at the origin with angle α to another wedge with vertex at the origin with angle $n\alpha$. For this transformation to be invertible, the angle α must be less than $2\pi/n$. Similarly, if $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^{1/n}$, after taking an appropriate branch, will map a wedge of angle α with vertex at the origin to another wedge with vertex at the origin and angle α/n, assuming that $\alpha < 2\pi$. Wedges have three boundaries, two of which are lines and the third of which is a circular arc; on the circular arc, a map $z \mapsto z^p$ ($p = n$ or $p = 1/n$, as the case may be) will take a point $re^{i\theta}$ to a point $r^p e^{ip\theta}$, i.e., it will multiply the angle θ by p (where θ must be in the range corresponding to the chosen branch if $p = 1/n$), while on the lines the map acts by simple exponentiation, as in the example above.

2. Inversion. The map $z \mapsto z^{-1}$, as we have seen before (see §16 of these lecture notes), takes the punctured plane into itself, but 'turns it inside out' by mapping the region outside the unit circle to the region inside the unit circle.

3. The exponential function. Consider the map $z \mapsto e^z$. We have seen already (see the solutions to the first homework assignment) that this map takes lines parallel to the real axis to rays from the origin, and lines parallel to the imaginary axis to circles centred at the origin. Thus this map will take certain rectangular regions in the plane into annular wedges. (Going further, to whole annular regions or even a full disk, would involve extra considerations beyond the scope of our present discussion.)

4. The maps $z \mapsto \sin z$ and $z \mapsto \cos z$. The second of these has already been described in some detail in §16 of these lecture notes (cf. Goursat, §22); in particular, it was shown there that $z \mapsto \cos z$ takes the rectangle $\{x + iy \,|\, x \in (0, \pi),\ y \in (0, +\infty)\}$ into the lower half-plane $\{x + iy \,|\, y < 0\}$. Let us see a bit more carefully what it does on the boundary of this rectangle. This boundary consists of three lines, the half-line $x = 0$, $y \geq 0$, the line segment $y = 0$, $x \in [0, \pi]$, and the half-line $x = \pi$, $y \geq 0$. Let us consider these in turn. We have (see, e.g., §13 of these notes)

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y;$$

thus if $x = 0$ we have

$$\cos(iy) = \cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $[1, +\infty)$ in a one-to-one fashion. Similarly, if $x = \pi$, then we have

$$\cos(\pi + iy) = -\cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $(-\infty, -1]$. Finally, if $y = 0$, $\cos z$ will be just the ordinary real-valued function $\cos x$, which takes the interval $[0, \pi]$ into the interval $[-1, 1]$ (though in reverse order, i.e., it is decreasing on that interval). All told, then, the half-line $x = 0$, $y \geq 0$ will map to the interval $[1, +\infty)$, the segment $y = 0$, $x \in [0, \pi]$ will map to the interval $[-1, 1]$ (where the point 1 is mapped from the intersection $x = y = 0$ of these two parts of the boundary) and finally the half-line $x = \pi$, $y \geq 0$ will map to the interval $(-\infty, -1]$, with as before the point -1 is mapped from the intersection $x = \pi$, $y = 0$ of these last two parts of the boundary. This suggests that this map may be useful if we are interested in finding solutions to problems on the lower half-plane whose initial data can be broken down in some way across the three intervals $(-\infty, -1]$, $[-1, 1]$, and $[1, +\infty)$.

We note one other result: if $x = \pi/2$, then we have

$$\cos(x + iy) = \cos(\pi/2 + iy) = -i \sinh y,$$

from which we see that the half-line $x = \pi/2$, $y \geq 0$ is mapped to the negative imaginary axis. This means, incidentally, that the two rectangles

$$\{x + iy \,|\, x \in (0, \pi/2),\, y > 0\}, \quad \{x + iy \,|\, x \in (\pi/2, \pi),\, y > 0\}$$

map to the fourth and third quadrants, respectively (since $\cos x > 0$ for $x \in (0, \pi/2)$ and $\cos x < 0$ for $x \in (0, \pi/2)$). Thus this map can also be used for problems on a quarter-plane.

Similarly, let us consider the map $z \mapsto \sin z$. We have (ibid.)

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

In this case we take as our domain the rectangular strip $\{x + iy \,|\, x \in (-\pi/2, \pi/2),\, y > 0\}$, since $\sin x$ is invertible on $(-\pi/2, \pi/2)$ but not on any strictly larger interval. Let us consider the values of sin on the three boundary lines as we did for cos. If $x = -\pi/2$, then we have

$$\sin(-\pi/2 + iy) = -\cosh y,$$

while if $x = \pi/2$ then we have

$$\sin(\pi/2 + iy) = \cosh y;$$

thus these two lines map to the segments $(-\infty, -1]$ and $[1, +\infty)$, similarly to what we found for cos (except note that the order is reversed). Similarly, if $y = 0$ then $\sin z$ is just the ordinary real-valued sine function $\sin x$, which maps the interval $[-\pi/2, \pi/2]$ to the interval $[-1, 1]$. Thus, again, the boundary is mapped onto the entire real line. Since $\cos x$ and $\sinh y$ are both positive everywhere on the rectangle, we see that sin maps the rectangle into the upper half-plane; and this map is actually onto. Again, as with cos, we see that the midline $x = 0$, $y \geq 0$ maps to the positive real axis, and we see further that the rectangles

$$\{x + iy \,|\, x \in (-\pi/2, 0),\, y > 0\}, \quad \{x + iy \,|\, x \in (0, \pi/2)\, y > 0\}$$

map to the second and first quadrants, respectively (again, which quadrant is the image of which rectangle can be determined from the sign of x).

Summary:

- We give a proof of L'Hôpital's rule for analytic functions.
- We then show how another class of boundary conditions for Laplace's equation transforms under analytic maps, and give an example.
- [For the material on analytic continuation, please see the pre-class notes.]

39. L'Hôpital's rule for analytic functions. We prove the following version of L'Hôpital's rule for analytic functions. Suppose that f and g are analytic near a point a, and that both f and g have a zero of order m at a. Then

$$\lim_{z\to a}\frac{f(z)}{g(z)} = \lim_{z\to a}\frac{f^{(m)}(z)}{g^{(m)}(z)}. \tag{1}$$

(In the terminology usually used to discuss L'Hôpital's rule in elementary calculus courses, this is a limit of type 0/0.)

To see this, note that since f and g have zeroes of order m at a, there are functions $\phi(z)$ and $\gamma(z)$ which are both analytic and nonzero near a and satisfy

$$f(z) = (z-a)^m\phi(z), \qquad g(z) = (z-a)^m\gamma(z).$$

Moreover, we claim that $\phi(a) = f^{(m)}(a)/m!$ and $\gamma(a) = g^{(m)}(a)/m!$. The proof of these two equations is the same so we show only the first one. There are two distinct ways of doing this. The one we used in lecture involved Taylor series and is as follows. Suppose that the Taylor series for ϕ at a is

$$\sum_{k=0}^{\infty} a_k(z-a)^k,$$

where we know that $a_0 = \phi(a) \neq 0$. Then the Taylor series for f at a is

$$\sum_{k=0}^{\infty} a_k(z-a)^{k+m}.$$

But this series must equal

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n;$$

equating coefficients of like powers, we see that

$$a_0 = \frac{1}{m!}f^{(m)}(a),$$

as claimed.

The second method is much quicker and involves the Cauchy integral formula: if C is a sufficiently small circle around a, then we have

$$\begin{aligned}\phi(a) &= \frac{1}{2\pi i}\int_C \frac{\phi(z')}{z'-a}\,dz'\\ &= \frac{1}{2\pi i}\int_C \frac{f(z')/(z'-a)^m}{z'-a}\,dz' = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{m+1}}\,dz'\\ &= \frac{1}{m!}f^{(m)}(a),\end{aligned}$$

by the Cauchy integral formula for derivatives.

Given this, the proof of (1) is easy:

$$\lim_{z\to a}\frac{f(z)}{g(z)} = \lim_{z\to a}\frac{(z-a)^m\phi(z)}{(z-a)^m\gamma(z)} = \lim_{z\to a}\frac{\phi(z)}{\gamma(z)} = \frac{\phi(a)}{\gamma(a)} = \frac{f^{(m)}(a)}{g^{(m)}(a)},$$

since the factors of $m!$ will cancel.

We can also use the inner workings of the above proof to evaluate limits, which is often much easier than applying the result itself, as the following example shows.

EXAMPLE. Let $n \in \mathbf{Z}$ be positive. Find

$$\lim_{x\to 0} \frac{(\sin x)^{2n}}{(1-\cos x)^n}.$$

Let us consider the corresponding complex limit, i.e., replace the real number x in the limit above by a complex number z, since if the resulting limit exists then certainly the original limit does as well. Now note that $\sin z$ has a zero of order 1 at 0, so that we may write

$$\sin z = z\phi(z)$$

for some function ϕ which will be analytic and nonzero at $z = 0$ (and hence everywhere in this case, though that is not important). In fact, of course, the function ϕ will simply be the function

$$\begin{cases} \sin z/z, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

which we have seen many times already. We have moreover that $\phi(0) = \frac{d}{dz}\sin z\big|_{z=0} = 1$. Further, since

$$\cos z = 1 - \frac{1}{2}z^2 + \cdots, \tag{2}$$

we see that $1 - \cos z$ has a zero of order 2 at 0, so we may write

$$1 - \cos z = z^2\gamma(z),$$

where by inspection of the series (2) we have $\gamma(0) = \frac{1}{2}$. (This could also be obtained by differentiating as in the proof above, of course; in that case the factor of $1/2$ comes from the $1/m!$.) Thus we may write

$$\begin{aligned} \lim_{z\to 0} \frac{(\sin z)^{2n}}{(1-\cos z)^n} &= \lim_{z\to 0} \frac{z^{2n}[\phi(z)]^{2n}}{z^{2n}[\gamma(z)]^n} \\ &= \lim_{z\to 0} \frac{[\phi(z)]^{2n}}{[\gamma(z)]^n} = \frac{[\phi(0)]^{2n}}{[\gamma(0)]^n} = 2^n. \end{aligned}$$

To apply L'Hôpital's rule to this directly would have required us to differentiate n times, which would be extremely messy. This demonstrates the utility of power series manipulations and other concepts (such as the order of a zero) which we have studied in this course in elucidating the behaviour of analytic functions.

40. Transformation of Neumann boundary conditions by analytic maps. So far we have considered only problems involving so-called *Dirichlet* boundary conditions, i.e.,

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = h. \tag{3}$$

This means, of course, that we seek a function u which satisfies Laplace's equation inside the region D and is equal to the function h on the boundary of D; note that the Laplacian of u need not be defined on the boundary of D. Another type of boundary condition we could give is that the normal deriviative of u on the boundary be equal to some given function. In other words, let $\mathbf{n}$ denote the unit outward normal to the boundary curve ∂D (this is defined in an analogous way to how one defines the outward unit normal to a region in three-dimensional space when one discusses the divergence theorem in multivariable calculus); then, writing

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$$

(i.e., we *define* $\partial u/\partial n$ to be the quantity on the right-hand side), we may consider the problem

$$\Delta u = 0 \text{ on } D, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = h, \tag{4}$$

where again h is some function on the boundary ∂D. Note first of all that this problem will not have a unique solution on D, since if u is any solution then clearly so also is $u+C$, where C is any real number; thus in order to get a unique solution we need to specify some other piece of information about u, for example, its value at some point. We shall assume that given the value of u at any point, (4) has a unique solution. We shall also restrict attention to the case $h = 0$.

Let us see whether our method of transforming the problem with an analytic function is applicable to problem (4). Thus consider a region E and an analytic function $f : E \to D$ satisfying the conditions given in §38 of these lecture notes, and consider problem (4) with $h = 0$. We claim that if $v = u \circ f$, then v satisfies the problem

$$\Delta v = 0 \text{ on } E, \qquad \left.\frac{\partial v}{\partial N}\right|_{\partial E} = 0, \tag{5}$$

where $\partial/\partial N$ denotes the derivative of v in the direction normal to E. Let us see why this is so. That v satisfies $\Delta v = 0$ has already been shown; thus we need only consider the boundary condition. Let $a \in E$. Let $\mathbf{N}$ denote the outward unit normal to E at a; then

$$\left.\frac{\partial v}{\partial N}\right|_a = \mathbf{N} \cdot \nabla v(a) = \mathbf{N} \cdot \nabla(u \circ f)(a).$$

Now let $b = f(a)$ and consider $\nabla u(b)$. If $\nabla u(b) = 0$, then it is not hard to show that $\nabla(u \circ f)(a) = 0$ (using the chain rule); thus in this particular case $\partial v/\partial N|_a$ will be zero, as claimed. Thus now suppose that $\nabla u(b) \neq 0$. Since

$$\left.\frac{\partial u}{\partial n}\right|_b = \mathbf{n} \cdot \nabla u(b) = 0,$$

where $\mathbf{n}$ denotes the outward unit normal to D at $b = f(a)$, we see that $\nabla u(b)$ must be perpendicular to $\mathbf{n}$. But now $\nabla u(b)$ must also be perpendicular to the level curve, call it C, of u which passes through b, i.e., to the set of points (x, y) satisfying $u(x, y) = u(b)$; since we are in the plane this implies that this level curve must be tangent to $\mathbf{n}$ and hence perpendicular to the boundary curve ∂D. But since f^{-1} is assumed to be analytic and therefore conformal (its derivative will be nonzero since it is invertible), and maps ∂D to ∂E, this means that the curve C' to which f^{-1} maps C must be perpendicular to the boundary ∂E at the point a. But this curve is just the level curve of v through a, for (x, y) satisfies $v(x, y) = v(a)$ if and only if

$$u[f(x, y)] = u[f(a)] = u(b),$$

i.e., if and only if $f(x+iy) = x'+iy'$ for some point (x', y') in C, which is equivalent to $x+iy = f^{-1}(x'+iy')$, i.e., that (x, y) lie on C'. Thus the level curve of v through a must be perpendicular to the boundary ∂E, which means as before that the gradient of v at a must be perpendicular to the normal vector $\mathbf{N}$, and hence

$$\left.\frac{\partial v}{\partial N}\right|_a = \mathbf{N} \cdot \nabla v(a) = 0,$$

so that v satisfies the boundary condition in (5), as claimed.

Before going into the example, recall that the map $z \mapsto \sin z$ was described at the end of §38 in the previous set of lecture notes.

Example. Let $U = \{(x, y)|y > 0\}$ denote the upper half-plane. Solve the following problem:

$$\Delta u = 0 \text{ on } U, \qquad u|_{\partial U} = \begin{cases} 0, & x \in (-\infty, -1) \\ 1 & x \in (1, \infty) \end{cases}, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial U} = 0 \text{ if } x \in [-1, 1]; \tag{6}$$

in other words, we require u to be 0 on the real axis except on the interval $[-1, 1]$, and we require $\partial u/\partial n$ to be zero on this interval. Now geometrically it is clear that $\partial u/\partial n = -\partial u/\partial y$, so that this is equivalent to requiring $\partial u/\partial y$ to vanish on this interval.

Now of the conformal maps listed at the end of §38, the only one which maps onto the upper half-plane is $z \mapsto \sin z$, where we take z to lie in the rectangle

$$R = \{(x, y)|x \in (-\pi/2, \pi/2),\ y > 0\}.$$

Thus we consider transforming the problem (6) using this map. Let $v(z) = u[\sin(z)]$; then v must satisfy the problem

$$\nabla v = 0 \text{ on } R, \qquad v(-\pi/2, y) = 0, \qquad v(\pi/2, y) = 1, \qquad \left.\frac{\partial v}{\partial N}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2), \tag{7}$$

where here $\partial/\partial N$ denotes the outward normal derivative of v; but as before this is just $-\partial/\partial y$, so that this last condition gives simply

$$\left.\frac{\partial v}{\partial y}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2).$$

Thus we must now find a function v which satisfies (7). Since the only real class of solutions we have on rectangles are the linear ones, let us see whether this problem has a linear solution; in other words, let us see whether we can find numbers a, b, and c such that

$$v(x, y) = a + bx + cy$$

(which will automatically satisfy $\Delta u = 0$) satisfies the boundary conditions. The conditions on $x = \pm\pi/2$ give

$$0 = v(-\pi/2, y) = a - b\pi/2 + cy, \qquad 1 = v(\pi/2, y) = a + b\pi/2 + cy,$$

whence we see immediately that $c = 0$ (as otherwise the right-hand sides of these expressions would depend on y while the left-hand sides clearly do not) and are thus left with the system

$$\begin{aligned} a - \frac{\pi}{2}b &= 0 \\ a + \frac{\pi}{2}b &= 1. \end{aligned}$$

Adding these equations gives $a = 1/2$, while subtracting the first from the second gives $b = 1/\pi$; thus the function

$$v = \frac{1}{2} + \frac{1}{\pi}x$$

will indeed satisfy the boundary conditions on $x = \pm\pi/2$, a fact which can readily be verified by direct substitution. Note that this solution satisfies also $\partial v/\partial y = 0$ everywhere, and hence in particular on the segment $\{(x, 0) \,|\, x \in (-1, 1)\}$, which is the final boundary condition in (7). Thus v is the solution to (7).

This means that the solution to the original problem will be $u = v \circ \arcsin z$. Now if we write a point in the plane in complex notation, then we have $v(x + iy) = \frac{1}{2} + \frac{1}{\pi}x = \frac{1}{2} + \frac{1}{\pi}\mathrm{Re}\,(x + iy)$; thus we need to determine $\mathrm{Re}\,\arcsin z$, where z is in the upper half-plane and the branch of arcsin taken is that which maps into the rectangle R. We could determine this using the formula for arcsin which we found previously, but it seems easier to proceed in a different way. Let $z = a + ib$ be in the upper half-plane, and let $w = x + iy \in R$ satisfy $\sin w = z$; in other words,

$$\sin w = \sin x \cosh y + i \cos x \sinh y = a + ib,$$

and we note that $b > 0$, which implies that $x \in (-\pi/2, \pi/2)$ and $y > 0$. We wish to find $\mathrm{Re}\, w = x$. Now as we just saw, $b > 0$ implies $\cos x > 0$. Further, $\sin x = 0$ exactly when $a = 0$, and in this case we must have $x = 0$. Thus for us $\mathrm{Re}\, arcsin 0 = 0$, and we may assume that $a \neq 0$. Now let $\alpha = \sin^2 x$. Then $\cos^2 x = 1 - \alpha$, so that we have

$$\frac{a^2}{\alpha} + \frac{b^2}{\alpha - 1} = \frac{a^2}{\sin^2 x} - \frac{b^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1;$$

if we multiply through by $\alpha(\alpha - 1)$, this gives

$$\begin{aligned} \alpha^2 - \alpha &= a^2(\alpha - 1) + \alpha b^2 = \alpha(a^2 + b^2) - a^2 \\ \alpha^2 - (1 + a^2 + b^2)\alpha + a^2 &= 0 \end{aligned}$$

whence applying the quadratic formula, we obtain

$$\alpha = \frac{1}{2}\left[1 + a^2 + b^2 \pm \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]. \tag{8}$$

That the quantity inside the square root is always positive can be seen as follows:

$$\begin{aligned}(1 + a^2 + b^2)^2 - 4a^2 &= 1 + 2a^2 + 2b^2 + a^4 + 2a^2b^2 + b^4 - 4a^2\\ &= b^2(b^2 + 2) + a^4 - 2a^2 + 1 + 2a^2b^2 = b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2,\end{aligned}$$

which is a sum of positive (since $b \neq 0$) and nonnegative quantities and therefore positive. Now note that both roots above are nonnegative, since $\sqrt{(1 + a^2 + b^2)^2 - 4a^2} \leq 1 + a^2 + b^2$. We claim that the + root in (8), which we denote by α_+, must be greater than 1. To see this, note that since $b > 0$

$$4 - 4(1 + b^2) = -4b^2 < 0;$$

thus

$$(1+b^2+a^2)^2 - 4a^2 > (1+b^2+a^2)^2 - 4(a^2+b^2) = (1+b^2+a^2)^2 - 4(1+a^2+b^2) + 4 = [(1+b^2+a^2) - 2]^2,$$

so

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > \left|1 + b^2 + a^2 - 2\right|.$$

Now if $1 + b^2 + a^2 > 2$, then clearly $\alpha_+ > 1$; while if $1 + b^2 + a^2 \leq 2$, then by the above inequality we have

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2 - (1 + b^2 + a^2), \qquad (1 + b^2 + a^2) + \sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2,$$

and again $\alpha_+ > 1$. Thus in any event we must have $\alpha_+ > 1$, as desired. Since we require $\alpha = \sin^2 x$ for x real, we are only interested in values of α that lie in $[0, 1]$, and we therefore reject the value α_+ and take $\alpha = \frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]$. We claim that $\alpha < 1$. This can be seen as follows:

$$\begin{aligned}2\alpha = 1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2} &= \frac{(1 + a^2 + b^2)^2 - [(1 + a^2 + b^2)^2 - 4a^2]}{1 + a^2 + b^2 + \sqrt{(1 + a^2 + b^2)^2 - 4a^2}}\\ = \frac{4a^2}{1 + a^2 + b^2 + \sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2}} &< \frac{4a^2}{1 + a^2 + b^2 + a^2 - 1} < \frac{4a^2}{2a^2} = 2,\end{aligned}$$

giving $\alpha < 1$, as desired. (Here we have used $b > 0$ to conclude that $\sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2} > \sqrt{(a^2 - 1)^2} = |a^2 - 1| \geq a^2 - 1$.) This gives, finally, then,

$$x = \operatorname{Re} \arcsin z = \pm \arcsin \sqrt{\frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]},$$

where here arcsin denotes the ordinary inverse sine of a number in the interval $[0, 1)$, lying in $[0, \pi/2)$. We will take the + sign for $a \geq 0$ and the − for $a \leq 0$; $a = 0$ clearly gives $x = 0$, so in this case it does not matter which sign we take. Our solution is then finally

$$u(x, y) = \frac{1}{2} + \frac{1}{\pi} \cdot \begin{cases} \arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \geq 0 \\ -\arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \leq 0 \end{cases}. \tag{9}$$

It is instructive to see how this solution satisfies the boundary conditions. Suppose $y = 0$; then the formula above gives[3]

$$1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2} = 1 + x^2 - \sqrt{(1 - x^2)^2} = 1 + x^2 - |1 - x^2|.$$

[3] Technically, of course, we should be considering rather the *limit* as $y \to 0^+$. The formula above will clearly be continuous at $y = 0$, however, if it is defined at $y = 0$; and thus it suffices to show that it is defined at $y = 0$, which is afforded by our following calculation.

Now if $x \in [-1,1]$, this gives $1+x^2-(1-x^2)=2x^2$, while if $x \in (-\infty,-1)\cup(1,+\infty)$ it gives instead $1+x^2-(x^2-1)=2$. Thus

$$\arcsin\sqrt{\frac{1}{2}\left[1+x^2+y^2-\sqrt{(1+x^2+y^2)^2-4x^2}\right]} = \begin{cases} \arcsin|x|, & x\in[-1,1] \\ \frac{\pi}{2}, & x\in(-\infty,-1)\cup(\infty,1) \end{cases},$$

where the second line follows from $\arcsin 1 = \pi/2$. If we substitute this back into (9), we thus obtain (using the fact that arcsin is odd)

$$\begin{aligned} u(x,0) &= \frac{1}{2}+\frac{1}{\pi}\begin{cases} -\frac{\pi}{2}, & x\in(-\infty,-1) \\ \arcsin x, & x\in[-1,1] \\ \frac{\pi}{2}, & x\in(1,+\infty) \end{cases} \\ &= \begin{cases} 0, & x\in(-\infty,-1) \\ \frac{1}{2}+\frac{1}{\pi}\arcsin x, & x\in[-1,1] \\ 1, & x\in(1,+\infty) \end{cases}, \end{aligned}$$

which clearly satisfies the boundary conditions on $(-\infty,-1)\cup(\infty,1)$. That the remaining condition on $(-1,1)$ holds, we leave to the intrepid reader; it follows fairly simply from the observation that

$$\frac{\partial}{\partial y}\left[1+x^2+y^2-\sqrt{(1+x^2+y^2)^2-4x^2}\right] = 2y-2y(1+x^2+y^2)[(1+x^2+y^2)^2-4x^2]^{-1/2},$$

and the observation that this quantity vanishes as $y\to 0^+$ for all $x\in(-1,1)$. (This is clear, since the expression inside the exponent is positive for all such x even when $y=0$, by the foregoing.)

MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR AUGUST 10 – 14

Due Friday, August 14, at 11:59 PM EDT.

1. [15 marks; modified version of Question 3 from last week's assignment.] Suppose that f is a nonzero function which is analytic on the entire complex plane. ('Nonzero' here means that there is some point in the complex plane at which f is not zero. It does not mean that f has no zeros in the plane.) Let C_R denote the (full) circle of radius R centred at the origin. Is it possible to have

$$\lim_{R\to\infty} \int_{C_R} |f(z)|\, ds = 0?$$

(The integral here is an arclength integral from multivariable calculus.) If not, prove it; otherwise, give an example. [Hint: check the proof of Liouville's Theorem (the one in the lecture notes)!]

We may proceed as in the proof of Liouville's Theorem in the lecture notes. Let $z_0 \in \mathbf{C}$[1 mark], and let $R > |z_0| + 1$[2 marks]. By the Cauchy integral formula, we have, letting C_R denote the circle of radius R centred at the origin[1 mark],

$$|f(z_0)| = \left|\frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}\,dz\right| \text{[1 mark]} \le \frac{1}{2\pi}\int_{C_R}\left|\frac{f(z)}{z-z_0}\right| ds\text{[2 marks]} \le \frac{1}{2\pi}\int_{C_R}|f(z)|\,ds\text{[1 mark]},$$

since if z is on C_R then $|z - z_0| \ge 1$[2 marks]. But if we now take R to infinity, this last integral must vanish[1 mark], giving $|f(z_0)| = 0$[1 mark], or $f(z_0) = 0$[1 mark]. But $z_0 \in \mathbf{C}$ was arbitrary[1 mark], so f must be identically zero[1 mark].

[Marking: as above. Another method would be to show that all of the coefficients of the Taylor series of f vanish at the origin – this would be much closer to the proof of Liouville's Theorem in Goursat. On the other hand, the vanishing of the integral does *not* imply directly that the modulus of f must be bounded – if you said that you may have gotten very few marks.]

2. [10 marks] (a) Show that for all $z = x + iy \in \mathbf{C}$,

$$|\sin z| \le e^{|y|}.$$

Let $z = x + iy$; then

$$|\sin z| = \left|\frac{e^{iz} - e^{-iz}}{2i}\right| \text{[1 mark]} \le \frac{1}{2}\left(\left|e^{ix+y}\right| + \left|e^{-ix+y}\right|\right) \text{[1 mark]} \le \frac{1}{2}\left(e^{-y} + e^{y}\right) \text{[1 mark]} \le e^{|y|}\text{[1 mark]}.$$

(b) Using part (a) and Rouché's Theorem, determine how many zeros the function

$$3z^8 + \sin z$$

has in the unit disk. [It is worth spending some time thinking about whether the same procedure can be applied on arbitrarily large disks. But you do not need to say anything about that in your solution.]

If $z = x + iy$ is any point on the unit circle[1 mark], we ahve $|y| \le 1$[1 mark], so

$$\left|3z^8\right| = 3 \ge e \ge e^{|y|} \ge |\sin z|, \text{[1 mark]}$$

and by Roché's Theorem the function $3z^8 + \sin z$ must have as many zeroes as $3z^8$[1 mark] (counting multiplicities[1 mark]), i.e., 8.[1 mark]

[Marking: as above.]

3. [10 marks] (a) Find a polynomial solution to the following problem on the unit disk $D = \{z \,|\, |z| < 1\}$:

$$\Delta u = 0, \qquad u|_{\partial D} = \cos\theta,$$

where θ is the usual polar coordinate on the plane.

We look for a linear solution of the form $u = a + bx + cy$[1 mark]. On the unit circle, we have $x = \cos\theta$, $y = \sin\theta$[1 mark]; thus we require

$$a + b\cos\theta + c\sin\theta = \cos\theta, \text{[1 mark]}$$

from which we see that $a = c = 0$, $b = 1$[1 mark]. Thus $u = x$ will solve the given problem.[1 mark]

(b) Use your solution to (a) and a conformal transformation to solve the following problem on the *exterior* of the unit disk, i.e., the set $E = \{z \mid |z| > 1\}$:

$$\Delta u = 0, \qquad u|_{\partial E} = \cos\theta, \qquad u \to 0 \text{ as } |z| \to \infty.$$

(Note that $\partial E = \partial D$, both being just the unit circle.)

The function $z \mapsto 1/z$ will take the interior of the unit circle (without the origin) to the exterior[1 mark]; thus the function (setting $z = x + iy$)

$$v(x, y) = u(1/z) = u\left(\frac{x - iy}{x^2 + y^2}\right) = \frac{x}{x^2 + y^2}\text{[1 mark]}$$

will be harmonic on the exterior of the unit disk[1 mark]. Since on the unit disk it satisfies

$$v = \frac{\cos\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta, \text{[1 mark]}$$

and if $z = R(\cos\theta + i\sin\theta)$, $R > 0$, it is

$$v = \frac{R\cos\theta}{R^2} = \frac{1}{R}\cos\theta\text{[1 mark]}$$

which goes to zero as $|R| = z \to \infty$, this v must be the solution to our problem.
[Marking: as above.]

4. [15 marks] Solve the following problem on the lower half-plane $H = \{x + iy \mid y < 0\}$:

$$\Delta u = 0, \qquad u|_{\partial H} = \begin{cases} \pi, & x < -1 \\ \cos^{-1} x, & x \in (-1, 1) \\ 0, & x > 1 \end{cases}$$

(Note that ∂H, the boundary of H, is just the real axis.)

We transform this problem using the map $z \mapsto \cos z$[1 mark] from the rectangle $R = \{x + iy \mid x \in (0, \pi), y > 0\}$[1 mark] to the lower half-plane. We must determine the new boundary data. The line $x = 0$, $y > 0$ is mapped to the interval $(1, +\infty)$, on which u is 0; thus the transformed data will also be 0 there[1 mark]. Similarly, the line $x = \pi$, $y > 0$ is mapped to the segment $(-\infty, -1)$, on which u is π; thus the transformed ata must be 1 there[1 mark]. Finally, on the segment $y = 0$, $x \in (0, \pi)$, we have

$$\cos z = \cos x,$$

i.e., the point $(x, 0)$ is mapped to $(\cos x, 0)$[1 mark], so

$$u(\cos z) = u(\cos x, 0) = \arccos\cos x = x\text{[1 mark]}$$

since we are using the branch of arccos which takes $[-1, 1]$ into $[0, \pi]$, and $x \in (0, \pi)$. Thus the transformed problem is

$$\Delta v = 0 \text{ on } R, \qquad \mathbf{v}|_{\partial R} = \begin{cases} 0, & x = 0 \\ x, & y = 0 \\ \pi, & x = \pi \end{cases}.\text{[1 mark]}$$

We seek a linear solution to this problem; thus we write $v = a + bx + cy$[1 mark], and try to solve for a, b, and c, using the boundary conditions. These give

$$\begin{aligned} a + cy &= 0 \\ a + bx &= x \qquad \text{[2 marks]} \\ a + b\pi + cy &= \pi; \end{aligned}$$

ince these must hold for all x and y, we have $a = c = 0$, $b = 1$[2 marks], so $v = x$[1 mark]. More carefully, we have $v(x, y) = x$, or $v(x + iy) = x$, so $v(z) = \operatorname{Re} z$. Thus $u(z) = (v \circ \arccos)(z) = \operatorname{Re} \arccos z$[1 mark]. We may express this in terms of $\operatorname{Re} \arcsin z$ as follows[1 mark]. We have the formula

$$\sin(w + \pi/2) = \cos w;$$

thus if $w = \arccos z$ we have

$$z = \cos w = \sin(w + \pi/2),$$

so $w = \arcsin z - \pi/2$, at least if we take the right branch of arcsin. Now the branch of arccos we use maps into $[0, \pi]$, while the branch of arcsin used in the notes maps into $[-\pi/2, \pi/2]$; thus we must replace arcsin by $\pi - \arcsin$, meaning that we have finally the solution

$$u(z) = \frac{\pi}{2} - \operatorname{Re} \arcsin z.$$

[Marking: as above. This exercise is quite close to the example on pp. 3 – 6 of the August 13 lecture notes on the course website.]

1. [9 marks] Suppose that f is a function which is analytic on the entire complex plane, and that there is a constant $C > 0$ such that $|f(z)| \leq CR$ whenever $|z| = R$. If $f(0) = 0$ and $f'(0) = 1$, what is f?

We note that f may be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{[2 marks]}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(z')}{z'^{\,n+1}}\, dz', \text{[1 mark]}$$

C_R being a circle of radius R about the origin. But now

$$|a_n| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z')}{z'^{\,n+1}}\, dz' \right| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z')|}{R^{n+1}}\, ds \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{CR}{R^{n+1}} R\, dt \text{[2 marks]} = \frac{C}{R^{n-1}} \text{[1 mark]}$$

which goes to zero as $R \to \infty$ if $n \geq 2$[1 mark]. Since a_n does not depend on R, we must have $a_n = 0$ for $n \geq 2$[1 mark]; thus $f(z) = z$[1 mark].

2. [7 marks] How many zeroes does the function $z^n e^z + \frac{1}{2} \sin z$, also on the unit disk?

We note that on the unit circle, $\frac{1}{8}|\sin z| \leq \frac{1}{8} e^{|y|} \leq e/8$[2 marks], while $|z^n e^z| = |e^z| = e^x \geq e^{-1} > \frac{1}{8} e$[2 marks], so that by Rouché's Theorem[1 mark], $z^n e^z + \frac{1}{8} \sin z$ has the same number of zeroes in the unit disk as $z^n e^z$[1 mark], namely n[1 mark].

3. [6 marks] Using the Taylor series for e^z around $z = 0$, find the Laurent series for $e^{1/z}$ around $z = 0$. Use this to determine $\int_C e^{1/z}\, dz$, where C is any circle centred at the origin.

We have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{[1 mark]},$$

so for $z \neq 0$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \text{[1 mark]}.$$

Thus

$$\int_C e^{1/z}\, dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^{-n}\, dz = 2\pi i \text{[2 marks]},$$

since only the $n = 1$ term contributes[2 marks], as we have seen many times during the course.

Evaluate the following integrals.

4. [10 marks]

$$\int_{-\infty}^{+\infty} \frac{e^{-ix}}{(x^2 + 4x + 8)^2}\, dx.$$

Since we have a factor of e^{-ix} in the numerator, we must close in the lower half-plane. We will use the contour shown in the figure[1 mark]. Now $z^2 + 4z + 8 = 0$ gives $z = -2 + \frac{1}{2}(16 - 32)^{1/2} = -2 \pm 2i$[1 mark], so in the lower half-plane we have only one pole, at $-2 - 2i$[1 mark]. Now since for R sufficiently large we have

$$\frac{1}{|z^2 + 4z + 8|^2} \leq \frac{1}{(R^2 - 4R - 8)^2}$$

and this goes to zero as $R \to \infty$[1 mark], we have

$$\int_{C_R} \frac{e^{-iz}}{(z^2 + 4z + 8)^2}\, dz \to 0$$

as $R \to \infty$ by the Jordan Lemma[1 mark] applied on the lower half-plane. Thus

$$\lim_{R\to\infty}\int_{L_R}\frac{e^{-ix}}{(x^2+4x+8)^2}\,dx = -2\pi i \mathrm{Res}_{-2-2i}\frac{e^{-iz}}{(z^2+4z+8)^2}\text{[1 mark]}$$
$$= -2\pi i\,\frac{d}{dz}\frac{e^{-iz}}{(z+2-2i)^2}\bigg|_{z=-2-2i}\text{[1 mark]}$$
$$= -2\pi i\left[\frac{-ie^{-iz}}{(z+2-2i)^2}-2\frac{e^{-iz}}{(z+2-2i)^3}\right]\bigg|_{z=-2-2i}\text{[1 mark]}$$
$$= \frac{2\pi i}{-16}\left[ie^{2i-2}+2\frac{e^{2i-2}}{-4i}\right]\text{[1 mark]}$$
$$= -\frac{\pi i}{8}e^{2i-2}\left[\frac{3}{2}i\right] = \frac{3\pi}{16}e^{2i-2}\text{[1 mark]}.$$

5.

[15 marks]

$$\int_0^{+\infty}\frac{\cos x^4-(1+\sqrt{2})\sin x^4}{1+x^8}\,dx.$$

We follow the method used in the homework and work with

$$\int_0^{+\infty}\frac{e^{ix^4}}{1+x^8}\,dx\text{[1 mark]}.$$

We wish to close along a wedge; we will choose the angle θ so that z^4 is in the upper half plane for $\arg z \in [0,\theta]$ while $\left(e^{i\theta}z\right)^8 = z^8$[1 mark]. This last gives $e^{8i\theta}=1$, or $\theta = n\pi/4$, while the former requires $\theta \le \pi/4$; thus we take $\theta = \pi/4$[1 mark]. Thus we close using the contour shown in the figure.[1 mark]

Now

$$\int_{C_R}\frac{e^{iz^4}}{1+z^8}\,dz = \int_0^{\pi/4}\frac{e^{ie^{4it}R^4}}{1+R^8e^{8it}}iRe^{it}\,dt\text{[1 mark]}.$$

For $t\in[0,\pi/4]$, e^{4it} will have a nonnegative imaginary part; in particular, $\operatorname{Im} e^{4it} = \sin 4t \ge \frac{2}{\pi}4t$ for $t\in[0,\pi/8]$ by the Jordan inequality[1 mark]. Thus

$$\left|\int_0^{\pi/4}\frac{e^{ie^{4it}R^4}}{1+R^8e^{8it}}\,dt\right| \le \int_0^{\pi/4}\frac{e^{-R^4\sin 4t}}{R^8-1}\,dt = 2\int_0^{\pi/8}\frac{e^{-R^4\sin 4t}}{R^8-1}\,dt\text{[2 marks]}$$
$$\le 2\int_0^{\pi/8}\frac{e^{-\frac{8}{\pi}R^4t}}{R^8-1}\,dt \le \frac{\pi}{4R^4(R^8-1)}\left(1-e^{-R^4}\right)\to 0\text{[1 mark]}$$

as $R\to\infty$. Further,

$$\int_{L'_R}\frac{e^{iz^4}}{1+z^8}\,dz = -\int_0^R\frac{e^{-it^4}e^{i\pi/4}}{1+t^8}\,dt\text{[1 mark]},$$

so (note that $z^8+1=0$ gives $z=e^{i\pi/8+n\pi/4}$, so inside the wedge we have only one pole, at $e^{i\pi/8}$)

$$2\pi i\mathrm{Res}_{e^{i\pi/8}}\frac{e^{iz^4}}{1+z^8} = \lim_{R\to\infty}\left(\int_{L_R}\frac{e^{iz^4}}{1+z^8}\,dz+\int_{L'_R}\frac{e^{iz^4}}{1+z^8}\,dz\right) = \lim_{R\to\infty}\int_0^R\frac{e^{it^4}-e^{i\pi/4}e^{-it^4}}{1+t^8}\,dt\text{[1 mark]}.$$

Now

$$e^{it^4}-e^{i\pi/4}e^{-it^4} = \cos t^4 + i\sin t^4 - \left(\frac{1}{\sqrt{2}}\cos t^4+\frac{1}{\sqrt{2}}\sin t^4 + i\left[-\frac{1}{\sqrt{2}}\sin t^4+\frac{1}{\sqrt{2}}\cos t^4\right]\right)$$
$$= \left(1-\frac{1}{\sqrt{2}}\right)\cos t^4 - \frac{1}{\sqrt{2}}\sin t^4 + i\left[\left(1+\frac{1}{\sqrt{2}}\right)\sin t^4-\frac{1}{\sqrt{2}}\cos t^4\right]$$
$$= \left(1-\frac{1}{\sqrt{2}}\right)\left[\cos t^4-(\sqrt{2}+1)\sin t^4\right] + i\frac{1}{\sqrt{2}}\left[\left(\sqrt{2}+1\right)\sin t^4-\cos t^4\right]\text{[1 mark]}$$

so the integral we want is $1/(1-1/\sqrt{2}) = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$ times the real part of the above limit. Now the residue may be computed as follows:

$$\operatorname{Res}_{e^{i\pi/8}} \frac{e^{iz^4}}{1+z^8} = \frac{e^{ie^{i\pi/2}}}{8e^{7i\pi/8}} = \frac{1}{8e}\left[\cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8}\right] \text{[2 marks]},$$

and we have finally

$$\int_0^\infty \frac{\cos x^4 - (1+\sqrt{2})\sin x^4}{1+x^8}\,dx = \frac{\pi}{8e}\sin\frac{7\pi}{8}\left(1+\frac{1}{\sqrt{2}}\right) \text{[1 mark]}.$$

6.

[20 marks]

$$\int_0^{+\infty} \frac{x^{-\alpha}}{x^4+3x^2+2}\,dx,$$

where $\alpha \in (0,1)$ and the exponential denotes the standard version of this function on positive real numbers. [Hint: while this can be done with a keyhole contour, there is another contour which requires fewer computations.]

We will close on an indented contour wedge. We pick the angle in the same fashion as in 5; thus we want $e^{2i\theta} = 1$, so $\theta = n\pi$, and we take $\theta = \pi$[2 marks] to minimise the number of poles within the contour. Thus we have the contour in the figure[1 mark]. We shall take the branch of the exponential function with a cut along the negative imaginary axis and an angle between $-\pi/2$ and $3\pi/2$.

Now $z^4 + 3z^4 + 2 = 0$ gives $z^2 = -\frac{3}{2} \pm \frac{1}{2} = -2, -1$, so $z = \pm i\sqrt{2}, \pm i$[2 marks], and we have only $z = i\sqrt{2}, i$ within the contour[1 mark]. At these points we have the residues

$$\operatorname{Res}_i \frac{z^{-\alpha}}{z^4+3z^2+2} = \operatorname{Res}_i \frac{z^{-\alpha}}{(z^2+2)(z^2+1)} = \frac{i^{-\alpha}}{1\cdot 2i} = \frac{e^{-i\alpha\frac{\pi}{2}}}{2i} = -\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} \text{[2 marks]},$$
$$\operatorname{Res}_{i\sqrt{2}} \frac{z^{-\alpha}}{z^4+3z^2+2} = \frac{(i\sqrt{2})^{-\alpha}}{2i\sqrt{2}(-1)} = \frac{2^{-\alpha/2}e^{-i\alpha\frac{\pi}{2}}}{-2i\sqrt{2}} = i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2} \text{[2 marks]}.$$

Further, we claim that

$$\int_{C_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz, \int_{C'_\epsilon} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz \to 0 \quad \text{as} \quad R\to\infty, \epsilon \to 0^+.$$

We have

$$\left|\int_{C_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz\right| = \left|\int_0^\pi \frac{R^{-\alpha}e^{-i\alpha t}}{R^4e^{4it}+3R^2e^{2it}+2}\,ds\right| \le \int_0^\pi \frac{R^{-\alpha}}{R^4-3R^2-2}R\,dt$$
$$\le \frac{R^{1-\alpha}\pi}{R^4-3R^2-2} \to 0 \text{ as } R\to\infty \text{[2 marks]},$$
$$\left|\int_{C'_\epsilon} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz\right| = \left|\int_0^\pi \frac{\epsilon^{-\alpha}}{\epsilon^4e^{4it}+3\epsilon^2e^{2it}+2}\epsilon\,dt\right| \le \frac{\epsilon^{1-\alpha}\pi}{2-3\epsilon^2-\epsilon^4} \to 0 \text{ as } \epsilon\to 0^+ \text{[2 marks]}.$$

Thus we will have

$$\lim_{\epsilon\to 0^+}\lim_{R\to\infty}\int_{L'_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz + \int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz = 2\pi i\left[-\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} + i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2}\right] \text{[2 marks]}.$$

Now (since $-L'_R$ can be parameterised by $-t$, $t\in[\epsilon, R]$)

$$\int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz = \int_\epsilon^R \frac{(-t)^{-\alpha}}{t^4+3t^2+2}\,dt = \int_\epsilon^R \frac{e^{-i\alpha\pi}t^{-\alpha}}{t^4+3t^2+2}\,dt = e^{-i\alpha\pi}\int_{L_R} \frac{z^{-\alpha}}{z^4+3z^2+2}\,dz \text{[1 mark]};$$

thus we have

$$\int_0^\infty \frac{x^{-\alpha}}{x^4+3x^2+2}\,dx = \frac{2\pi i}{1+e^{-i\alpha\pi}}\left[-\frac{i}{2}e^{-i\alpha\frac{\pi}{2}} + i2^{-(\alpha+3)/2}e^{-i\alpha\pi/2}\right] \text{[1 mark]}$$
$$= \frac{\pi}{2\cos\frac{1}{2}\alpha\pi}\left[1-2^{-(\alpha+1)/2}\right] \text{[2 marks]}.$$

7. [11 marks] Use the fact that $1/(z+1)^2$ is analytic on the right half-plane to solve the following problem on the wedge $D = \{(x,y)\,|\,x>0,\ -x\le y\le x\}$:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = -\frac{4x^2}{(1+4x^4)^2}.$$

We see that, writing $z = x+iy$,

$$\frac{1}{(1+z)^2} = \frac{1}{[(1+x)+iy]^2} = \frac{1}{(1+x)^2-y^2+2i(1+x)y}$$
$$= \frac{(x+1)^2-y^2-2i(1+x)y}{[(1+x)^2-y^2]^2+4(1+x)^2y^2} = \frac{(x+1)^2-y^2-2i(1+x)y}{[(1+x)^2+y^2]^2}.\text{[2 marks]}$$

Now let us try to convert the original problem to one on the right half-plane by using the map $z\mapsto z^{1/2}$[2 marks]. The boundary conditions will transform as follows:

$$v|_{x=0,y\ge 0} = u|_{y=x}(\sqrt{x/2},\sqrt{x/2}) = -\frac{2y}{(1+y^2)^2}, \text{[2 marks]}$$
$$v|_{x=0,y\le 0} = u|_{y=-x}(\sqrt{x/2},-\sqrt{x/2}) = \frac{2y}{(1+y^2)^2}, \text{[2 marks]}$$

so

$$v = -\frac{2(1+x)y}{[(1+x)^2+y^2]^2}\text{[2 marks]},$$

and

$$u = v\circ z^2 = v(x^2-y^2,2xy) = -\frac{4(1+x^2-y^2)xy}{[(1+x^2-y^2)^2+4x^2y^2]^2}$$

is the desired solution.[1 mark]

[It was not until after the marking was commenced that the mistake in the above problem was discovered (note that the transformed initial data should have been the same thing on both $y\ge 0$ and $y\le 0$, which would require a different sign for the two half-line boundaries in the original initial data). The marking was carried out in such a way as to avoid penalising anyone for this error in the problem, essentially as follows: 2 marks for working out the function $1/(1+z)^2$, 2 marks for knowing that the imaginary part of an analytic function is harmonic, 2 marks for the correct conformal transformation, 2 marks for each of the boundary conditions, and 1 mark for knowing that the final solution should be $v\circ f^{-1}$ for f the conformal map and v the solution of the transformed problem.]

1. [9 marks] Suppose that f is a function which is analytic on the entire complex plane, and that there is a constant $C > 0$ such that $|f(z)| \leq CR$ whenever $|z| = R$. If $f(0) = 0$ and $f'(0) = 1$, what is f?
2. [7 marks] How many zeroes does the function $z^n e^z + \frac{1}{2} \sin z$, also on the unit disk?
3. [6 marks] Using the Taylor series for e^z around $z = 0$, find the Laurent series for $e^{1/z}$ around $z = 0$. Use this to determine $\int_C e^{1/z}\, dz$, where C is any circle centred at the origin.

Evaluate the following integrals.

4. [4 marks]

$$\int_{-\infty}^{+\infty} \frac{e^{-ix}}{(x^2 + 4x + 8)^2}\, dx.$$

5.

$$\int_0^{+\infty} \frac{\cos x^4 - (1 + \sqrt{2}) \sin x^4}{1 + x^8}\, dx.$$

6.

$$\int_0^{+\infty} \frac{x^{-\alpha} \log x}{x^4 + 3x^2 + 2}\, dx,$$

where $\alpha \in (0, 1)$ and the exponential and logarithm denote the standard versions of these functions on positive real numbers. [Hint: you do not need to use a keyhole contour! Also, this is closely related to a certain homework problem.]

7.

$$\int_0^{2\pi} \frac{\cos x}{5 + 2\cos x + \sin x}\, dx.$$

8. Use the fact that $1/(z + 1)^2$ is analytic on the right half-plane to solve the following problem on the wedge $D = \{(x, y) \,|\, x > 0,\, -x \leq y \leq x\}$:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \frac{2(x^2 + y^2)}{[4(x^2 + y^2)^2 + 1]^2}.$$

Summary:

- We motivate the definition of the Poisson kernel.

101. Poisson kernel . In lecture we attempted to give a *derivation* of the Poisson kernel. While the main idea was correct, there were a few errors in detail and interpretation, and when those are corrected it turns out that what was given can *motivate* the definition of the Poisson kernel, but does not really serve as a proof. We go through this motivation anyway.

Let D be the disk of radius r centred at the origin, $C = \partial D$ its boundary (the unit circle centred at the origin), and consider the following problem:

$$\Delta u = 0, \qquad u|_{\partial D} = h, \tag{1}$$

where Δ is the Laplacian, $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, and h is a function on C. Now suppose that there is a function f which is analytic on the complex plane and satisfies $\operatorname{Re} f|_{\partial D} = h$; since $\operatorname{Re} f$ must be harmonic everywhere on the plane, and in particular on D, we see that $u = \operatorname{Re} f$ is a solution to problem (1). Now the Cauchy integral formula allows us to write

$$f(x+iy) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z' - [x+iy]}\,dz'. \tag{2}$$

If we define

$$z^* = \frac{r^2}{\overline{z}}$$

[note that this corrects an error in the lecture, where z^* was mistakenly given as r^2/z], then z^* will be outside of C so that we will have

$$\int_C \frac{f(x'+iy')}{z' - z^*}\,dz' = 0 \tag{3}$$

Thus we may subtract this integral from (2). Now we may parameterise C as

$$z'(t) = re^{i\theta}, \qquad \theta \in [0, 2\pi];$$

let us write also $x + iy = r_0 e^{i\theta_0}$ for some θ_0. Then (2) becomes, after subtracting (3),

$$\begin{aligned}
f(x+iy) &= \frac{1}{2\pi i}\int_0^2 \pi \left[\frac{1}{re^{i\theta} - r_0 e^{i\theta_0}} - \frac{1}{re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}}\right] f(re^{i\theta}) ire^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2}{r_0}e^{i\theta_0} + r_0 e^{i\theta_0}}{\left(re^{i\theta} - r_0e^{i\theta_0}\right)\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} re^{i\theta} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-(r^2 - r_0^2)\frac{r}{r_0}e^{i(\theta_0+\theta)}}{\left(re^{i\theta} - r_0e^{i\theta_0}\right)\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2-r_0^2}{r_0^2}e^{i(\theta_0+\theta)}}{\left(\frac{r}{r_0}e^{i\theta} - e^{i\theta_0}\right)\left(e^{i\theta} - \frac{r}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{r^2-r_0^2}{r_0^2}}{\left(\frac{r}{r_0} - e^{i(\theta_0-\theta)}\right)\left(\frac{r}{r_0} - e^{i(\theta-\theta_0)}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0e^{i(\theta_0-\theta)}\right|^2} f(re^{i\theta})\,d\theta.
\end{aligned}$$

If we take the real part of this, then, since everything in the integrand is real except for f, and $\operatorname{Re} f|_{\partial D} = h$, we have

$$\begin{aligned}
u(x,y) = \operatorname{Re} f &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0 e^{i(\theta_0-\theta)}\right|^2} h(r\cos\theta, r\sin\theta)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)} h(r\cos\theta, r\sin\theta)\,d\theta. \qquad (4)
\end{aligned}$$

As noted above, the foregoing does not actually *prove* that given a continuous function h the above function u will give a solution to problem (1); however, this can be proved by other means, though we shall not do so here. We shall however give an example.

Example. Let us start with a trivial example:

$$\Delta u = 0, \qquad u|_{\partial D} = 1.$$

Clearly the solution to this is 1. Using the integral formula (4), we have

$$\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\,d\theta. \qquad (5)
\end{aligned}$$

Now it turns out that the integrand here has an explicit antiderivative. To determine it, we work with the integral

$$\int \frac{1}{a - b\cos\theta}\,d\theta, \qquad (6)$$

where we assume $a > b \geq 0$. Note that, since $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2 = 2\cos^2\theta/2 - 1$,

$$a - b\cos\theta = (a+b) - b(1+\cos\theta) = (a+b) - 2b\cos^2\frac{\theta}{2},$$

so that the integral (6) may be rewritten as

$$\int \frac{1}{(a+b) - 2b\cos^2\frac{\theta}{2}}\,d\theta = \int \frac{\sec^2\frac{\theta}{2}}{(a+b)\sec^2\frac{\theta}{2} - 2b}\,d\theta.$$

Let us now make the substitution $v = \tan\theta/2$, $dv = \frac{1}{2}\sec^2\theta/2\,d\theta$; then this integral becomes, since $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned}
2\int \frac{1}{(a+b)(1+v^2) - 2b}\,dv &= \frac{2}{a+b}\int \frac{1}{v^2 + \frac{a-b}{a+b}}\,dv \\
&= \frac{2}{a+b}\cdot\left[\frac{a+b}{a-b}\right]^{1/2} \tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right] \\
&= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right],
\end{aligned}$$

from which we obtain finally

$$\int \frac{1}{a - b\cos\theta}\,d\theta = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2}\tan\frac{\theta}{2}\right].$$

Now for us $a = r^2 + r_0^2$ while $b = 2rr_0$, so $a + b = (r+r_0)^2$, $a - b = (r-r_0)^2$, and $\sqrt{a^2-b^2} = r^2 - r_0^2$, and we obtain

$$\int \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\,d\theta = 2\tan^{-1}\left[\frac{r+r_0}{r-r_0}\tan\frac{\theta}{2}\right].$$

Note that this formula is only valid on intervals which do not contain any odd integer multiple of π; for example, $(-\pi, \pi)$, $(\pi, 3\pi)$, and so on. This is because $\tan\frac{\theta}{2}$ is not defined at odd integer multiples of π. Thus to evaluate our original integral (5) we must split the interval $[0, 2\pi]$ into two pieces, $[0, \pi)$ and $(\pi, 2\pi]$, and treat each piece as an improper integral. The two integrals we thus get are

$$\frac{1}{2\pi}\int_0^{\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\, d\theta = \frac{1}{2\pi}\lim_{\theta\to\pi^-} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_0^{\theta} = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2},$$
$$\frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\, d\theta = \frac{1}{2\pi}\lim_{\theta\to\pi^+} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_{\theta}^{2\pi} = \frac{1}{2\pi}(0 - (-\pi)) = \frac{1}{2},$$

and finally we obtain $u = 1$, as we found originally by inspection.

hat its real part approaches h as the point (x, y) approaches the curve C *radially*, i.e., along lines directed outwards from the origin. (Note that it is *not* the case that f itself approaches h as z approaches the boundary: in fact it is not hard to show that an analytic function which is real on the boundary of a region must actually be constant throughout the region (try it!).) Let us see whether we can rewrite the real part of f as a real integral. It is convenient to consider a slight modification of this integral. The function f as defined by the above integral will also be analytic outside of the curve C. If we set

Summary:

- We give a basic introduction to the concept of *analytic continuation*, and relate it to our discussion of the logarithm.

102. Analytic continuation. Suppose that we have a function f which is analytic on some region containing a point a. Since f is analytic near a, we may expand it in a Taylor series near a and write

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(z-a)^k, \tag{1}$$

and the series will converge to f on the largest disk centred at a on which f is analytic. (A good example to keep in mind here might be something simple like $f(z) = 1/z$, expanded around some $a \neq 0$.) Suppose that the series (1) converges on the disk D, necessarily centred at a. Now instead of starting with the function f, suppose that we instead start with the series (1); in other words, suppose that we simply given a series

$$\sum_{k=0}^{\infty} a_k (z-a)^k, \tag{2}$$

which we know converges on the disk D. Let us denote its sum by $f_1(z)$ when $z \in D$ (note that we must have $f_1(z) = f(z)$ where f is analytic). Suppose that D has radius $r > 0$, and let b be a point in D with $|b - a| = r - \epsilon$, where ϵ is small; in other words, b is close to the boundary of D. Now since $b \in D$, the series (2) must converge to an analytic function at b; we may expand thus expand f_1 in a Taylor series about b and write

$$f_1(z) = \sum_{k=0}^{\infty} b_k (z-b)^k. \tag{3}$$

This series will also converge on a disk, say D', which must have radius at least ϵ (since the disk of radius ϵ about b will be contained in the disk D by the triangle inequality). However, D' may have radius bigger than ϵ; in that case, we say that it gives an *extension* of the function f_1 to the disk D'.

Now clearly there is nothing to keep us from continuing on in this fashion, by taking further points in D or in D' and expanding the functions around those points to see whether we get further extensions. A treatment of this is given in Goursat, Chapter IV; sections 83, 84, and 86 in particular are germaine to what we shall do here, but we do not have time to go into all of the details and shall content ourselves with a particular example.

103. A particular example Let us consider the branch of the complex logarithm obtained by taking a cut along the negative real axis and requiring the angle to lie in thet set $(-\pi, \pi)$, and denote the resulting branch by L; in other words, for $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$, we define

$$L(re^{i\theta}) = \log r + i\theta.$$

This function is analytic at $z = 1$, and thus has a power series expansion there – which we shall not give as finding it is one of the practice problems on the final review sheet; let us denote it by

$$\sum_{k=0}^{\infty} a_k (z-1)^k. \tag{4}$$

Now this series will converge to L on the largest disk centred at 1 on which L is analytic, which is easily seen to be the disk of radius 1 centred at $z = 1$, since L has a singularity at $z = 0$. Let us call this disk D_1. Since $L(re^{i\cdot 0}) \to -\infty$ as $r \to 0^+$, we see that the series (4) cannot converge on any larger disk.

Now let $b = 1 + i/2$; note that $b \in D_1$, so the series (4) converges at b. Thus we may expand the resulting function about b, obtaining

$$\sum_{k=0}^{\infty} b_k (z-b)^k. \tag{5}$$

Now of course on D_1 the series (4) converges to L, so we are really just expanding the function L around $z = b$ in this case. Thus, once again, the series will converge to L on the largest disk about b which does not include any singularity of L; we see that, as above, this will be the disk D_2 of radius $\sqrt{5/4}$ about b.

Continuing one step further, let us let $c = -1/10 + i/2$; then $|b - c|^2 = 11^2/10^2 = 121/100 < 5/4$, so $c \in D_2$. Thus we may expand the function obtained from the series (5) about $z = c$, obtaining the series

$$\sum_{k=0}^{\infty} c_k (z - c)^k; \tag{6}$$

as before we are actually expanding the function L, so we know that the series will converge to L on any disk about c on which L is analytic. This time, though, there is an extra twist. As above, clearly the largest disk on which the series (6) can converge is the disk, call it D_3, of radius $|c| = \sqrt{\frac{51}{100}}$ about c. However, clearly the point $-1/10$ satisfies $|c - (-1/10)| = 1/2 < \sqrt{\frac{51}{100}}$; in other words, $-1/10 \in D_3$. But $-1/10$ was on the branch cut of L – in other words, the function L *is not defined* at the point $-1/10$!

Let us investigate what is going on here in more detail. Let L_2 denote the branch of Log with a cut along the negative *imaginary* axis and with the angle required to lie in $(-\pi/2, 3\pi/2)$. Then clearly $L = L_2$ on the fourth, first, and second quadrants (i.e., where $\theta \in (-\pi/2, \pi)$); thus the series (4) and (5) are also the Taylor series for L_2 about $z = 1$ and $z = b$, respectively. Similarly, the series (6) is the Taylor series for L_2 about $z = c$, and will converge to L_2 on the largest disk about c on which L_2 is defined. Now geometrically it is clear that this disk is indeed the disk D_3 – in other words, the series (6) will indeed converge everywhere on D_3. It will converge to the function L_2 there. Now, as noted, $L_2 = L$ for $\theta \in (-\pi/2, \pi)$; but a little thought shows that $L_2 - L = 2\pi i$ if $\theta \in (\pi, 3\pi/2)$ – they are distinct branches over that interval.

This means, in particular, that for any $z = re^{i\theta} \in D_3$ with $\theta \in (\pi, 3\pi/2)$, the series in (6) will converge to a value equal to $L(z) + 2\pi i$. Now note that the difference of limits

$$\lim_{\theta \to \pi^-} L(re^{i\theta}) - \lim_{\theta \to -3\pi/2^+} L(re^{i\theta}) = 2\pi i$$

for any r; in other words, the function L has a jump discontinuity equal to $2\pi i$ across the branch cut – it decreases by $2\pi i$ as we cross the branch cut. But this means that if we add $2\pi i$ to L as we cross the branch cut, the result will be continuous (as long as we stay close to the branch cut, of course!) – and this is exactly what the series (6) accomplishes! In other words, given only local information about the branch L, the series was somehow able to pick out the branch L_2 which should follow L across the branch cut in order to have a function which is analytic on both sides of the branch cut.

Now we know of course that there is no branch of the logarithm which is analytic on the entire punctured plane, so the process of extension given above must run into some irresolvable obstacle at some point. Let us see what that is. Suppose that we continue expanding the function L_2 on yet another disk D_4 which is counterclockwisely further around the origin than D_3; we are evidently able to do so. If we then continue, using disks say D_5, D_6, etc., at some point we shall run into the same situation with L_2 which we had with L: the series will converge across the branch cut, but to another branch, call it L_3, on the other side of the cut, i.e., for $\theta > 3\pi/2$. As before, we will have $L_3 - L_2 = 2\pi i$ for points just across the branch cut; but $L_2 = L$ there, so $L_3 - L = 2\pi i$. Now we may take L_3 to be the branch with a cut along the positive imaginary axis and the angle θ required to lie in $(3\pi/2, 7\pi/2)$. Continuing to expand as before, we will eventually arrive back at the point $z = 1$. However, at that point we will be expanding the function L_3 instead of L, and that means that the series will converge to $L_3(1) = L(1) + 2\pi i = 2\pi i$, not $L(1) = 0$!

Note the close analogy of this procedure to what we discussed much earlier in the course about how integrating $1/z$ around the origin to get $\operatorname{Log} z$ will increase the value by $2\pi i$ – exactly the value determined here. The procedure here is however far more general, and in particular could be applied to the root functions, giving the different branches in cyclic succession as we continue expanding around the origin. (In other words, if we start with one branch of – say – the cube root function, say that which gives $1^{1/3} = 1$, and then expand it in Taylor series which circle the origin as here, then when we come back to the point 1 again we will have instead $e^{2\pi i/3}$; if we circle again, we will have $e^{4\pi i/3}$; and if we circle once more – making three times in total – we will come back to the original value, 1. With the logarithm, we would continue adding $2\pi i$ each time, which means we will never return to the original value no matter how many times we circle.) Similarly, we could apply this to the function in the last problem on the term test: in particular, the last part shows that if we were to start with the function on one side of the branch cut, and then expand it in series on disks which wrapped around the branch cut to the other side, by the time we came back to the original point there would be a difference of $-\pi/(2\sqrt{2})$.

Those who have seen – or will yet see – covering spaces should note the similarity here to the construction of the universal cover of the circle, or the punctured plane (which is homotopically equivalent, in fact there is an obvious deformation retract of the punctured plane onto the circle): we start at a certain point and then start taking curves, reducing by homotopy; since a closed loop once around the origin is not homotopic to a point, the endpoint of this curve is taken to represent a distinct point from its initial point. Similarly, a closed loop *twice* about the origin is not homotopic to either of these paths, meaning that *its* endpoint is yet another distinct point, and so on. Another way of putting all of this together is that, should we define the logarithm on the universal cover of the punctured plane instead of the punctured plane itself, it would become a single-valued analytic function. Taking a branch would then correspond to restricting the domain to some piece of this universal cover for which the covering map is a homeomorphism onto a cut plane. (What we just noted about root functions shows that something similar is true for them, but instead of needing to use the universal cover of the punctured plane, we need to use, for the nth-root function, the n-sheeted cover.) Similarly, functions with more complicated branch points – such as the function on the term test, which has four branch points; or the arctangent function, which has two – can be defined on covers of the multiply-punctured plane. The fact that if we traverse a loop around all four points we come back to the same value means however that we do not get the universal cover of the quadruply-punctured plane in this case, but rather some other set, an elucidation of which is however beyond the knowledge of the present author, who will therefore retire before he says anything more wrong than he already has.

104. A specific example Let us consider the complex logarithm, and try to find its Taylor series expansion about $z = 1$; in particular, let us see if we can determine what the radius of convergence of that power series must be. Immediately there is a problem: shouldn't we have to specify which branch of the logarithm we are using? – after all, the logarithm itself is a multivalued function but power series always give single-valued functions! On the other hand, what do we need to compute the Taylor series of a function at $z = 1$? Let us let f denote the 'logarithm' (whatever that means in the end, e.g., a particular branch or whatever we end up deciding on); then what we need is

$$f(1), \quad f'(1), \quad f''(1), \cdots.$$

Now $f(1)$ will depend on the branch, but since all of the different branches differ by only a constant value, all branches defined at $z = 1$ will have equal values for the higher derivatives! Moreover, even though $f(1)$ depends on the branch, we still know that no matter which branch we choose, as long as it is defined at $z = 1$, we will have $f(1) = 2n\pi$ for some $n \in \mathbf{Z}$. Thus, there is a sequence of numbers a_1, a_2, $\cdots$ such that for any branch of Log defined at $z = 1$, there will be some $n \in \mathbf{Z}$ such that this branch equals

$$2n\pi + \sum_{k=1}^{\infty} a_k (z-1)^k \tag{7}$$

(where the sequence a_k is given, of course, by

$$a_k = \frac{1}{k!} \frac{d^k}{dz}^k \operatorname{Log} z \bigg|_{z=1},$$

though we are not too interested in this fact right here). Now whether the series (7) converges for a particular value of z is clearly independent of the value of n; thus in determining its radius of convergence we may use any branch we like. Now from what we have seen in lecture, the series (7) will converge on the largest disk centred at $z = 1$ on which the branch we are dealing with is analytic. Since all branches of the logarithm have a singularity at the origin, the largest possible radius of convergence for the series (7) is clearly 1. That this is in fact actually its radius of convergence can be seen by taking the branch of the logarithm with a cut along the negative real axis and an angle required to lie in $(-\pi, \pi)$: clearly the origin is the nearest point on the cut to $z = 1$, and since this branch will be analytic everywhere else, its Taylor series (7) (here $n = 0$, but as noted above this is not important) will converge on the disk of radius 1 centred at $z = 1$, call it D. This is then the same for all other branches of the log function which are defined at $z = 1$.

But now it seems that we have a problem! Suppose that we take a branch cut along a line radially outwards from the origin and making a very small angle α (positive or negative) with the positive real axis: clearly the resulting function is defined at $z = 1$, so by the foregoing, its Taylor series at $z = 1$ must be given by (7) for some $n \in \mathbf{Z}$, and must therefore converge on D. But as long as the angle α is between $-\pi/2$ and $\pi/2$, the branch cut we have chosen will clearly intersect D, meaning that the branch is *not* analytic everywhere on D. So evidently we have a case of a Taylor series which converges even across a singularity of the function it is suppose to represent!

With a little more thought, though, this is actually not that confusing. Let us pick some specific numbers to make things concrete. Thus let L_1 denote the branch of Log obtained by making a cut along the line $\theta = \pi/4$ and requiring the angle to lie in the set $(-7\pi/4, \pi/4)$, and let L_2 denote the branch of Log obtained by making a cut instead along the line $\theta = \pi/2$ and requiring the angle to lie in $(-3\pi/2, \pi/2)$. Then clearly both of these branches are defined at $z = 1$, and $L_1(1) = L_2(1) = 0$, so that they have the same Taylor series, given by (7) with $n = 0$. Moreover, if we write out the definitions of these two functions more carefully, we see that for $\theta \in (-3\pi/2, \pi/4)$ we have

$$L_1(re^{i\theta}) = L_2(re^{i\theta}) = \log r + i\theta,$$

while for $\theta \in (-7\pi/4, -3\pi/2)$ we have

$$L_1(re^{i\theta}) = \log r + i\theta$$

but, since the corresponding angle in $(-3\pi/2, \pi/2)$ is $2\pi + \theta$, we have

$$L_2(re^{i\theta}) = \log r + i(2\pi + \theta);$$

in other words, the difference is given by (for $\theta \in (-7\pi/4, \pi/4)$, $\theta \neq -3\pi/2$)

$$L_1(re^{i\theta}) - L_2(re^{i\theta}) = \begin{cases} 0, & \theta \in (-3\pi/2, \pi/4) \\ -2\pi i, & \theta \in (-7\pi/4, -3\pi/2). \end{cases}$$

Now since the branch cut for L_2 does not intersect the disk D, the series (7) will not only converge on D, it will actually converge to $L_2(z)$ everywhere on D. Thus we can see that the series (7) will converge to $L_1(z)$ as long as the argument of z lies between $-\pi/2$ and $\pi/4$, while it will converge to $L_1(z) + 2\pi i$ if the argument of z is greater than $\pi/4$ (in both cases, of course, we assume that $z \in D$ as otherwise (7) does not converge at all).

Summary:

- We give a more careful treatment of the method of solving Laplace's equation using linear solutions.

105. Linear solutions to boundary value problems for Laplace's equation. Basically, when we are solving a boundary-value problem, we are trying to find, by whatever method, a function u which satisfies two conditions: (i) it must satisfy Laplace's equation, i.e., its Laplacian must vanish; (ii) its value on the boundary of the region must equal the specified function. 'By whatever method' means that we do not care how we obtained the function, only that it satisfies these two conditions; in other words, the method does not need to be constructive or computational in any way. (This is all right since there are general theorems which guarantee that the solutions to problems like this are unique, at least when the boundary data is sufficiently nice.) Obviously, then, one way of 'finding' the solution would be to try one function after another until (hopefully) the correct one is found. Since there are infinitely many different functions we might need to try, though, that isn't a very practical idea. On the other hand, most functions one could think of writing down will certainly not satisfy condition (i); for example, $\sin x$ certainly doesn't, nor does $\sin x \sin y$, etc.. So maybe a good starting point would be to try to find a collection of functions (ideally all functions, though in practice that is again too many) which satisfy Laplace's equation, and then see if maybe we can somehow find one of them which also has the correct values on the boundary. Now one especially simple class of functions which satisfy Laplace's equation are the linear functions, $g(x, y) = a + bx + cy$: this is because we can calculate:

$$\frac{\partial g}{\partial x} = a, \qquad \frac{\partial g}{\partial y} = b,$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x} a = 0, \qquad \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y} b = 0,$$

so

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

Now that we know that these functions all satisfy Laplace's equation, if we are trying to solve a boundary-value problem we only need to find *numbers* a, b, and c such that the boundary conditions are satisfied. In other words, we are going to substitute the linear solution $a + bx + cy$ into the boundary conditions in (ii) and try to solve for the numbers a, b, and c. (For anything other than very special boundary conditions, of course, this will not be possible, because the linear solutions are too special; but for the problems on this assignment this is possible at some point.) Generally we solve for the numbers a, b and c either 'by inspection' or by substituting in values for x and y to obtain a system of equations that they must satisfy, which we then try to solve.

Consider the following trivial examples on the unit square $D = \{(x, y) \,|\, x, y \in [0, 1]\}$:

EXAMPLE 1. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = 1.$$

If we try a linear solution in this case, then we would say, let us see whether a solution of the form $u = a + bx + cy$ can solve this problem. It clearly solves $\Delta u = 0$; now the condition $u|_{\partial D} = 1$ means $a + bx + cy = 1$ whenever $(x, y) \in \partial D$. Thus, for example, if we let $x = 1$, $y = 0$ (which is clearly a point in ∂D), we get $a + b = 1$; if we let $x = 1$, $y = 1$, we get $a + b + c = 1$, which means $c = 0$; if we let $x = 0$, $y = 0$, we get $a = 1$, which also means that $b = 0$ since $a + b = 1$. Thus the only linear function which could possibly satisfy the boundary conditions is $u = 1$. But this function clearly does actually satisfy the boundary condition on all of ∂D, and since it satisfies Laplace's equation (as it had to since it was a particular example of the class of linear functions, each one of which is a solution to Laplace's equation), it must be the desired solution.

EXAMPLE 2. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = x.$$

Let us again try a linear solution: thus we wish to determine whether there are constants a, b, and c such that $u = a + bx + cy$ (which must satisfy Laplace's equation) satisfies the boundary condition. In this case things get a bit more interesting. Suppose that $y = 0$; then we have, for all $x \in [0, 1]$, that $a + bx = x$.

If we set $x = 0$, this gives $a = 0$, so $bx = x$; if we set $x = 1$ (or, for that matter, if we let x be any nonzero number), this gives $b = 1$. So far, then, we know that we must have $u = x + cy$. Now if $x = 0$ and $y \neq 0$, say $y = 1$, then we have by the boundary condition that $cy = c = 0$. Thus the only linear solution that could possibly satisfy the boundary conditions is $u = x$. Now this does actually clearly satisfy the boundary conditions; and since it also satisfies Laplace's equation, it must be the desired solution.

Note however that the solution cannot always be read off from the boundary conditions as in the two examples above. For example, the boundary condition in example 2 could have been expressed as follows: $u|_{\partial D} = 0$, $x = 0$, 1, $x = 1$, x, $y = 0$ or $y = 1$, which we might not immediately recognise – but the method above would give $u = x$ as the only solution regardless. For a more involved example, consider the following problem:

EXAMPLE 3. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \begin{cases} x, & y = 0 \\ x + 1, & y = 1 \\ y, & x = 0 \\ y + 1, & x = 1 \end{cases}$$

Let us see whether we can find a linear solution; thus suppose that $u = a + bx + cy$. The first of the boundary conditions gives $a + bx = x$ for $x \in [0, 1]$; if $x = 0$ this gives $a = 0$, while if $x = 1$ this gives $b = 1$. Thus we already know that if there is such a solution, it must be of the form $u = x + cy$. Now consider the third of the boundary conditions ($u = y$ when $x = 0$); this gives $cy = y$, which, setting $y = 1$, gives $c = 1$. Thus the only possible solution would be $u = x + y$. But now we need to show that this does indeed satisfy the other two boundary conditions. At $y = 1$ this expression gives $u(x, 1) = x + 1$, which is the correct expression; and at $x = 1$ it gives $u(1, y) = 1 + y = y + 1$, which is again the correct expression. Thus $u = x + y$ satisfies the boundary conditions, and since it satisfies Laplace's equation, it must be the desired solution.

These are of course rather simple examples, but they demonstrate the technique. More complicated examples were given in the lecture notes (see August 11, Section 38, pp. 3 – 4, August 13, Section 40, p. 4); it may be helpful to restudy these in the light of the above explanations.

The same technique is applicable to the problems on the last homework assignment: we posit that u takes a certain form, and then determine the coefficients by matching that form to the given boundary conditions.

APPENDIX I. REVIEW OF MULTIVARIABLE CALCULUS AND LINEAR ALGEBRA

I. MULTIVARIABLE CALCULUS

1. Parametric curves. A *plane parametric curve*[1] is a curve in the plane which can be described by two equations

$$x = x(t), \qquad y = y(t), \qquad\qquad (t \in [a, b])$$

for some interval $[a, b]$; in other words, for every point (x, y) on the curve, there is some value $t \in [a, b]$ such that $x = x(t)$ and $y = y(t)$. (Note that this t need not be unique.) More informally, if we view t as a dynamical quantity, the point $(x(t), y(t))$ 'traces out' the entire curve as t varies from a to b. It is often convenient to represent the point $(x(t), y(t))$ by a single function, often called $\gamma(t)$ (the Greek letter gamma), so that $\gamma(t) = (x(t), y(t))$. We shall use γ (without t) to refer to the entire curve, considered as a single object. When necessary to distinguish between the *function* $\gamma(t)$ and the plane curve this function represents, we shall call the latter the *image* of γ.

A curve is called *closed* when (in the notation of the previous paragraph) $\gamma(a) = \gamma(b)$. A closed curve which does not intersect itself (i.e., for which the value of t mentioned above *is* unique) is called a *Jordan curve*. A Jordan curve γ is said to be oriented *counterclockwise* if, as t increases from a to b, the point $\gamma(t)$ traces out the curve in a counterclockwise direction, and similarly to be oriented *clockwise* if this point traces out the curve in a clockwise direction.[2] We note for future use that if D is a connected region of the plane, then its boundary curve is always a Jordan curve. This result has a converse in the so-called *Jordan curve theorem* which we shall mention later on in the course.

General parametric curves can display pathological behaviour, even when $x(t)$ and $y(t)$ are both continuous.[3] In this course we shall deal exclusively with so-called *piecewise-smooth curves*, defined as follows. A parametric curve γ is said to be *piecewise-smooth* on an interval $[a, b]$ if (i) it is continuous on $[a, b]$ and (ii) there are points $t_0 = a < t_1 < \cdots < t_n = b$ such that on each subinterval (t_i, t_{i+1}), $i = 0, \cdots, n-1$, the *derivative* $\gamma'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$[4] exists, and is continuous and nonzero. (Condition (ii) amounts to saying that $x(t)$ and $y(t)$ are continuously differentiable on (t_i, t_{i+1}), and that $x'(t)$ and $y'(t)$ never vanish simultaneously. This last requirement is necessary to avoid 'corners'; see the practice problems!)

A piecewise smooth curve has a well-defined length. Recall that the *length* of a parametric curve γ defined on some interval $[a, b]$ and such that γ' is continuous there is given by

$$\int_a^b |\gamma'(t)|\, dt,$$

where $|\cdot|$ denotes the length of a vector. This definition can be extended to a piecewise-smooth curve in an obvious way: if $t_0, t_1, \ldots, t_n$ are the points given in the definition of piecewise-smoothness, then we define the length of γ to be[5]

$$\int_{t_0}^{t_1} |\gamma'(t)|\, dt + \int_{t_1}^{t_2} |\gamma'(t)|\, dt + \cdots + \int_{t_{n-1}}^{t_n} |\gamma'(t)|\, dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\gamma'(t)|\, dt.$$

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[1] Parametric curves can, of course, also be considered in three (and even arbitrary) dimensions. In this course, though, we shall only need them in two.

[2] Note that this definition would not make sense for a self-intersecting curve: for example, no matter how you trace out a figure-eight, the upper part will be oriented one way while the lower part will be oriented another.

[3] For example, one can find a parametric curve which – at least if we are allowed to replace the bounded interval $[a, b]$ by the whole real line – essentially fill out an entire two-dimensional region!

[4] While we shall not need to make this distinction in this course, it bears pointing out that, technically speaking, points and vectors are not identical, and when one must distinguish between them, a curve γ gives a point for each t while its derivative gives a vector.

[5] To be precise, the integrals here should be understood as improper integrals obtained by integrating from something slightly greater than t_i to something slightly less than t_{i+1}, and then taking the limit as these endpoints approach those two values, respectively; but this is generally not something which needs to be made explicit in practice, and we shall generally pass over it in silence in similar cases in the future.

The main use we shall make of parametric curves is in *line integrals* (see §3 below), and also in describing how two points in the plane are connected. This latter will become clearer as we progress through the course. The fact that two real numbers are essentially only connected in one way, while two complex numbers can be connected in multiple ways, some of which may be distinct (in an appropriate sense), is part of what makes complex analysis interesting.

2. Partial derivatives. Suppose that we have a function f defined on a region of the plane, which we suppose has Cartesian coordinates (x, y). We define its *partial derivatives* with respect to x and y to be

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h},\\ \frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{(f(x,y+h)-f(x,y)}{h}.\end{aligned}$$

We recall that in multivariable calculus we saw that the *existence* of both partial derivatives still allowed for quite a bit of pathological behaviour. It turns out that for functions of a complex variable there are additional requirements on the partial derivatives in order for the function to have a single *complex* derivative, and that these requirements, though simple, lead to far-reaching results which rule out all such pathological behaviours.

Recall that if a function f has a local extremum at a point where its partial derivatives exist, then they must both vanish.

Some examples of partial derivatives are given in the review problems.

[This paragraph is an aside for students who have had MAT237 or MAT257, or who have otherwise learned how to view the derivative as a linear map. In this class we shall be interested in complex-valued functions of a complex variable; since the set of complex numbers is a two-dimensional vector space over the real numbers, this means that we are in essence considering functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ (or, in essence, a vector field on $\mathbf{R}^2$). Thus the derivative of such a function, in the multivariable-calculus sense, should be a linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$ approximating the original function at the point of differentiation. It turns out that the requirement that a complex derivative exists requires that this map be a composition of an isotropic scaling (i.e., multiplication by a single real number) and a rotation. This is the basis for the study of functions of a complex variable as *conformal maps*, namely functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ which preserve angles.]

3. Line integrals and vector fields. Suppose that γ is a piecewise-smooth curve on an interval $[a, b]$ ($\gamma(t) = (x(t), y(t))$), and that f is a continuous function defined on some set containing the image of γ. Then we define three different types of *line integral* along γ, as follows. Let $t_0, t_1, \cdots, t_n$ be the points given in the definition of piecewise-smoothness; then we define

$$\begin{aligned}\int_\gamma f\,dx &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x(t),y(t))\,x'(t)\,dt\\ \int_\gamma f\,dy &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x(t),y(t))\,y'(t)\,dt\\ \int_\gamma f\,ds &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x(t),y(t))\,|\gamma'(t)|\,dt,\end{aligned}$$

and call these *the line integrals of f along γ with respect to x, y, and arclength*, respectively.

Recall that a *vector field* on a region of $\mathbf{R}^2$ is a function which to every point in its domain associates a vector in $\mathbf{R}^2$; in other words, it can be written as a function $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y)$ and $Q(x, y)$ are functions defined on the region called (naturally) the *components* of the vector field. If the vector field $\mathbf{F}$ is defined on a region containing γ, then we may combine the line integrals with respect to x and y of the components of $\mathbf{F}$ to define a new line integral, as follows:

$$\int_\gamma P(x,y)\,dx + \int_\gamma Q(x,y)\,dy = \int_\gamma \mathbf{F}(x,y)\cdot d\mathbf{x}.$$

We call this the line integral of the vector field along the curve γ. Recall the following *fundamental theorem of calculus for line integrals*: If $\mathbf{F} = \nabla f$ for some function f, i.e., if $\mathbf{F}$ is a gradient, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = f(\gamma(b)) - f(\gamma(a)),$$

and this integral is therefore independent of the choice of path γ. This notion of *path-independence* (this is a standard term, though in our current setting it would be more natural to call it *curve-independence*!), namely that the line integral along a certain curve only depends on the end-points of the curve and not on the curve itself, is of central importance in the study of analytic functions of a complex variable. Recall that it is equivalent to the requirement that the line integral along any closed curve be zero. This is in turn related to *Green's theorem*, which states that for any vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and any closed curve γ bounding a connected region D and oriented counterclockwise,

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where the latter is an *area integral* over the region D. This is a special case of *Stokes's theorem*, which we shall not need in its full generality but which we state here because it provides useful notation: If S is any (sufficiently smooth) connected surface in $\mathbf{R}^3$ with boundary curve C, and S and C are oriented consistently,[6] then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dA.$$

Here the second integral is a *surface integral* and $\mathbf{n}$ represents the *unit normal* to the surface S, but we shall not need these things in this class. The *curl* of a vector field can be defined heuristically as $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$; if $\mathbf{F}$ is a vector field on $\mathbf{R}^2$ then the curl can be taken to be the single number

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

appearing in Green's theorem. For us this is the only case for which we shall need to use the curl (and we shall not need to use it much even here).

Note now that Green's theorem tells us that line integrals of a vector field are path-independent exactly when the curl of that vector field is zero. Such a vector field is called *conservative*, though we shall only need this term only occasionally. We have seen that a vector field which is the gradient of a function is conservative; on a so-called *simply connected region* – by which we mean a region 'without holes', or, more precisely, whose boundary is a *single* Jordan curve – the converse is also true. We shall see that these results have analogues in the theory of functions of a complex variable, though the results generally are not quite exact copies.

II. LINEAR ALGEBRA

4. Matrices. In this course we shall not need much from the results of linear algebra, but mostly a familiarity with its concepts. Recall that a *matrix of size m by n* is a two-dimensional array of numbers

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

and is called *square* if $m = n$. The *product* of matrices $[a_{ij}]$ and $[b_{jk}]$ of sizes m by n and n by ℓ is defined to be the matrix $[c_{ik}]$ of size m by ℓ given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

[6] For us, this just means that were γ oriented *clockwise* we would need to introduce an extra minus sign on the right-hand side of Green's theorem.

Recall that the *identity matrix of size n by n*

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

has the property that $AI = A$ and $IB = B$ for any matrices A and B of size m by n and n by ℓ, respectively. If a matrix $A = [a_{ij}]$ is square of size n by n, then its *inverse* (when it exists) is a matrix A^{-1} of size n by n satisfying

$$AA^{-1} = A^{-1}A = I.$$

In general, finding an inverse matrix is hard. For two-by-two matrices, however, there is a simple formula which is often useful, given by *Cramer's rule*: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

as long as $ad - bc \neq 0$. The quantity $ad - bc$ is called the *determinant* of the matrix A; the notion of determinant can be defined for a square matrix of any size, but as the general definition is complicated and we shall not need it in this course we pass over it for the moment.

Recall that a matrix of size m by n can be viewed as giving a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. In particular, a 2 by 2 matrix can be viewed as a linear transformation on the plane. Two particularly important and simple examples are *isotropic scaling* and *rotation*. The first is just multiplication by a single scalar and corresponds to the matrix ($\lambda \neq 0$)

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \text{which has inverse} \qquad \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The second is a bit more complicated. Consider rotation of the plane by an angle θ counterclockwise around the origin. Since vector addition and scalar multiplication in the plane can be defined in terms of geometric pictures which are transformed rigidly by such a rotation, we see that this rotation must be linear; thus it suffices to determine its effect on the basis vectors $\mathbf{i}$ and $\mathbf{j}$ of the plane. If we rotate the vector $\mathbf{i}$ by an angle θ counterclockwise around the origin, a little geometry makes it clear that we obtain the vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, while if we rotate $\mathbf{j}$ the same way we obtain the vector $-\sin\theta i + \cos\theta\mathbf{j}$; thus the matrix giving this transformation is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We note two interesting properties of this matrix: first, its determinant is

$$\cos\theta \cdot \cos\theta - (-\sin\theta) \cdot \sin\theta = \cos^2\theta + \sin^2\theta = 1;$$

secondly, its inverse is (by the general formula above)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

which is just the original matrix with θ replaced by $-\theta$! This makes good sense since the inverse to a counterclockwise rotation by θ is a clockwise rotation by θ, which is essentially just a counterclockwise rotation by $-\theta$.

APPENDIX II. BASIC DEFINITIONS AND PROPERTIES OF COMPLEX NUMBERS.

1. Basic definitions. A *complex number* is an abstract quantity $z = a + ib$, where a and b are real numbers and i is an abstract quantity which we require to satisfy $i^2 = -1$.[1] We require that these numbers satisfy all of the usual properties of arithmetic; thus if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we have

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$
$$z_1 \cdot z_2 = a_1a_2 - b_1b_2 + i(a_1b_2 + b_1a_2.$$

We define the *conjugate* of a complex number $z = a + ib$ to be the complex number

$$\overline{z} = a - ib,$$

and note that the product

$$z\overline{z} = a^2 + b^2$$

is always real and nonnegative. By the Pythagorean theorem, $\sqrt{z\overline{z}}$ is the distance from the origin to the point (a, b), and we call this quantity the *modulus*[2] (or, sometimes, the *absolute value* or even the *length*) of the complex number z, and denote it as

$$|z| = \sqrt{z\overline{z}}.$$

The function $|\cdot|$ satisfies all of the usual properties of the absolute value function on real numbers:

$$|z_1z_2| = |z_1||z_2|, \qquad |z_1 + z_2| \le |z_1| + |z_2|.$$

Since geometrically $|\cdot|$ represents distance from the origin, a set of the form

$$\{z||z - z_0| < R\}$$

is a circle of radius R centred at the point corresponding to z_0.

The ratio of two complex numbers can be determined as follows. Let $z_1 = a+ib$, $z_2 = c+id$ be complex numbers with $z_2 \neq 0$; then

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a + ib)(c - id)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Let $z = a + ib$ be a complex number, and consider the corresponding point (a, b) in the plane. Let this point have a polar representation (r, θ), where r is the distance to it from the origin and θ is the angle from the positive x axis to a ray from the origin to the point, measured counterclockwise. Then we may write

$$a = r\cos\theta, \qquad b = r\sin\theta,$$

so that we have

$$z = r(\cos\theta + i\sin\theta).$$

Here, clearly, $r = |z|$. The angle θ is called the *argument* of the complex number z and is defined only up to a multiple of 2π. From the definition of the complex exponential below it follows that we may write equivalently

$$z = re^{i\theta}.$$

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are any two complex numbers, then it is not hard to show that their product has the polar representation

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)},$$

[1] For those who are familiar with fields, we note that we may view the set of complex numbers as the quotient field $\mathbf{R}[x]/(x^2 + 1)$, which is quite natural since we wish it to be the real field $\mathbf{R}$ with a root of the equation $x^2 + 1 = 0$ attached. In this quotient field the equivalence class of x plays the role of i.

[2] Plural, *moduli*.

i.e., that moduli multiply while arguments add. Geometrically, multiplying complex numbers amounts to multiplying lengths and adding angles. Further,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)},$$

i.e., dividing corresponds to dividing lengths and subtracting angles. If m is any positive integer, then, we have

$$z^m = \underbrace{z \cdot z \cdots z}_{m\,\text{times}} = r^m e^{im\theta}.$$

If m is a negative integer, we define

$$z^m = \frac{1}{z^{|m|}},$$

and it is simple to show that in this case also

$$z^m = r^m e^{im\theta},$$

i.e., that this formula holds for all nonzero integer exponents m. For nonzero z we define $z^0 = 1$, and this formula then holds for all integer exponents.