Summary:

- We give an introduction to differentials, and consider how various integration and differentiation rules look like when written using this notation.

Changes in quantities

Suppose that we measure a certain quantity, call it x, as a function of time. (For example, x could be the position of a bug walking along the edge of a table, or the charge on a capacitor, or the amount of bacteria in a culture, etc..) Suppose we are interested in how x changes as a function of time. In particular, if we let t denote time measured in seconds, we might be interested in a quantity like $\frac{dx}{dt}$; or maybe we want to measure time in milliseconds: if we let τ denote that, then we might be interested in $\frac{dx}{d\tau}$. Since $\tau = \frac{1}{1000}t$, the chain rule tells us that these two quantities have a simple relationship:

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = 1000\frac{dx}{dt}.$$

Now suppose that we are interested in another quantity which has a more complicated relationship with t; for example, suppose that while we measure x we are simultaneously measuring a separate quantity s, which always increases in time. (Always decreases would work too; the point is that we need to be able to solve for t, given s, so s has to be one-to-one as a function of t.) If we are interested now in the rate of change of x with respect to s, the chain rule obligingly gives us the answer:

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}.$$

Now if ds is an infinitesimal change in s, then the amount x changes by when s increases by an amount ds is given by

$$x|_{s+ds} - x|_s = \frac{dx}{ds}ds.$$

(Here we use the evaluation symbol $|$ instead of functional notation (e.g., $x(s)$) since we want to think of x as a function of more than one independent variable; we write equality since we are taking ds to be infinitesimal.) If we use the chain rule to write out $\frac{dx}{ds}$, this formula becomes

$$x|_{s+ds} - x|_s = \frac{dx}{dt}\frac{dt}{ds}ds;$$

but now $\frac{dt}{ds}ds$ is just the amount by which t increases between s and $s + ds$, so the right-hand side can be written simply as $\frac{dx}{dt}dt$. What about the left-hand side? Well, if we denote the first point by t, then the second point will be $t + dt$; in other words, we have obtained the relation

$$x|_{s+ds} - x|_s = x|_{t+dt} - x|_t = \frac{dx}{dt}dt.$$

This suggests that there is something fundamental about the change in x under an infinitesimal change in its independent variable, which is independent of the particular choice of independent variable. We call this quantity dx and think of it as the infinitesimal change in x under an infinitesimal change in its independent variable. From the foregoing, we see that for any choice of independent variable s on which x depends differentiably,

$$dx = \frac{dx}{ds}ds.$$

(If you are having a bit of trouble visualising the relation between s and t, it might help to think of t as measuring time in some standard way, say in seconds, and s as measuring the passage of certain events, such as the distance a bug has walked along the edge of a table, or the amount of a certain quantity present in a reaction. As long as the dependence of s on t is differentiable, and either always increasing or always decreasing, the above method will work. In this case, while t measures a 'constant rate' time, s is 'measuring time' in terms of some 'nonconstant rate' quantity.)

Let us see where this gets us with the things in calculus which we have studied so far. First of all, the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

becomes simply

$$dy = \frac{dy}{du}du;$$

in other words, we can 'cancel' the differential du with the du in the denominator of $\frac{dy}{du}$ as though they were ordinary numbers. (The differential, of course, is *not* an ordinary number!) Similarly, the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

becomes

$$d(uv) = vdu + udv.$$

These two examples are rather trivial. More interesting examples come from indefinite integrals. We know that

$$\frac{d}{dx}\int f(x)\,dx = f(x);$$

in terms of differentials,

$$d\int f\,dx = f\,dx; \tag{1}$$

in other words, d cancels out $\int$. (This is essentially true the other way around, too: if x is a variable on which f depends differentiably,

$$\int df = \int \frac{df}{dx}dx = f(x) + C = f + C.)$$

Equation (1) is worth studying in more detail as it suggests that quantities like fdx may also have an interpretation. Suppose that F is an antiderivative of f with respect to x, so that $F'(x) = f(x)$; then we have

$$f(x)dx = F'(x)dx = \frac{dF}{dx}dx = dF.$$

In other words, fdx is the differential of any antiderivative (with respect to x) of $f(x)$. Note though that we have been careful to specify "with respect to x" at multiple points in the foregoing. This is because whether a function F is an antiderivative of a function f depends on our choice of independent variable: if u is another choice of independent variable, then

$$\frac{dF}{du} = \frac{dF}{dx}\frac{dx}{du} = f(x)\frac{dx}{du},$$

which will not in general be equal to $f(u)$. As a specific example, suppose that $f(x) = \cos x$, $F(x) = \sin x$, $u = \sqrt{x}$; then clearly $F'(x) = f(x)$, but since $x = u^2$, we have $F(u) = \sin u^2$, so $F'(u) = 2u\cos u^2 = 2uf(u)$, which is not the same as $f(u)$.

This is actually related to what goes on when we evaluate integrals by substitution. To see this, suppose that we have two different independent variables x and u; then we have

$$d\int fdu = fdu = f\frac{du}{dx}dx,$$

which if we integrate both sides gives $\int f\frac{du}{dx}dx = \int fdu$, exactly the standard formula for integration by substitution. To put it another way: an integral of f is basically a sum of infinitesimal increments fdu, where f measures how much of an increment we need in a change of independent variable u equal to du;

thus if we wish to work with x instead of u, we have to adjust f to give the increment needed in a change of independent variable x equal to dx, and the adjustment needed is to replace f by $f\frac{du}{dx}$.

More concisely, since (by the chain rule) $f du = f\frac{du}{dx}dx$, if f measures the amount of increment we need when working in terms of u, then $f\frac{du}{dx}$ measures the amount of increment we need when working in terms of x to get the same total increment after integrating.

It is precisely this factor of $\frac{du}{dx}$, or in other words, the fact that we cannot write $f du = f dx$, which makes many integration problems intractable by substitution. For example, consider the integral

$$\int \sin x^2 \, dx.$$

The obvious substitution is $u = x^2$. Were we able to write $f dx = f du$, then this integral would become $\int \sin u \, du = -\cos u + C = -\cos x^2 + C$ and we would be done. However, we have instead

$$f du = f\frac{du}{dx}dx = 2xf dx,$$

meaning that $-\cos x^2 + C$ is not an antiderivative of $f(x) = \sin x^2$, but (as can readily be verified) of $2x \sin x^2$.

Finally, we may rewrite our integration-by-parts formula using differentials by adapting the product rule as written above:

$$uv = \int d(uv) = \int u dv + \int v du,$$

whence

$$\int u \, dv = uv - \int v \, du.$$

Introducing a particular independent variable x and writing $dv = \frac{dv}{dx}dx$, $du = \frac{du}{dx}dx$, we obtain the formula as given in the book:

$$\int uv' \, dx = uv - \int vu' \, dx.$$