Example of using Taylor series to solve a differential equation

Remember that we discussed solving the equation $y' = x + y$ using Taylor polynomials. This process gave us an approximate solution because we always had one more term on the right-hand side than on the left-hand side – there was always an error term which we could not balance between the two sides. Now the higher the degree of the Taylor polynomial, the higher the degree of this term. At least if x is close to 0, this suggests that if we take the Taylor polynomial of high enough degree, we can find a solution which is as accurate as we like. Thus if we replace the Taylor polynomial with a Taylor *series*, perhaps we can get an exact solution. Let us see how this would work.

Suppose that we assume our solution has a Taylor series expansion

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(Remember that writing the solution out this way does not assume anything particular about the coefficients a_n – in particular, some or many of them could be zero.) We want this to satisfy $y' = x + y$. To see whether this holds, we differentiate both sides:

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \\ x + y &= x + \sum_{n=0}^{\infty} a_n x^n = a_0 + (a_1 + 1)x + a_2 x^2 + \cdots. \end{aligned}$$

If we set these two expressions equal, we obtain the equations

$$\begin{aligned} a_1 &= a_0, \\ 2a_2 &= a_1 + 1, \\ 3a_3 &= a_2, \\ 4a_4 &= a_3, \\ &\vdots \\ (n+1)a_{n+1} &= a_n, \\ &\vdots \end{aligned}$$

whence we see that, given a_0, we can find all higher coefficients.

We shall work out a particular case of this, as well as this method applied to other equations, in class.

It is worthwhile to step back for a moment and compare this method to what we did previously when we were working only with polynomials. Beyond the second equation, all equations in the above list are of the form $(n+1)a_{n+1} = a_n$; in other words, if a_n is nonzero, then of necessity a_{n+1} must be nonzero also. This clearly implies that, as long as $a_2 \neq 0$, all of the coefficients a_n for $n \geq 2$ must also be nonzero; in other words, we need *infinitely many* nonzero coefficients, and hence *infinitely many* nonzero terms, in order to exactly satisfy the differential equation. Our approximation method previously was to look for Taylor polynomials of a certain degree, say n; this amounts to setting $a_{n+1} = a_{n+2} = \cdots = 0$, which means that it is not possible to satisfy the equation $(n+1)a_{n+1} = a_n$ and we will have the ‘error term’ $a_n x^n$ on the right-hand side. In the present case, on the other hand, we can keep going indefinitely, meaning that, at least formally, there is no error term since all of the equations above will be satisfied. More careful work would of course be required to show that the resulting series converges to a solution, but we pass over that here.