Summary:
- We discuss how to use Taylor series to solve differential equations.
- We also discuss the issues that arise when we attempt to solve nonlinear equations using Taylor series.

Using Taylor series to solve linear differential equations

Please see the pre-class notes for some general comments. Here we restrict ourselves to working out another specific example.

EXAMPLE. Solve the differential equation $y' = e^x + y$ using Taylor series.

We will expand around $x = 0$ (for no particular reason in this particular case – since here we are given no initial data, so there is no distinguished point about which we are required to solve – other than that the Taylor series for e^x, which we shall need to find the Taylor series for y, is simplest around $x = 0$). Let us assume that y has the Taylor series expansion

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then as usual

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,$$

and since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

the differential equation gives

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}\left(a_n + \frac{1}{n!}\right)x^n.\end{aligned}$$

Since two Taylor series are equal if and only if all of their coefficients are equal, we must have

$$(n+1)a_{n+1} = a_n + \frac{1}{n!},$$

which gives that

$$a_{n+1} = \frac{1}{n+1}\left(a_n + \frac{1}{n!}\right) = \frac{a_n}{n+1} + \frac{1}{(n+1)!}.$$

This will allow us to determine a_1 in terms of a_0, a_2 in terms of a_1, etc., and hence to ultimately determine all coefficients in terms of a_0; in other words, a_0 will be an undetermined constant similar in principle to a constant of integration. (a_0 would be fixed by giving $y(0)$, for example.) Now we would like to find a closed form for the coefficients a_n, i.e., an expression as a function of n which does not involve any other values of a_n (no recursion). The above recurrence relation suggests rather strongly that a_n must involve $\frac{1}{n!}$ somehow; thus we see whether we can find an expression for $n! \cdot a_n$. If we multiply the above recurrence relation by $(n+1)!$, we find that

$$(n+1)!a_{n+1} = n!a_n + 1;$$

thus the sequence $n! \cdot a_n$ must just be the sequence a_0, $a_0 + 1$, $a_0 + 2$, etc.; i.e., we must have

$$n! \cdot a_n = a_0 + n,$$

so that

$$a_n = \frac{a_0 + n}{n!},$$

which for $n > 0$ becomes

$$\frac{a_0}{n!} + \frac{1}{(n-1)!}.$$

(Recall again that $0! = 1$.) For $n = 0$ it gives of course just $a_0 = a_0$. If we substitute this back into y, we obtain

$$\begin{aligned} y = a_0 + \sum_{n=1}^{\infty} a_n x^n &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_0}{n!} + \frac{1}{(n-1)!} \right) x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n \\ &= a_0 e^x + \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} = a_0 e^x + x e^x. \end{aligned}$$

We can check that this expression is correct by differentiating:

$$\frac{d}{dx} \left(a_0 e^x + x e^x \right) = a_0 e^x + e^x + x e^x = a_0 + x e^x + e^x = y + e^x.$$

As in this example, the process of finding the general form for the coefficient a_n as well as determining the sum of the series is difficult (and, in general, not possible).

Nonlinear equations

You may have heard mention made of ‘nonlinearity’, ‘nonlinear problems’, etc., in various discussions of scientific or mathematical problems. Here we shall discuss a particular problem to see how nonlinearity can make things more tricky.

EXAMPLE. Solve the differential equation $(y')^2 = x + y$ using Taylor series.

Again, we shall expand around $x = 0$ (since in this case the expansion of x is just x; as an exercise, it is very worthwhile working out what the expansion of x is around a general value a!) and assume that y has the expansion

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then as before we have[1]

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

[1] This raises an interesting point. Remember that we term functions equal to the sum of their Taylor series – at least on some interval – to be analytic; in other words, an analytic function is determined entirely by its sequence of Taylor coefficients. Suppose that a function f has Taylor series with coefficients $\{a_n\}_{n=0}^{\infty}$. Then the derivative f' has Taylor series with coefficients $\{(n+1)a_{n+1}\}_{n=0}^{\infty}$. Now the operations of translating a sequence and multiplying by a constant factor are certainly far easier than the operation of differentation; in other words, representing functions by Taylor series makes the derivative operator look much simpler. Integration can also be represented similarly, up to some technical points involving the constant of integration and such. While we shall not make further use of this notion in this course, the idea of representing a function in terms of sequences of coefficients is a most fruitful one which comes up in many other parts of mathematics and science; for example, Fourier series and wavelet representations, which you will likely run into if you do any data processing, are two other examples.

Now we may find the Taylor series for $(y')^2$ about $x = 0$ by squaring this series, as follows:

$$\begin{aligned}
(y')^2 &= \left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)^2 \\
&= a_1\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right) + 2a_2x\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right) + 3a_3x^2\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right) + \cdots \\
&= a_1^2 + 2a_1a_2x + 3a_1a_3x^2 + \cdots \\
&\qquad + 2a_2a_1x + 4a_2^2x^2 + \cdots \\
&\qquad\qquad + 3a_3a_1x^2 + \cdots \\
&= a_1^2 + 4a_1a_2x + (6a_1a_3 + 4a_2^2)x^2 + \cdots,
\end{aligned}$$

where $\cdots$ represents terms of order higher than 2. This gives the first few terms of the expansion of the left-hand side of the above differential equation, while the right-hand side is given before as

$$x + y = a_0 + (a_1 + 1)x + a_2x^2 + \cdots;$$

equating these two expressions gives the equations

$$\begin{aligned}
a_1^2 &= a_0, \\
4a_1a_2 &= a_1 + 1, \\
6a_1a_3 + 4a_2^2 &= a_2, \\
&\vdots
\end{aligned}$$

Now a_0 can be found from the initial condition as before; we have $a_0 = y(0) = 1$. The equations can then be solved to determine the coefficients a_n.