Summary:

- We discuss Taylor series and the relation between their coefficients and the derivatives of the functions they represent.
- Then we discuss methods of determining new Taylor series from known ones.

Review of Taylor polynomials

Recall that back at the start of the course we talked about the possibility of representing functions by polynomials. This is discussed in detail in the lecture notes for January 6; here we simply give a brief recap. Recall that the idea was to find a polynomial whose derivatives up to a certain degree at a certain point (say 0) were all equal to those of the original function at that point. This led us to the formula for the nth Taylor polynomial of a function f about $x = 0$:

$$\begin{aligned} P_n(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n \\ &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)x^k. \end{aligned}$$

(We note two things here: by convention $f^{(0)} = f$; more importantly, by definition, $0! = 1$.[1]) Taylor polynomials around points other than 0 can be treated similarly; in general, we have the following formula for the nth degree Taylor polynomial about $x = a$:

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k. \tag{1}$$

If we are given a Taylor polynomial, we can determine the centre point, that is, the point around which we expand, by determining the value of a for which the series is a power series in $x - a$. Also, if we are given a Taylor polynomial of degree n of a function f, we can determine the kth derivative of f *at the point* $x = a$ by multiplying the coefficient of $(x - a)^k$ by $k!$; in other words, if we are given

$$P_n(x) = \sum_{k=0}^{n} C_k(x-a)^k,$$

then we know that for $k = 0, 1, \ldots, n$, $f^{(k)}(a) = k!C_k$. (This is by comparison with the definition of the Taylor polynomial above, since two polynomials are equal if and only if their coefficients are equal.) Note however that this method does not allow us to calculate the derivatives (or even the value) of f at any point other than a.

Taylor series

As we saw at the start of the course, in general, higher order Taylor polynomials give better approximations to the function. This raises the question as to whether we can somehow take the order of the polynomial 'to infinity', with the hope that in that limit the approximation would become exact and we would get the full, exact function.

If we look at the formula in equation (1) above, this would amount to considering the infinite power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k.$$

[1] There is a good reason for this definition. The factorial of a number n can be defined as the number of ways of arranging n objects; thus if I have one object, there is only one way of arranging it, if I have two there are 2 (if I label them a and b, I can arrange them either as ab or as ba), if there are three objects then there are 6 ways of doing the arranging (abc, acb, bac, bca, cab, cba), and in general if there are $n > 0$ objects then there are $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$ ways of doing the arranging. Now if there are 0 objects then there is also exactly one way of doing the arranging (slightly tongue-in-cheek, one might say that way can be expressed as, Don't do it, i.e., don't do anything at all). But for us it is sufficient to consider this as a special-case definition.

We note a few things right away. First of all, for this procedure to have any chance of working, the function f must possess derivatives of all orders. Second, we note that the question of whether the series above is equal to the function f can be split into two parts: first, whether the series converges at all; second, whether it converges to f. In this course we shall focus on the first question; section 10.4 in the textbook discusses the second question but is not part of the course this year.[2] Now Taylor series are just power series, which means we can use the methods we talked about on Monday (see the notes for March 23) to determine the set of x for which they converge, and that it will always be an interval centred on $x = a$.

There are examples in the textbook, and more in the TopHat. Here we just make a few further comments. First of all, we note that representation of a function by a power series is always unique; in other words, any representation of a function by a power series will be the Taylor series of that function. This will be particularly useful in the next section, but even here it allows us to see that the relationship between a function and a power series goes both ways: if we are given a function, we can compute its Taylor series around a point a by finding its derivatives of all orders at a; and if we are given a power series around a point a which represents some function, then we know that the power series must be the Taylor series of that function, so we can find its derivative of any order, say n, simply by multiplying the coefficient of its nth order term by $n!$.

We note one point which could cause confusion. When thinking about Taylor series we think of all terms from zeroth order (i.e., the constant term) up to be present; in cases where a certain term does not appear (for example, x^2 in the series for $\sin x$ around $x = 0$), we consider that the term is there but has a zero coefficient. Thus when we speak of the nth order term of a power series we always mean the term involving $(x - a)^n$, *not* the nth nonzero term. (Because of this, we do not generally talk about the nth term of a power series as that could be ambiguous: we either say the nth order term, or the nth nonzero (or non-vanishing) term.)

Constructing new Taylor series from old

In general, it is very difficult to find a formula for the nth derivative of a function, except in very simple cases. Now most functions we write down are constructed out of simpler functions by the processes of addition, subtraction, multiplication, division, and composition, so it would be nice if we could figure out how these processes are reflected in Taylor series. This is made possible by the uniqueness of power series expansions just mentioned. Let us demonstrate by a simple example.

Suppose that I have two functions $f(x)$ and $g(x)$ whose Taylor series about a point a (note that this must be the same point for both functions!) are $\sum_{n=0}^{\infty} C_n(x-a)^n$ and $\sum_{n=0}^{\infty} D_n(x-a)^n$. This means that there is some interval $(a - r, a + r)$[3] on which

$$\sum_{n=0}^{\infty} C_n(x-a)^n = f(x), \qquad \sum_{n=0}^{\infty} D_n(x-a)^n = g(x).$$

[2]Most of the more-or-less natural functions one can write down have Taylor series which, when they converge at all, converge to the original function. A function which has this property is called *analytic*. In particular, sums, differences, products, quotients (as long as the denominator is nonzero), and compositions of analytic functions are all analytic. While not at all part of the present course, it is not difficult to write down an example of a function which has derivatives of all orders but is not analytic; the standard example is the following:

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0 & x \leq 0 \end{cases};$$

it is not particularly difficult, though extremely long, to show that the function f as defined has derivatives of all orders at $x = 0$, and that they are all zero. However, $e^{-\frac{1}{x}}$ is not zero anywhere, meaning that the series does not converge to the function on any open interval around $x = 0$. Roughly, this is because as $x \to 0^+$, $-\frac{1}{x} \to -\infty$, and the exponential function at infinity goes to zero so quickly that the graph of $e^{-\frac{1}{x}}$ is in some sense 'infinitely flat' as $x \to 0^+$. While we do not need these functions for anything we do here, they give rise to so-called C^∞ functions of compact support, which are very important in more advanced parts of mathematics (I use them all the time in my research, for example).

[3]Here r can be no larger than the smaller of the radii of convergence of the Taylor series for f and g.

But by general properties of series, this means that the sum of the two series, which is the same as the series of the sums of the general terms, must converge to $(f+g)(x)$ on the interval $(a-r, a+r)$, i.e., that

$$\sum_{n=0}^{\infty}(C_n + D_n)(x-a)^n = \sum_{n=0}^{\infty} C_n(x-a)^n + \sum_{n=0}^{\infty} D_n(x-a)^n = f(x) + g(x).$$

This makes perfect sense when we recall how C_n and D_n are computed from the derivatives of f and g: we have

$$C_n = \frac{1}{n!} f^{(n)}(a), \qquad D_n = \frac{1}{n!} g^{(n)}(a),$$

so

$$C_n + D_n = \frac{1}{n!}\left(f^{(n)}(a) + g^{(n)}(a)\right) = \frac{1}{n!}\,\frac{d^n}{dx^n}(f+g)\bigg|_{x=a}.$$

More interesting series result when we consider composition. Let us consider the function $f(x) = \sin x^2$, and find its Taylor series about 0. We know that the Taylor series for $\sin x$ around $x = 0$ is[4]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

It is not hard to show that the radius of convergence of this series is infinite, so that the series converges for all x. It can also be shown that the series converges to $\sin x$ everywhere on its interval of convergence. How does this help us? Well, if $x \in \mathbf{R}$, then certainly also $x^2 \in \mathbf{R}$; and since the above series converges to $\sin x$ for all real x, we must have also

$$\begin{aligned}\sin x^2 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x^2\right)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}.\end{aligned}$$

But this last series is just a power series! This means that we have constructed a power series representation of $\sin x^2$, which must therefore be the Taylor series expansion for $\sin x^2$.

We can also multiply Taylor series to obtain the Taylor series of a product, or work out the Taylor series of more complicated compositions than that given here.

Another manipulation we can do with Taylor series is to differentiate. For this, we need the following result: Suppose that the function $f(x)$ has a Taylor series $\sum_{n=0}^{\infty} C_n(x-a)^n$ which is convergent on an open interval $(a-r, a+r)$. Then the Taylor series of its derivative $f'(x)$ about $x = a$ converges on this same interval, and can be obtained by term-by-term differentiation; i.e., on $(a-r, a+r)$ we have

$$f'(x) = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$$

(where we write $n = 1$ instead of $n = 0$ since the term with $n = 0$ would be zero as it would be the derivative of the constant term from $f(x)$, and the derivative of a constant is, of course, always zero). The same is true for integration, at least as long as we stay within the interval of convergence. In other words, in the above case, we have also that for x in $(a-r, a+r)$

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1};$$

note that this is just the term-by-term integral of the series for f, since

$$\int_a^x (t-a)^n\,dt = \frac{(t-a)^{n+1}}{n+1}\bigg|_a^x = \frac{(x-a)^{n+1}}{n+1}.$$

Examples are in the textbook and the TopHat.

[4]You should memorise the Taylor series about $x = 0$ for the functions $\sin x$, $\cos x$, and e^x, and also the geometric series (which is just the Taylor series for the function $\frac{1}{1-x}$ around $x = 0$).