Summary:
- We give an example of applying the ratio test to a power series, and talk about radius of convergence matters in general.

An extended example

Consider the power series
$$\sum_{n=0}^{\infty} \frac{n^3+n^2}{4n^9+10} x^n.$$
What kind of mathematical object is this? Well, evidently it involves an independent variable x, so it looks like it might define a function. What do we need to get a function? We need a domain, a range, and a rule. Usually, of course, we are simply given a rule and assume that the domain is the set of real numbers for which that rule makes sense.

So could the above expression give a function in some sense? It certainly does look like it could give a rule, at least for some values of x: given x, the rule would say to find the sum of the series. What would the domain be? The domain would be the set of x for which the rule makes sense; in other words, the set of x such that the series above converges. Let us see a specific example of how we would determine whether the series above converges for a specific value of x.

EXAMPLE. Determine whether the above series converges for (a) $x = 0$; (b) $x = \frac{1}{2}$; (c) $x = \frac{3}{2}$.

(a) If $x = 0$, then all terms in the above series are zero[1] since the coefficient when $n = 0$ is just 0. Thus the series is simply a sum of infinitely many zero terms, and hence must equal 0 and in particular must converge. (More precisely, the partial sums S_n must be 0 for all n, and hence their limit must also be 0.)

(b) This case requires a bit more work. We have the series
$$\sum_{n=0}^{\infty} \frac{n^3+n^2}{4n^9+10} \left(\frac{1}{2}\right)^n.$$
We consider applying the ratio test. Let us let $b_n = \frac{n^3+n^2}{4n^9+10}\left(\frac{1}{2}\right)^n$ denote the nth term of the series; then to apply the ratio test we must investigate the fraction
$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^3+(n+1)^2}{4(n+1)^9+10}\left(\frac{1}{2}\right)^{n+1}\frac{4n^9+10}{n^3+n^2}\left(\frac{1}{2}\right)^{-n}\\ &= \frac{(n+1)^3+(n+1)^2}{n^3+n^2}\frac{4n^9+10}{4(n+1)^9+10}\cdot\frac{1}{2}.\end{aligned}$$
We now wish to evaluate the limit of the absolute value of this quantity as $n \to +\infty$. We note that in this case $\frac{b_{n+1}}{b_n}$ is always positive, so we don't need to worry about taking the absolute value. To find the limits of the above fractions, we can simply factor out the leading-order terms, as follows:
$$\frac{(n+1)^3+(n+1)^2}{n^3+n^2} = \frac{(n+1)^3}{n^3}\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}};$$
now as $n \to +\infty$
$$\frac{(n+1)^3}{n^3} = \left(1+\frac{1}{n}\right)^3 \to 1, \qquad 1+\frac{1}{n+1} \to 1, \qquad 1+\frac{1}{n} \to 1,$$
so we conclude that
$$\lim_{n\to+\infty} \frac{(n+1)^3+(n+1)^2}{n^3+n^2} = 1.$$

[1] Note that when dealing with power series, we make the convention that $x^0 = 1$ for all x, even $x = 0$; recall that in general 0^0 is not defined so this is just a convention in that case.

Similarly,

$$\frac{4n^9+10}{4(n+1)^9+10} = \frac{n^9}{(n+1)^9}\frac{4+\frac{10}{n^9}}{4+\frac{10}{(n+1)^9}},$$

which also converges to 1 as $n \to +\infty$. (In fact, one has the general result that $\lim_{n\to+\infty} \frac{p(n+1)}{p(n)} = 1$ for any nonzero polynomial $p(n)$, but on a test we should write these limits out as above to show that we understand what we are doing.)

Going back to our original ratio, then, we see that

$$\lim_{n\to+\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to+\infty} \frac{(n+1)^3+(n+1)^2}{n^3+n^2}\frac{4n^9+10}{4(n+1)^9+10}\cdot\frac{1}{2} = 1\cdot 1\cdot \frac{1}{2} = \frac{1}{2}.$$

Since this limit is less than 1, the ratio test tells us that the series converges when $x = 2$.

(c) This case is very similar to (b). We are now dealing with the series

$$\sum_{n=0}^{\infty} \frac{n^3+n^2}{4n^9+10}\left(\frac{3}{2}\right)^n.$$

As before, let $b_n = \frac{n^3+n^2}{4n^9+10}\left(\frac{3}{2}\right)^n$; then we are interested in computing the limit of the fraction

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^3+(n+1)^2}{4(n+1)^9+10}\left(\frac{3}{2}\right)^{n+1}\cdot\frac{4n^9+10}{n^3+n^2}\left(\frac{3}{2}\right)^{-n}\\ &= \frac{(n+1)^3+(n+1)^2}{n^3+n^2}\frac{4n^9+10}{4(n+1)^9+10}\cdot\frac{3}{2}.\end{aligned}$$

We note that this is exactly the same as the previous fraction except that the value of $\frac{1}{2}$ has changed to $\frac{3}{2}$! Since we know that the limit of each of the first two fractions is 1, the limit of the entire ratio must be $\frac{3}{2}$. By the ratio test, then, the series does not convege.

To generalise off of parts (b) and (c), let us consider the series

$$\sum_{n=0}^{\infty} \frac{n^3+n^2}{4n^9+10}x^n;$$

if we let again $b_n = \frac{n^3+n^2}{4n^9+10}x^n$ denote the nth term of the series, then the ratio we need to apply the ratio test is as before (we now include the absolute value signs since x may not be positive)

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)^3+(n+1)^2}{n^3+n^2}\frac{4n^9+10}{4(n+1)^9+10}\cdot|x|,$$

and as before the limit will be $|x|$. Thus by the ratio test the series will converge when $|x| < 1$ and diverge when $|x| > 1$, while the ratio test fails when $|x| = 1$, i.e., when $x = \pm 1$. In this case we must go back to the original series and try a different test. When $x = 1$ the series is

$$\sum_{n=0}^{\infty} \frac{n^3+n^2}{4n^9+10};$$

this looks similar to a p-series with $p = 6$, so we expect it to converge, and we can verify this by using the integral test and the comparison test for improper integrals (alternatively, by applying the comparison test for series and noting that

$$\frac{n^3+n^2}{4n^9+10} < \frac{2n^3}{4n^9} = \frac{1}{2n^6},$$

and the series $\sum \frac{1}{2n^6}$ converges). When $x = -1$ the series is

$$\sum_{n=0}^{\infty} \frac{n^3 + n^2}{4n^9 + 10}(-1)^n;$$

now the sum of the absolute values of the terms is just the series for $x = 1$, and since that series converges this above series must converge absolutely, and hence must converge. (Note that in general it is *not* the case that convergence at one endpoint implies convergence at the other endpoint!) Thus the given series converges on the interval $[-1, 1]$ and diverges everywhere else.

We call $[-1, 1]$ the *interval of convergence* for the series, and also say that the *radius of convergence* is 1.[2]

In general

Let us see how this works in the general case. Thus suppose that we have a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

and suppose that the coefficients a_n are always nonzero (at least for sufficiently large n). Then we may apply the ratio test to the above series: in analogy with the example above, we set $b_n = a_n x^n$ and find

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \left|\frac{x^{n+1}}{x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x|.$$

Now suppose that the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit, call it L. If $L = 0$, then the above calculation shows that for all real x,

$$\lim_{n\to+\infty} \left|\frac{b_{n+1}}{b_n}\right| = 0 \cdot |x| = 0,$$

so the series must converge by the ratio test, for all real x. In this case, we say that the interval of convergence is $(-\infty, \infty)$ (or the whole real line), and say that the radius of convergence is $+\infty$.

If $L \neq 0$, then we have instead

$$\lim_{n\to+\infty} \left|\frac{b_{n+1}}{b_n}\right| = L \cdot |x|,$$

and the ratio test tells us that the series will converge when $L|x| < 1$ and diverge when $L|x| > 1$. If we let $R = \frac{1}{L}$, then the series converges when $|x| < R$ and diverges when $|x| > R$; we call R the *radius of convergence*. The ratio test will however fail when $|x| = R$, i.e., when $x = \pm R$, and for these points we must use other tests to investigate convergence of the series (e.g., integral test or p-series test or divergence test). In any event, the set of x on which the series must converge will be one of the four intervals

$$(-R, R), \qquad [-R, R), \qquad (-R, R], \qquad [-R, R],$$

and we call this set the *interval of convergence.*

If the limit of $\left|\frac{a_{n+1}}{a_n}\right|$ does not exist, then in general we would need other techniques to investigate the convergence of the series. There is, however, one special case where we can still say something in general:

[2]The use of the term "radius" may seem a bit out of place since we are talking about one-dimensional sets, and circles – which are the geometric objects with which we most readily associate the word "radius" are two-dimensional sets. The idea is that in either case the "radius" is essentially half of the width of the object. More substantively, it turns out that power series can be considered where x is a complex number, and these converge on actual disks in the complex plane, whose radius (for series like the one here where all coefficients are real) is exactly the radius of convergence of the series determined using real numbers (as we are doing here). Thus the word "radius" is literally applicable. But complex numbers are beyond the scope of this course so we shall not pursue this matter further.

suppose that the sequence diverges to infinity, meaning that it is eventually larger than any real number; in this case the power series converges only when $x = 0$ and we say that the radius of convergence is $R = 0$ and the interval of convergence is $\{0\}$.

Note that in all cases the radius of convergence is determined entirely by the coefficients a_n.

Even more generally, we can consider the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n;$$

we call this a *power series about* $x = c$. We may compute the radius of convergence using the coefficients a_n exactly as in the previous paragraphs; the interval of convergence will now in general be of the form $(-R+c, R+c)$, et cetera.

Examples are given in the TopHat problems.