Summary:

- We continue with our discussion of series, and talk about tests for series convergence.

PROPERTIES OF SERIES. DIVERGENCE AND INTEGRAL TESTS.

In the notes for last class we gave an introduction to infinite series. In these notes we will continue this study by giving some properties satisfied by infinite series as well as some tests for their convergence.

Infinite series satisfy the basic algebraic properties of finite series: if the series $\sum a_n$ and $\sum b_n$ converge[1], then

$$\begin{aligned}\sum a_n + b_n &= \sum a_n + \sum b_n \\ \sum k a_n &= k \sum a_n\end{aligned}.$$

Note that, just as with ordinary finite series, the sum of a product is not the product of the sums. (This was in the TopHat question just to innoculate us against any misconceptions!)

Additionally, if $\sum a_n$ converges, then we must have $\lim_{n\to+\infty} a_n = 0$. (Recall that this is not necessarily true for functions with convergent improper integrals: the best we can say there is that if $\int_a^\infty f(x)\,dx$ exists, and $\lim_{x\to+\infty} f(x)$ exists, then $\lim_{x\to+\infty} f(x) = 0$. In the case of series, we have that the limit $\lim_{n\to+\infty} a_n$ must exist and equal zero whenever the infinite sum $\sum a_n$ converges.) This gives rise to the so-called *divergence test*:

$$\text{If } \lim_{n\to+\infty} a_n \neq 0, \text{ then } \sum a_n \text{ does not converge.}$$

When this test is applicable, it is usually the easiest one to use, and therefore one we should always think of when asked about the convergence of a series we don't recognise.

We also have the so-called *integral test*, which can be described in words as follows (please see the textbook for illustrations). Suppose that we have a series $\sum a_n$, and that we also have a function $f(x)$ which satisfies $a_n = f(n)$ (at least for n sufficiently large; recall that the starting values of a sequence do not affect convergence of the series at all). Then the integral test tells us the following[2]:

> If the *function* $f(x)$ is continuous, decreasing everywhere, and positive everywhere, and the integral $\int_a^\infty f(x)\,dx$ converges (this is independent of the value a, as long as f is continuous everywhere), then the series $\sum a_n$ also converges.

Note that all of the conditions are on the *function* f, not the *sequence* a_n. In particular, it is not enough for the sequence to simply be decreasing and positive. (An example of this is given in Friday's TopHat.) Also note that the test does *not* say anything about the value of the infinite sum $\sum a_n$, only that it converges. (Indeed, given that we are allowed to choose a essentially arbitrarily in evaluating the integral $\int_a^\infty f(x)\,dx$, there is no way that the value of the integral could always be connected to the value of the series. But even if we choose the value of a to be the same as the starting point for the series, there is no reason for the integral and the series to be equal.) If we are careful with starting points, we can sometimes use the value of the integral to put an upper bound on the value of the series; see examples 3 and 4 of section 9.3 of the textbook for an illustration of this point.

The integral test together with what we know about convergence of integrals gives rise to the convergence rule for so-called p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1 \text{ and diverges if } p \leq 1.$$

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[1]As the textbook discusses, convergence of a series only depends on the ultimate behaviour of the series; in other words, whether or not the series converges does not depend on the initial value for the index. Because of this, we simplify our notation by leaving off explicit bounds on the index, since the lower bound doesn't matter and the upper bound will always be ∞.

[2]Conditions required to hold 'everywhere' really only need to hold for x sufficiently large, since what happens for small x does not affect the convergence of the series or the integral – assuming, of course, that f is continuous everywhere so that the integral isn't improper at some finite x!

RATIO TEST.

Recall the comparison test for improper integrals:

- If $0 \leq f(x) < g(x)$ on $[a, +\infty)$, and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
- If $0 \leq g(x) < f(x)$ on $[a, +\infty)$, and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

In other words: smaller than convergent is convergent, greater than divergent is divergent (assuming that all functions involved are positive!). For series, we have an exactly analogous result:

- If $0 \leq a_n < b_n$, and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n < a_n$, and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

In other words, assuming again that all sequences involved are positive, we have the same rule: smaller than convergent is convergent, greater than divergent is divergent. This is called the *comparison test*.

For us, the main use of the above test will be its application to the so-called *ratio test*, which is probably one of the easiest tests to use to prove convergence:

Suppose that the sequence a_n is nonzero and that the limit $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then:

- If $L < 1$, the series $\sum a_n$ converges.
- If $L > 1$, the series $\sum a_n$ diverges.
- If $L = 1$, the series $\sum a_n$ could either converge or diverge.

In the third case above, we say that the test *is inconclusive* (or *fails*, though inconclusive is probably a slightly better term). Examples of the use of this test are given in Friday's TopHat.

Note that the quantity L only depends on the absolute values $|a_n|$ of the sequence, not on their sign. In other words, if $L < 1$ in the ratio test, we actually get convergence of $\sum |a_n|$, which turns out to be somewhat stronger than convergence of $\sum a_n$ alone. If a series $\sum a_n$ is such that $\sum |a_n|$ converges, then we say that the original series is *absolutely convergent*; if $\sum |a_n|$ is divergent but $\sum a_n$ is convergent, then we say that the series is *conditionally convergent*. Perhaps the simplest example of a conditionally convergent series is $\sum_{n=1}^\infty \frac{(-1)^n}{n}$. It is easy to show that this series converges using the alternating series test (discussed in section 9.4 of the textbook), but we know from our work with p-series above that $\sum_{n=1}^\infty \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^\infty \frac{1}{n}$ is actually divergent.

As a summary, we now have three ways of checking for divergence of a series $\sum a_n$ (though only the last two are able to prove convergence):

1. The DIVERGENCE TEST: If $\lim\limits_{n\to+\infty} a_n$ does not exist or exists but does not equal 0, then $\sum a_n$ diverges.
2. The RATIO TEST: If $\lim\limits_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$, while if $L = 1$ it may either converge or diverge.
3. The INTEGRAL TEST: If $a_n = f(n)$, where $f(x)$ is a function which is positive and decreasing, then $\sum a_n$ converges if $\int_b^\infty f(x)\,dx$ converges, and $\sum a_n$ diverges if $\int_b^\infty f(x)\,dx$ diverges.

Note the hierarchy of 'critical values': for the divergence test, the critical value is 0 (if the limit of a_n is not equal to zero, the series diverges); for the ratio test, it is 1 (if the limit of ratios is greater than 1, the series diverges); for the integral test there is no 'critical value' – or, somewhat tongue-in-cheek, the 'critical value' is ∞: as long as the integral is finite, the series will converge. It is important to not confuse these!