Summary:
- We discuss convergence of sequences, and give solutions to the related TopHat questions.
- We then work a (slightly contrived) example problem introducing infinite series.
- Finally, we give a more detailed treatment of infinite series, and go over infinite geometric series in detail.

CONVERGENCE OF SEQUENCES

On Friday, we talked about sequences, and saw an example of a sequence which converged to $\sqrt{2}$. Why are we interested so much in what a sequence converges to? Is that the only thing about the sequence we really care about? It turns out that in a lot of cases, the answer is actually yes: We use sequences mostly to find approximations to other quantities. In the example on Friday, we found fractions approximating $\sqrt{2}$; we have already talked about using polynomials to find approximations to functions, and we shall see shortly that for many of the functions we are familiar with (like exponential and trigonometric functions) these polynomials can be made into sequences (or, more precisely, *series*) whose limit is exactly equal to the function we started with.

In short, when we talk about sequences by themselves, what they converge to is about the only thing we ultimately care about. (When we talk about series, which are sums of sequences, we are concerned about the *sequence of partial sums*, to be defined later, and its convergence.)

Now, colloquially, the word ‘converge’ means something like ‘come together with’, ‘come close to’. When we say that a sequence ‘converges’, we mean that its values are ‘coming close together with’ its limit value. More precisely, we mean that as we go further and further out along the sequence, the values become arbitrarily close to the limit value.[1] We have seen something similar to this when we talked about improper integrals: there we were concerned with limits of *functions* as the independent variable became arbitrarily large. Now going ‘further and further out along the sequence’ means that we are letting the sequence index n, which corresponds to the independent variable of a function, become arbitrarily large. Thus this current situation is a close analogue to what we have seen before.

Let us do some examples.

EXAMPLES. Determine the limits of the following sequences, if they exist.

(a) $x_n = \frac{1}{n}$.

We know that as n gets large, $\frac{1}{n}$ gets very small, i.e., very close to zero. This means that the sequence converges to 0. (Hopefully this is not too big of a surprise; we know after all that the limit of $\frac{1}{x}$ as $x \to +\infty$ is also 0.)

(b) $x_n = \frac{1+n}{1-n}$.

We can treat this problem the way we would if we were working with the limit of a function, by cancelling the largest term from the numerator and denominator:

$$\lim_{n\to+\infty} \frac{1+n}{1-n} = \lim_{n\to+\infty} \frac{\frac{1}{n}+1}{\frac{1}{n}-1} = \frac{\lim\limits_{n\to+\infty} \frac{1}{n}+1}{\lim\limits_{n\to+\infty} \frac{1}{n}-1} = \frac{1}{-1} = -1.$$

(c) $x_n = \frac{1+e^n}{1-2e^n}$.

This can be handled exactly like (b):

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{1+e^n}{1-2e^n} = \lim_{n\to+\infty} \frac{e^{-n}+1}{e^{-n}-2} = \frac{\lim\limits_{n\to+\infty} e^{-n}+1}{\lim\limits_{n\to+\infty} e^{-n}-2} = \frac{1}{-2} = -\frac{1}{2}.$$

[1] Note carefully that this does *not* require that the values become ‘closer and closer’ to the limit value, which would imply that each value is closer to the limit than the previous one. This is true for some sequences, but fails in general. We only need to know that the sequence values can be made as close as we like to the limit value; more precisely, that beyond a certain point all sequence values will be as close as we like to the limit value.

(d) $x_n = \frac{1+(-1)^{n+1}}{2}$.

Here we have an example of a sequence which does not immediately resemble a function. Since $(-1)^{n+1}$ continually bounces between 1 and -1, it is not immediately clear what to do with this sequence. We write out a few terms to try to see what is happening:

$$\begin{aligned} x_1 &= \frac{1+(-1)^2}{2} = \frac{2}{2} = 1, \\ x_2 &= \frac{1+(-1)^3}{2} = 0, \\ x_3 &= \frac{1+(-1)^4}{2} = \frac{2}{2} = 1, \\ x_4 &= \frac{1+(-1)^5}{2} = 0, \end{aligned}$$

and we see that x_n is 1 if n is even and 0 if n is odd. While this means that all even sequence values will be close (in fact, equal!) to 1 and all odd sequence values will be close (again, equal!) to 0 if n is large, since convergence means that *all* sequence values after a certain point must be getting arbitrarily close to the (single) limit value, and 0 is not arbitrarily close to 1, this sequence does not converge. We say it *diverges.*

(e) $x_n = \frac{1+e^n}{1+e^{-n}}$.

In this case, there is nothing common between the numerator and denominator to be cancelled (we could cancel the e^n anyway, but that would not be the best thing to do in this case). We instead investigate the behaviour of the numerator and denominator separately. If n gets very big, e^{-n} will be very small, which means that the denominator will be very close to 1. If n is very big, e^n will also be very big, which means that the numerator will also be very big. Pulling this all together, we see that x_n itself will also be very big. In some sense, x_n goes to infinity; but for our purposes we just say it *diverges.*

INFINITE GEOMETRIC SERIES.

Let us review our derivation of the expression for the sum of a (finite) geometric series from Friday's class. We have (using a for the common ratio in line with the TopHat problems; apologies for conflicting with the textbook's choice of r)

$$\left(\sum_{k=0}^{n} a^k\right) \cdot (1-a) = 1 + a + a^2 + a^3 + \cdots + a^n \\ -\left(a + a^2 + a^3 + \cdots + a^n + a^{n+1}\right) = 1 - a^{n+1},$$

so if we divide both sides by $1-a$ we obtain

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

(One could also prove this by mathematical induction; we leave this as an exercise.) This differs slightly from the expression in the textbook since we sum all the way to $k = n$ while the textbook sums only up to $k = n-1$. Now let us work the TopHat problem from Monday.

EXAMPLE. At the start of term, your instructor made a point of coming to class early. Suppose that he initially came to the classroom 20 minutes before the start of class. As the term progessed, however, you start to notice that he arrives later and later each day. Suppose that on the second day of class he arrives 18 minutes before the start of class, i.e., 2 minutes later than he did the first day, and that each day he arrives later by an amount equal to $\frac{7}{8}$ of the amount from the previous day; in other words, on the third day he was later by an amount $2 + 2 \cdot \frac{7}{8}$, on the third day by an amount $2 + 2 \cdot \frac{7}{8} + 2 \cdot \left(\frac{7}{8}\right)^2$, and so forth.

(a) How long before class starts will your professor arrive on the 10th day of class?

For definiteness, let us suppose that we are talking about the author of these notes, meaning that class starts at 9.10 and he arrived initially 20 minutes early, so at 8.50. Then on the second day of class he would have arrived at 8.52, on the third day at 8.53:45, and so on. The expressions above certainly look like the first few terms in a geometric series, so let us see if we can figure out how to express them carefully in terms of one. Let us let t_n denote the number of minutes after 8.50 when your instructor arrived on day n; thus $t_1 = 0$, $t_2 = 2$, $t_3 = 2 + 2 \cdot \frac{7}{8}$, and we see that in general, for $n \geq 2$,

$$t_n = \sum_{k=0}^{n-2} 2 \cdot \left(\frac{7}{8}\right)^k.$$

Using our general formula, then, we have (noting that the sum here extends only to $n - 2$, not all the way to n)

$$t_n = 2 \cdot \frac{1 - \left(\frac{7}{8}\right)^{n-1}}{1 - \frac{7}{8}} = 16 \cdot \left(1 - \left(\frac{7}{8}\right)^{n-1}\right).$$

(While we derived this formula only for $n \geq 2$, it is in fact true for $n = 1$ also, as the quantity in parentheses above will be 0 in that case.) Thus on the 10th day of class your instructor will arrive $16 \cdot \left(1 - \left(\frac{7}{8}\right)^9\right) \approx 11.189$ minutes after 8.50, i.e., a bit after 9.00.

(b) How long after class starts will your professor arrive on the 20th day of class? The 30th?

We can use the above formula:

$$t_{20} = 16 \cdot \left(1 - \left(\frac{7}{8}\right)^{19}\right) \approx 14.734,$$

$$t_{30} = 16 \cdot \left(1 - \left(\frac{7}{8}\right)^{29}\right) \approx 15.667,$$

which means that on day 20 your instructor will arrive at about 9:05, while on day 30 he will arrive at almost 9:06.

(c) So far, your instructor hasn't technically been late yet. What would happen if the term were infinitely long?

In this case, we are interested in what would happen to the limit of the above sequence t_n as $n \to +\infty$. (Note that in this case we are not really looking at an approximation, but rather a long-time behaviour.) Now we note first of all that, since $\frac{7}{8} < 1$, when n is very large $\left(\frac{7}{8}\right)^n$ will be very small: if we think of the corresponding function $\left(\frac{7}{8}\right)^x$, we have an exponential function with base less than 1, and we know that all functions of that sort go to zero as $x \to +\infty$; thus in the limit as $n \to +\infty$ it will actually go to zero. This means that the limit of t_n can be found as follows:

$$\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} 16 \cdot \left(1 - \left(\frac{7}{8}\right)^n\right) = 16\left(1 - \lim_{n\to+\infty} \left(\frac{7}{8}\right)^n\right) = 16(1 - 0) = 16,$$

so that in the limit of an infinitely long term your instructor would arrive at precisely 9.06.

INFINITE SERIES. The above problem gives us an example of an *infinite series.* An infinite series is both something constructed from a sequence, and a sequence itself, and the main reason for our introducing sequences in the first place. (In other words, we shall not have that much more to say about sequences which is not somehow related or applied to series.) In order to understand this hierarchy, let us consider the above problem again, except for simplicity let us change n slightly and define simply

$$t_n = \sum_{k=0}^{n} 2 \cdot \left(\frac{7}{8}\right)^k.$$

In this case, the only immediately obvious sequence is the sequence t_n itself. However, if we think for a moment we see that we actually have another sequence, which we might call d_n (d for 'difference'):

$$d_n = 2 \cdot \left(\frac{7}{8}\right)^n.$$

In other words, t_n is simply the sum of the first $n+1$ terms in the sequence d_n:

$$t_n = \sum_{k=0}^{n} d_k.$$

Now we saw in the last example that $\lim_{n\to+\infty} t_n = 16$; how about d_n? Actually we already computed its limit too: $\lim_{n\to+\infty} d_n = 0$. So what exactly is going on here?

There is a nice analogy between these matters and improper integrals. Recall that we made the following definition:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx;$$

in other words, we first integrated up to some finite point b and then took the limit as that point went to infinity. Recall also that we said that, if $\lim_{x\to+\infty} f(x)$ exists, then for $\int_a^{+\infty} f(x)\,dx$ to exist, we must have $\lim_{x\to+\infty} f(x) = 0$. In the above case, we had the limits

$$\lim_{n\to+\infty} \sum_{k=0}^{n} d_k = 16,$$
$$\lim_{n\to+\infty} d_n = 0;$$

in other words, we are again summing to a certain finite point (in this case, n rather than b) and then taking the limit as that finite point tends to infinity; and we also have that the limit of the thing being summed (compare to $f(x)$ in the integral case!) is equal to 0.

Taking the limit of the sequence t_n above gives us what is called an *infinite series*: by taking n larger and larger, we are summing more and more terms, so by taking the limit it makes sense to think that we are summing somehow *infinitely many* terms. We denote this in a way very similar to our notation for improper integrals:

$$\sum_{k=0}^{\infty} d_k = 16.$$

We are now ready to consider infinite series in general (we shall go over this in class tomorrow also). Suppose that we have a sequence a_n (this will be the analogue of the sequence d_n above). We then form another sequence S_n, which we call the *sequence of partial sums*, as follows (this is analogous to the sequence t_n above):

$$S_n = \sum_{k=0}^{n} a_k.$$

Now S_n is itself a sequence, and we can ask about its convergence. Suppose that we are able to show that $\lim_{n\to+\infty} S_n = S$; then we write

$$\sum_{k=0}^{\infty} a_k = S,$$

since in this case (exactly as in the example above) by taking n larger and large, S_n will represent a sum of more and more terms from the sequence a_n, and thus in the limit we are somehow summing 'infinitely many' terms. In this case, we say that the series $\sum_{k=0}^{\infty} a_k$ *converges.*[2]

Now from our analogy, it appears that finding the sequence of partial sums given the sequence a_n is somewhat analogous to finding an integral (in fact, an indefinite integral); since integration is hard, we

[2]Strictly speaking, this is bad terminology, since if the series $\sum_{k=0}^{\infty} a_k$ does not converge it is not clear that it is even defined, so why are we even writing it out? While this point is completely valid, the practice of writing $\sum_{k=0}^{\infty} a_k$, even when it might not converge, is extremely common, so we shall continue to do it here. In practice, it rarely leads to confusion if we are careful.

expect that finding an expression for the general term S_n will also be hard. Unfortunately, it is worse than that: as far as I know, there are almost no cases – even for very simple sequences a_n – for which one can find an explicit expression for S_n. This means that it is not possible, in general, to determine whether a particular series converges by writing out an explicit expression for S_n and then trying to find the limit. It also means that in general it is extremely difficult to find an exact expression for the sum of an infinite series, even if we know it exists.[3] This means that our main concern with series will be to find indirect methods of determining when a series converges, and we shall generally stop once we know the series converges without asking what its sum is (except in special cases which we shall come to later). It turns out – and we shall see this shortly – that there are tests involving just the sequence a_n – not S_n, which we generally can't write out – which will tell us whether the series converges.

With that introduction, let us briefly work the last set of TopHat problems from Monday.

EXAMPLE. Let us consider a general infinite geometric series. We have the expression

$$S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

We consider two cases: $|a| < 1$ and $|a| \geq 1$. In the first case, we have clearly $\lim\limits_{n\to+\infty} a^{n+1} = 0$ (if $a = 0$ this is trivial; if $0 < a < 1$ this follows by thinking of exponential functions, as we did in the last example; if $-1 < a < 0$ this follows by writing $a^{n+1} = (-1)^{n+1}|a|^{n+1}$, and noting that $|a|^{n+1}$ goes to zero as n goes to infinity). Thus by the foregoing the series converges and we may write

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

Now suppose that $a = 1$. Then the denominator of the first expression above would be 0, meaning that the expression doesn't make any sense; if we look directly at the series, we see that in this case $S_n = \sum_{k=0}^{n} a^k = \sum_{k=0}^{n} 1 = n+1$, since we are adding $n+1$ terms, each of them equal to 1. Clearly, when n gets large, so does this expression, meaning that the series must diverge in this case. Now if $a = -1$, we may write

$$S_n = \frac{1-(-1)^{n+1}}{1-(-1)} = \frac{1+(-1)^n}{2},$$

which is just the sequence we met in part (d) of the first example above (except that n is shifted by 1); since that sequence did not converge, neither does this one.

If $|a| > 1$, then we may write

$$S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

However, in this case $\lim\limits_{n\to+\infty} a^{n+1}$ cannot exist (if $a > 0$ this number will go to infinity, while if $a < 0$ it oscillates between positive and negative numbers which are farther and farther away from 0), so the series must be divergent.

[3] As a very simple example, there is no known expression for the sum $\sum_{k=1}^{\infty} \frac{1}{k^3}$.