Summary:
- We consider a problem combining integration by parts with numerical approximation of an integral.

Suppose we are given the following function values:

$$h(0) = 0, \qquad h(1) = 1, \qquad h(2) = 2, \qquad h(3) = 3\frac{1}{2}, \qquad h(4) = 5,$$

and consider the integral

$$\int_0^4 x^4 h'(x)\,dx.$$

One way we might think of approaching this integral would be to use the values of h given above to approximate h', and then use this information to do a Riemann-sum approximation to the integral directly. Here we shall give a different method which does not require knowing anything directly about h'.[1] First we may perform the following integration by parts:

$$\begin{aligned} u &= x^4, & dv &= h'(x)\,dx \\ du &= 4x^3\,dx, & v &= h(x), \end{aligned}$$

from which we obtain

$$\int_0^4 x^4 h'(x)\,dx = x^4 h(x)\big|_0^4 - \int_0^4 h(x)\left(4x^3\,dx\right).$$

Now since $h(0) = 0$ and $h(4) = 5$, we have

$$x^4 h(x)\big|_0^4 = 4^4 h(4) - 0^4 h(0) = 256 \cdot 5 = 1280.$$

The remaining integral must be approximated numerically. If we use left sums, we have for our evaluation points

$$x_0 = 0, \qquad x_1 = 1, \qquad ,x_2 = 2, \qquad x_3 = 3,$$

whence we obtain

$$\begin{aligned} \int_0^4 4x^3 h(x)\,dx &\approx \sum_{k=0}^{3} 4x_k^3 h(x_k)\Delta x \\ &= 4\left(0^3 h(0)\cdot 1 + 1^3 h(1)\cdot 1 + 2^3 h(2)\cdot 1 + 3^3 h(3)\cdot 1\right) \\ &= 4\,(0 + 1 + 8\cdot 2 + 27\cdot 3.5) = 4\,(9 + 94.5) = 414. \end{aligned}$$

Similarly, if we use right sums, we have for our evaluation points

$$x_0 = 1, \qquad x_1 = 2, \qquad x_2 = 3, \qquad x_3 = 4,$$

whence we have

$$\begin{aligned} \int_0^4 4x^3 h(x)\,dx &\approx \sum_{k=0}^{3} 4x_k^3 h(x_k)\Delta x \\ &= 4\left(1^3 h(1)\cdot 1 + 2^3 h(2)\cdot 1 + 3^3 h(3)\cdot 1 + 4^3 h(4)\cdot 1\right) \\ &= 4\,(1 + 8 + 94.5 + 64\cdot 5) = 4\,(103.5 + 320) = 1694. \end{aligned}$$

[1] The question of which of these two methods is actually more accurate in practice is quite involved, and unfortunately lies outside the expertise of the present author.

If we take the average of these two, we obtain

$$\int_0^4 4x^3 h(x)\,dx \approx \frac{1}{2}(414 + 1694) = 1054.$$

Putting these three estimates back in to the integration-by-parts formula above, we obtain finally the three estimates (using the right endpoint, average, and left endpoint approximations, respectively)

$$\int_0^4 x^4 h'(x)\,dx \approx 1280 - 1694 = -414,$$
$$\int_0^4 x^4 h'(x)\,dx \approx 1280 - 1054 = 226,$$
$$\int_0^4 x^4 h'(x)\,dx \approx 1280 - 414 = 866.$$

We see that we do not have enough information to get a very precise value for the integral. What could be done about this? Were the function h an experimental quantity, we would have to go back and perform the experiment for more values of x; were the function h a numerically calculated quantity, we would have to calculate it for more values of x. If we have no way of determining h for additional values of x (for example, if we are dealing with historical data with no way of going back in time to get more of it), we might be able to tighten the bounds by taking other aspects of the problem into account, or by using a more accurate numerical procedure; but in general we would just have to make do with the results above.