The first and second fundamental theorems of calculus

We began our study of integration by discussing how we can determine distance travelled by taking the area under the curve giving the velocity as a function of time. Subsequently, we learned how to evaluate definite integrals by symbolically finding an antiderivative function and applying the *first fundamental theorem of calculus:*

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{1}$$

It is worthwhile to step back a bit and consider exactly what kind of process this formula represents. (What follows may seem a bit pedantic, but it is important that we clearly understand all of the steps in the process.) Suppose that we are told that the velocity of an object as a function of time is given by

$$v(t) = 6 + 10t$$

and asked to determine how far the object travels between $t = 1$ and $t = 5$. Since $v(t) > 0$ on $[1, 10]$, this distance is the same as the displacement over the time period, and is given by the integral

$$\int_1^5 v(t)\,dt.$$

Now by running the power rule $\frac{d}{dx}x^n = nx^{n-1}$ backwards, we conclude that an antiderivative of 6 is $6t$ and an antiderivative of $10t$ is $10 \cdot \left(\frac{1}{2}t^2\right) = 5t^2$; thus the function $f(t) = 6t + 5t^2$ is an antiderivative of $v(t)$, and by the first fundamental theorem of calculus, the integral above can be evaluated as follows:

$$\begin{aligned}\int_1^5 v(t)\,dt &= \int_1^5 (6 + 10t)\,dt = f(5) - f(1)\\ &= (6 \cdot 5 + 5 \cdot 25) - (6 \cdot 1 + 5 \cdot 1) = 155 - 11 = 144.\end{aligned}$$

Let us now consider the question of determining the distance travelled as a function of time. In other words, instead of considering the distance travelled between two fixed points of time, we will pick some initial time, say $t = 0$, and define a new function $s(t)$ to be the distance travelled by the object between the time 0 and the time t. Now a function is defined by three things: a domain, a range, and a rule. Generally the domain and range are clear from the rule, so let us focus on that. A rule will tell us, given a time t, how far the object has travelled between time 0 and time t. Just as above, this distance is equal to the area under the velocity-versus-time curve, which can be expressed as an integral; in other words, we can write something like

$$s(t) = \int_0^t v(u)\,du. \tag{2}$$

Immediately there is a problem: Previously we used t as our variable of integration but now we use u; why is that? A first answer, which is correct but somehow unsatisfying, is it really doesn't matter much what variable we use inside the integral since a definite integral is just a number and hence can't depend on it anyway: in other words, we could just as well have used u in the first integral above, so why not use it now? This actually points the way towards a more complete answer: t is our independent variable, which means that it can take on any value in the domain of s; but when we go to calculate s, we have to pick a specific value of t, which will then remain fixed. In the formula for s, though, the variable of integration will take on all values between 0 and t: in other words, even after we give t a specific value, the variable of integration is still a variable. This means that it cannot be t itself, so we have to use another letter. Beyond this, though, our first comment was right: it doesn't matter whether we call it u, or v, or x, or α, or $\aleph$, or something else.

Let us now go back to investigate the formula (2) more closely. This formula gives us the distance travelled by the object since time 0. Let us consider how we might apply it in practice. Since we know that $f(t) = 6t + 5t^2$ is an antiderivative of $v(t)$, we can apply the first fundamental theorem of calculus again to obtain

$$s(t) = \int_0^t v(u)\,du = f(t) - f(0) = 6t + 5t^2 - (6 \cdot 0 + 5 \cdot 0) = 6t + 5t^2.$$

Curiously, we find that $s(t) = f(t)$! Why should that be? After all, we defined $s(t)$ in equation (2) above to be a definite integral, which is simply an area and shouldn't 'know' (or care) anything about derivatives! If we stop and think for a moment, though, it starts to seem more reasonable: after all, velocity is the derivative of position, so if our formula for $s(t)$ is correct – which it is – we must have $s'(t) = v(t)$; in other words, $s(t)$ must be an antiderivative of $v(t)$. But we got $f(t)$ by looking for an antiderivative of $v(t)$; in other words, $f(t)$ also satisfies $f'(t) = v(t)$. Thus $s(t)$ and $f(t)$ are both antiderivatives of the same function, and hence must differ by at most a constant. Since $s(0) = f(0)$ (this was the only part of the above calculation which was a fluke), we must have $s(t) = f(t)$ everywhere.

It is worthwhile emphasising again just how unexpected this result should be. A definite integral is obtained by finding an area under a curve, which is a purely geometric problem with no intrinsic connection to limits or derivatives. On the other hand, finding an antiderivative is a purely analytic problem with no direct connection to geometry. There is no obvious connection between the two at all. It took the genius of Newton and Leibnitz to realise that these two processes are in fact closely related.

Let us go back and review the first fundamental theorem of calculus in the light of the foregoing. We see that the first fundamental theorem of calculus tells us that, if we can find an antiderivative of a function f, then we can use that antiderivative to find the area under the graph of f; in other words, finding an antiderivative (say by running our differentiation rules backwards, as we did above) allows us to find areas under curves.

What about formula (2)? Here we defined a function $s(t)$ by an area: in the definition of $s(t)$ there was no mention of running differentiation rules backwards, or of any other kind of antiderivative; in fact, there was no mention of derivatives or differentiation at all. Yet as we saw above, the function we obtained was in fact an antiderivative. This suggests that what we have here is a special case of a result which is complementary to that in the previous paragraph: finding the area under a curve allows us to find an antiderivative. This result is known as the *second fundamental theorem of calculus*.

More precisely, suppose that f is a continuous function on an interval $[a, b]$. Then the second fundamental theorem of calculus states that

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

(Here we use t as an integration variable, instead of x, for exactly the same reasons that forced us to use u as a variable of integration in (2) instead of t.) In other words, the function whose rule gives the area under the graph of f is in fact an antiderivative of f, just as we found in our specific example above.

The observant reader may have noticed by this point that we have actually been implicitly using this result all along: after all, the process of finding distance travelled given velocity is *prima faciae* a process of antidifferentiation, yet we have done it multiple times by finding areas. Thus this theorem, in a certain sense, doesn't tell us anything we did not more or less know.

In fact, the only new thing that the second fundamental theorem really tells us is that all continuous functions have an antiderivative. Let us see why this is so; in other words, let us assume that all continuous functions have antiderivatives, and use this to derive the second fundamental theorem. Thus let f be a continuous function on an interval $[a, b]$, and let F be an antiderivative of f. Then by the first fundamental theorem of calculus, we have

$$\int_a^x f(t)\, dt = F(x) - F(a).$$

But by definition F was an antiderivative of f, so $F'(x) = f(x)$; but since $F(a)$ is a constant, this gives

$$\frac{d}{dx} \int_a^x f(t)\, dt = \frac{d}{dx}(F(x) - F(a)) = F'(x) - 0 = f(x),$$

which is exactly the formula in the second fundamental theorem.

Note that this does *not* show that the second fundamental theorem can be derived from the first, since without the second fundamental theorem we would have no way of knowing whether the function f *had* any antiderivative; in other words, it would be entirely possible that there was a continuous function which was not the derivative of any other function. The main new result which the second fundamental theorem of calculus gives us is that this cannot in fact happen.

The other main use of the second fundamental theorem of calculus is related, and is as follows. Suppose that we have a function f for which we cannot find a symbolic antiderivative: for example, f may be too complicated, or it may not even be known entirely (perhaps its values are only known as the results of experimental measurements). How can we find an antiderivative in this case? The second fundamental theorem tells us that an antiderivative of f is given by

$$F(x) = \int_a^x f(t)\,dt.$$

Initially this looks like it might not be much help, since how are we to calculate the definite integral? The point is that a definite integral (unlike an indefinite integral, which is the same thing as an antiderivative) can be approximated numerically as long as the integrand is known. The word 'approximated' may give us pause: don't we prefer to know exact values in mathematics? Well, we consider e^x to be a perfectly well-defined and useful function, but can we write an exact number which is equal to $e^1 = e$? We can't, because e is irrational. The point is that for any of the transcendental functions (exponential, logarithmic, and trigonometric functions), if we wish to find a numerical value, we will still have to make do with an approximation. Thus this situation is really no different to that of the functions with which we are familiar. In other words, the second fundamental theorem of calculus gives us a way of computing an antiderivative which can be done numerically if necessary, and which is deductive, meaning that one can write out an algorithm for it which can be programmed into a computer, and hence can always be applied as long as f is known.

On the other hand, as we noted above, the second fundamental theorem really does not give us anything new computationally, since once we know that all continuous functions have antiderivatives the formula in the second fundamental theorem follows from the first. Thus the examples and problems we do in this section can be done using the first fundamental theorem by itself. We give two examples.

EXAMPLE 1. Suppose that a block of ice is melting at a rate $(2-t)^3$ cubic inches per minute, where t is measured in minutes.
(a) Suppose that there are initially 10 cubic inches of ice. How much ice is left after 1 minute?
(b) Suppose instead that we measure the amount of ice after 1 minute, and find that there are 5 cubic inches of ice remaining. How much ice was present initially?

Solution. Let us let $V(t)$ denote the amount of ice remaining after t minutes, in cubic inches. Since $(2-t)^3$ is the rate at which the ice is *melting*, the rate of change of $V(t)$ must be its negative; i.e., $V'(t) = -(2-t)^3$.

(a) By the first fundamental theorem of calculus,

$$V(t) - V(0) = \int_0^t V'(u)\,du,$$

so

$$\begin{aligned} V(t) &= \int_0^t V'(u)\,du + V(0) = \int_0^t -(2-u)^3\,du + 10 \\ &= \frac{1}{4}(2-u)^4\bigg|_0^t + 10 \\ &= \frac{1}{4}(2-t)^4 - \frac{1}{4}(2-0)^4 + 10 = \frac{1}{4}(2-t)^4 - 4 + 10 = \frac{1}{4}(2-t)^4 + 6. \end{aligned}$$

Thus the amount of ice remaining at $t = 1$ minute in this case is $V(1) = \frac{1}{4}(1)^4 + 6 = 6.25$ cubic inches.

(b) Here we may proceed analogously, obtaining the formula

$$\begin{aligned} V(t) &= \int_1^t V'(u)\,du + V(0) = \int_1^t -(2-u)^3\,du + 5 \\ &= \frac{1}{4}(2-u)^4\bigg|_1^t + 5 \\ &= \frac{1}{4}(2-t)^4 - \frac{1}{4} + 5 = \frac{1}{4}(2-t)^4 + 4.75, \end{aligned}$$

so that at time $t = 0$ there were initially $V(0) = \frac{1}{4}2^4 + 4.75 = 8.75$ cubic inches of ice. (It is worth noting that this could have been written down almost directly from our answer to (a): since in the situation in (a) there were 1.25 more cubic inches of ice at time $t = 1$ than there were in (b) at time $t = 1$, the same must have been true at $t = 0$ (since the rate of change of the volume was the same in both cases), meaning that initially in (b) there must have been $10 - 1.25 = 8.75$ cubic inches of ice.)

EXAMPLE 2. Find the following derivatives:

(a)

$$\frac{d}{dx}\int_0^{x^2} \frac{1}{1+t^4}\,dt.$$

(b)

$$\frac{d}{dx}\int_{\tan x}^{x^2} \frac{1}{1+t^4}\,dt.$$

Solution. Let us begin by setting $F(x) = \int_0^x \frac{1}{1+t^4}\,dt$; note that it is not possible (at least given what we know at this point) to find a simple expression for F in terms of known functions, but F is still a well-defined function, and is a particular antiderivative of the function $\frac{1}{1+x^4}$, i.e., $F'(x) = \frac{1}{1+x^4}$.

(a) By the first fundamental theorem of calculus, we have

$$\int_0^{x^2} \frac{1}{1+t^4}\,dt = F(x^2) - F(0),$$

so by the chain rule we have

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} \frac{1}{1+t^4}\,dt &= \frac{d}{dx}\big(F(x^2) - F(0)\big) = \frac{d}{dx}F(x^2) \\ &= 2xF'(x^2) = \frac{2x}{1+(x^2)^4} = \frac{2x}{1+x^8}.\end{aligned}$$

(b) This is analogous to (a); the only difference is that we will have one extra term. By the first fundamental theorem of calculus, we have

$$\int_{\tan x}^{x^2} \frac{1}{1+t^4}\,dt = F(x^2) - F(\tan x).$$

Thus by the chain rule we have

$$\begin{aligned}\frac{d}{dx}\int_{\tan x}^{x^2} \frac{1}{1+t^4}\,dt &= \frac{d}{dx}\big(F(x^2) - F(\tan x)\big) \\ &= \frac{d}{dx}F(x^2) - \frac{d}{dx}F(\tan x) = 2xF'(x^2) - \sec^2 x F'(\tan x) \\ &= \frac{2x}{1+x^8} - \frac{\sec^2 x}{1+\tan^4 x}.\end{aligned}$$