Summary:

- We consider how to approximate the function $\sin x$ near $x = 0$.
- We look at the tangent-line approximation and note where it begins to fail, and why.
- We then see how using additional derivatives can provide a higher-order polynomial which gives a better approximation.
- Finally, we show how to do all of this for a general function, and discuss Taylor polynomials around arbitrary points.

How can we approximate functions?

Consider the function $\sin x$. We know $\sin 0 = 1$, $\sin \frac{\pi}{6} = \frac{1}{2}$, $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin \frac{\pi}{2} = 1$, and so on; but suppose we needed to calculate $\sin 1$: how could we go about doing that? Since we know we could use a computer or a calculator to find an approximate value for $\sin 1$, there has to be a way; but what is it?

One way[1] to approximate $\sin x$ is to use a tangent-line approximation. Let us investigate the accuracy of this approximation. The tangent line to the graph of $\sin x$ at $x = 0$ is the line which has the same value and slope as the graph of $\sin x$ at $x = 0$. Now $\sin 0 = 0$, and the slope of the graph of $\sin x$ at $x = 0$ is

$$\frac{d}{dx} \sin x \bigg|_{x=0} = \cos 0 = 1,$$

so the tangent line is just the line $y = 0 + x = x$. If we plot these two together (it is also interesting to make a table of values but we will not do that here), we ee that the approximation is good for small x, but by $x = 1$ the two curves are quite far apart. What can be done about this?

Let us recall the second derivative test: Suppose that f is a twice-differentiable function and x is some point in its domain. If $f''(x) > 0$, then f is concave up at x, while if $f''(x) < 0$ then f is concave down at x; if $f''(x) = 0$ then f can be concave up (think of $f(x) = x^4$), concave down ($f(x) = -x^4$), or neither ($f(x) = x^3$). Now 'concave up' means that the graph curves upwards from its tangent line, while 'concave down' means that the graph curves downwards from its tangent line; thus this suggests that f'' may tell us something about how the graph of f deviates from its tangent line – i.e., it may give us information which can be used to devise a more precise approximation.

In particular, perhaps we would obtain a better approximation to $\sin x$ if we used, instead of a polynomial of degree one, a higher-order polynomial whose second-, or even third- or fourth-, order derivatives matched those of $\sin x$ at $x = 0$. Let us see how we might go about doing this and investigate the result.

Let[2] $p(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ be a fifth-order polynomial, where a, b, c, d, e and f are constants to be determined. We shall determine these constants by requiring that the values $p(0)$, $p'(0)$, $p''(0)$, etc., are equal to, respectively, $\sin 0$, $\frac{d}{dx} \sin x\big|_{x=0}$, $\frac{d^2}{dx^2} \sin x\Big|_{x=0}$, etc.. We first note the following derivatives:

$$\begin{aligned}
p(x) &= a + bx + cx^2 + dx^3 + ex^4 + fx^5 \\
p'(x) &= b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 \\
p''(x) &= 2c + 6dx + 12ex^2 + 20fx^3 \\
p'''(x) &= 6d + 24ex + 60fx^2 \\
p^{(4)}(x) &= 24e + 120fx \\
p^{(5)}(x) &= 120f.
\end{aligned}$$

We note something interesting here: the nth derivative of $p(x)$, where $n \geq 0$, only involves the coefficients of the terms of degree n and higher in p. In other words, $p'(x)$ $(n = 1)$ only involves the coefficients from

[1]The matter of finding good methods for computing transcendental functions such as $\sin x$ on digital computers is very complicated, and using a power series as we discuss here is probably not the best in general. If anyone is interested in more discussion of this point, see, for example, Cody, W.J. and Waite, W., *Software Manual for the Elementary Functions.* Toronto: Prentice-Hall, 1980.

[2]In class we considered a fourth-order polynomial. This was probably due to an oversight on the instructor's part, since the highest-order term ends up being zero and a fourth-order polynomial therefore gives us no more information than a third-order one. Here we consider a fifth-order polynomial to rectify that.

terms of degree 1 and higher in p: it doesn't involve the term of degree 0 (a). Similarly, $p^{(5)}(x)$ only involves the coefficient in the highest-order term in p, namely f, since p has degree 5.

Now if we evaluate all of these derivatives at $x = 0$, we find

$$\begin{aligned} p(0) &= a \\ p'(0) &= b \\ p''(0) &= 2c \\ p'''(0) &= 6d \\ p^{(4)}(0) &= 24e \\ p^{(5)}(0) &= 120f. \end{aligned}$$

Note that while differentiating n times got rid of coefficients from $p(x)$ of order less than n, evaluating at $x = 0$ got rid of coefficients from $p(x)$ of terms of order *greater* than n: thus differentiating and then evaluating at 0 allows us to isolate single coefficients.

Also, you may recognise the sequence of numbers 1, 1, 2, 6, 24, 120: in fact it is just the sequence of factorials, $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$. This turns out to be true in a much more general setting, as we shall see shortly.

Recall now that our goal was to construct a polynomial whose derivatives at zero agreed with those of $\sin x$. Thus we now need to redo the calculations above with $\sin x$ in place of $p(x)$. We have first[3]

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d^2}{dx^2}\sin x &= -\sin x \\ \frac{d^3}{dx^3}\sin x &= -\cos x \\ \frac{d^4}{dx^4}\sin x &= \sin x \\ \frac{d^5}{dx^5}\sin x &= \cos x. \end{aligned}$$

This gives

$$\begin{aligned} \sin 0 &= 0 \\ \left.\frac{d}{dx}\sin x\right|_{x=0} &= 1 \\ \left.\frac{d^2}{dx^2}\sin x\right|_{x=0} &= 0 \\ \left.\frac{d^3}{dx^3}\sin x\right|_{x=0} &= -1 \\ \left.\frac{d^4}{dx^4}\sin x\right|_{x=0} &= 0 \\ \left.\frac{d^5}{dx^5}\sin x\right|_{x=0} &= 1. \end{aligned}$$

[3]Note that $\frac{d^4}{dx^4}\sin x = \sin x$. When we come back to discuss Taylor series later on in the course (what we have here are simply Taylor *polynomials*), we shall see that this is very useful.

We may now equate these to the derivatives of $p(x)$ obtained above. This gives

$$\begin{aligned} a &= 0 \\ b &= 1 \\ 2c = 0, \qquad c &= 0 \\ 6d = -1, \qquad d &= -\frac{1}{6} \\ 24e = 0, \qquad e &= 0 \\ 120f = 1, \qquad f &= \frac{1}{120}. \end{aligned}$$

Thus our polynomial p is

$$p(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

Note that the first term is just the tangent-line approximation, while the higher-order terms are corrections which came about since some of the higher-order derivatives of $\sin x$ were nonzero at $x = 0$, just as we guessed at the beginning.

Now let us see how well this new polynomial approximates $\sin x$. If we plot $\sin x$, its tangent line at $x = 0$, x, and the polynomial $p(x)$ together, we can see that this new polynomial is a much better approximation to $\sin x$ than the tangent-line approximation as we move away from 0.

The procedure above may be carried out in general. The result is called the *Taylor polynomial of degree n* of the function. We now indicate how this may be done. Let n be some positive integer, and suppose that f is a function with at least n derivatives. Then the method above can be used to find an nth degree polynomial which approximates f. Specifically, consider the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

As above, we require $p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$, and so on, all the way up to $p^{(n)}(0) = f^{(n)}(0)$. Thus we need to determine how to differentiate powers of x arbitrarily many times.

Suppose that k is a positive integer greater than or equal to 3. Then we have

$$\frac{d}{dx}x^k = kx^{k-1}, \qquad \frac{d^2}{dx^2}x^k = k(k-1)x^{k-2}, \qquad \frac{d^3}{dx^3}x^k = k(k-1)(k-2)x^{k-3},$$

and in general the mth derivative of x^k, where $m \le k$, will be

$$k(k-1)(k-2)\cdots(k-m+1)x^{k-m}.$$

Now if $m = k$, x^{k-m} means just the constant function 1; in other words, the kth derivative of x^k is just a constant. This means that the $k+1$th derivative of x^k must be zero, and also that all further derivatives of x^k must also be zero. What is going on here is exactly what we saw above when we noted that, for example, $p'(x)$ did not depend on a, and $p^{(5)}(x)$ only depended on f. In particular, this means that $p^{(k)}(0)$ cannot depend on any of the coefficients $a_0, a_1, \ldots, a_{k-1}$.

At the end of the day, though, we are only really interested in the derivatives of p at $x = 0$. Now as we noted in our example with $\sin x$ above, if we differentiate a power x^k less than k times and then evaluate at 0, the result is always 0. This means that $p^{(k)}(0)$ cannot depend on $a_{k+1}, \ldots, a_n$. Thus, just like in our example above, $p^{(k)}(0)$ only depends on a_k. If we look at the formula for $\frac{d^m}{dx^m}x^k$ above, we see that when $m = k$ it gives

$$\frac{d^k}{dx^k}x^k = k(k-1)(k-2)\cdots 2 \cdot 1 = k!.$$

Thus

$$p^{(k)}(0) = k! \cdot a_k,$$

so our requirement that $p^{(k)}(0) = f^{(k)}(0)$ gives

$$a_k = \frac{f^{(k)}(0)}{k!}$$

and the nth-orer Taylor polynomial of f (at 0) is

$$p(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n.$$

As discussed in section 10.1 of the textbook, we can also construct Taylor polynomial approximations to a function f around points other than 0. The formula for the Taylor polynomial around a point $x = a$ is very similar to the formula above:

$$p(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n.$$

This makes sense if one thinks about the formula for the tangent line to f at $x = a$ for a moment.