APM346 (Summer 2019), Term Test.

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— Dude, who asked you to write your own test?

— me, of course! Someone also told me to mark it afterwards! :)

Student Number: ______________________

Signature: ______________________

This test will run for 120 minutes, beginning at 7.00 PM EDT.

No aids of any form are allowed. Do not open the test until instructed to do so.

There are six questions on this test, for a total of 70 marks. The weighting is indicated on each question. You must show all of your work for credit.

You may use the back sides of the pages, as well as pages 15 and 16, to continue your solutions, as long as this is clearly indicated.

1. [10 marks] Solve on $\{(x,y) | 0 \le x \le 1,\ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \ u|_{y=0} = 0, \ u|_{y=1} = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}.$$

From the first two boundary conditions, we see that the solution may be expressed as a series [this is one of the standard series from the review sheet]

1 $$u = \sum_{n=1}^{\infty} \sin n\pi x \,(a_n \cosh n\pi y + b_n \sinh n\pi y).$$

The third boundary condition then gives

$$0 = u|_{y=0} = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

1 which means that $a_n = 0$ for all n. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y.$$

The final condition then gives

1 $$u|_{y=1} = \sum_{n=1}^{\infty} b_n \sinh n\pi \,\sin n\pi x = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1. \end{cases}$$

This gives (denoting the function on the right by $h(x)$)

$$\sinh n\pi \, b_n = \frac{(h(x), \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)} = 2\left(\int_0^{1/2} x \sin n\pi x \, dx + \int_{1/2}^1 (1-x) \sin n\pi x \, dx\right).$$

1 0.5 1

– since $\sin[n\pi(1-x)] = \sin(n\pi - n\pi x) = (-1)^{n+1} \sin n\pi x$, you might be able to use symmetry to simplify things here. At any rate this could explain why so many terms end up cancelling in your final answer below.

Now, using integration by parts,

1.5 $$\int x \sin n\pi x \, dx = \frac{-1}{n\pi} x \cos n\pi x + \frac{1}{n\pi}\int \cos n\pi x \, dx = -\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x + C$$

So

$$\sinh n\pi \, b_n = 2\left(-\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_0^{1/2} - \left[-\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x\right]\Big|_{1/2}^1 - \frac{1}{n\pi} \cos n\pi x \Big|_{1/2}^1\right)$$

2 $$= 2\left(-\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin\frac{n\pi}{2} - \left(-\frac{1}{n\pi}(-1)^n - \left[-\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin\frac{n\pi}{2}\right]\right) - \frac{1}{n\pi}\left((-1)^n - \cos\frac{n\pi}{2}\right)\right)$$

$$= -\frac{2}{n\pi} \cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} + \frac{2}{n\pi} \cos\frac{n\pi}{2} = \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2}.$$

0.5

Now $\sin\frac{n\pi}{2} = 0$ when n is even, while $\sin\frac{(2k+1)\pi}{2} = (-1)^k$; thus we have for u the expansion

0.5 $$u = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 \pi^2} \sin[(2k+1)\pi x] \sinh[(2k+1)\pi y] / \sinh[(2k+1)\pi].$$

2. [10 marks] Solve on $\{(x,y)|0 \le x \le 1,\ \ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = 0,\ \ u|_{x=1} = -\left.\frac{\partial u}{\partial x}\right|_{x=1},\ \ u|_{y=0} = 0,\ \ u|_{y=1} = x.$$

We start by looking for separated solutions ~~satisfying the first two boundary conditions~~, i.e., for solutions $u(x,y) = X(x)Y(y)$. This gives as usual

This is a good way of looking at things, though we haven't necessarily talked about it in class much — ok

.5 $\dfrac{X''}{X} + \dfrac{Y''}{Y} = 0, \quad X'' = -\lambda^2 X,\ \ Y'' = \lambda^2 Y,$

where we take $X''/X < 0$ (.5) so as to be able to fit the first two boundary conditions. Now applying these to the separated solution gives

$$X'' = -\lambda^2 X,\ \ X(0) = 0,\ \ X(1) = -X'(1),$$

so $X = a\sin\lambda x + b\cos\lambda x$; (.5) $X(0) = 0$ means $b = 0$, (.5) while $X(1) = -X'(1)$ means

$$a\sin\lambda = -a\lambda\cos\lambda, \quad \lambda = -\tan\lambda. \ \ .5$$

If $\lambda = 0$, the sol'n for X would be $a + bx$; $X(0) = 0$ gives $a = 0$, while $X(1) = -X'(1)$ gives $b = -b$, so $b = 0$. Thus we can ignore the $\lambda = 0$ solution and only take $\lambda > 0$. ($\lambda < 0$ would give the same equation and hence would just duplicate the functions we have here.)

This equation has infinitely many positive solutions, which we denote $\lambda_1, \lambda_2, \ldots$, in increasing order. Then the general solution to Laplace's equation satisfying the first two boundary conditions will be

1 $$u = \sum_{n=1}^{\infty} \sin\lambda_n x\,(a_n \sinh\lambda_n y + b_n \cosh\lambda_n y).$$

Note the similarity to the series sol'n in problem 1.

.5 $u|_{y=0} = 0$ gives $b_n = 0$ (as in problem 1), while $u|_{y=1} = x$ gives

.5 $$\sum_{n=1}^{\infty} a_n \sinh\lambda_n \sin\lambda_n x = x,$$

on [0,1]!

whence (we know from the homework that $\{\sin\lambda_n x\}$ is an orthogonal set)

Perhaps I should have put ~~a reminder about this on the test paper~~. On the other hand I'm pretty sure this came up on the practice problems.

1 $$a_n \sinh\lambda_n = \frac{(x, \sin\lambda_n x)}{(\sin\lambda_n x, \sin\lambda_n x)}.$$

(~~from~~ see problem 1) ok

1 Now $\int x\sin\lambda_n x\,dx = -\frac{1}{\lambda_n} x\cos\lambda_n x + \frac{1}{\lambda_n^2}\sin\lambda_n x + C$, so

1 $$(x, \sin\lambda_n x) = \int_0^1 x\sin\lambda_n x\,dx = -\frac{1}{\lambda_n} x\cos\lambda_n x + \frac{1}{\lambda_n^2}\sin\lambda_n x\Big|_0^1 = -\frac{1}{\lambda_n}\cos\lambda_n + \frac{1}{\lambda_n^2}\sin\lambda_n$$

$$= -\frac{1}{\lambda_n}\cos\lambda_n\left(1 - \frac{1}{\lambda_n}\frac{\sin\lambda_n}{\cos\lambda_n}\right) = -\frac{2}{\lambda_n}\cos\lambda_n,$$

good [the previous expression above would be sufficient though]

while

$$(\sin\lambda_n x, \sin\lambda_n x) = \int_0^1 \sin^2\lambda_n x\,dx = \int_0^1 \frac{1}{2}(1 - \cos 2\lambda_n x)\,dx = \frac{1}{2} - \frac{1}{4\lambda_n}\sin 2\lambda_n$$

1 $$= \frac{1}{2} - \frac{1}{2\lambda_n}\sin\lambda_n\cos\lambda_n = \frac{1}{2}(1 + \cos^2\lambda_n) = \cancel{\frac{1}{2}\sin^2\lambda_n},$$

You forgot the − sign in $\frac{1}{\lambda_n}\sin\lambda_n = -\cos\lambda_n$...

So (see p. 6)

$$\cancel{b_n}\, a_n \sinh\lambda_n = \frac{-\frac{2}{\lambda_n}\cos\lambda_n}{\frac{1}{2}\cancel{\sin^2\lambda_n}\,(1+\cos^3\lambda_n)} = -\frac{4\cancel{\cos}\lambda_n}{\lambda_n \sin^2\lambda_n} = \cancel{-\frac{4}{\lambda_n^2 \sin\lambda_n}} \quad -\frac{4\cos\lambda_n}{\lambda_n(1+\cos^3\lambda_n)}$$

and the solution is

1 $$u = \sum_{n=1}^{\infty} -\frac{4\cos\lambda_n}{\lambda_n^2\,\cancel{\sin\lambda_n}\,(1+\cos^3\lambda_n)} \sin\lambda_n x \,\frac{\sinh\lambda_n y}{\sinh\lambda_n}.$$

3. [5 marks] Solve on $\{(r,\theta,\phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \cos\theta, \quad u|_{r=2} = 0.$$

Since we are solving on a spherical shell, and the boundary data are azimuthally symmetric, we have that u can be written as

.5 $$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right).$$

The boundary conditions then give

.5 $$u|_{r=1} = \cos\theta = P_1(\cos\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)(a_\ell + b_\ell)$$

.5 $$u|_{r=2} = 0 = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + b_\ell 2^{-(\ell+1)}\right);$$ .5

by inspection, then, we have $a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0$ for all ℓ, while $a_\ell + b_\ell = 0$ if $\ell \neq 1$ and $a_1 + b_1 = 1$. .5 Thus if $\ell = 1$ we have

1 $$\begin{aligned} a_1 + b_1 &= 1 \\ 2a_1 + \cancel{\tfrac{1}{4}}\tfrac{1}{4} b_1 &= 0 \end{aligned} \qquad a_1 = -\cancel{\tfrac{1}{4}}\tfrac{1}{8} b_1 \qquad \tfrac{7}{8}\cancel{\tfrac{3}{4}} b_1 = 1 \qquad b_1 = \cancel{\tfrac{4}{3}}\tfrac{8}{7}, \quad a_1 = -\cancel{\tfrac{1}{3}} -\tfrac{1}{7}$$

while if $\ell \neq 1$

1 $$\begin{aligned} a_\ell + b_\ell &= 0 \\ 2^\ell a_\ell + 2^{-(\ell+1)} b_\ell &= 0 \end{aligned} \qquad a_\ell = -2^{-(2\ell+1)} b_\ell \qquad b_\ell\left(1 - \frac{1}{2^{2\ell+1}}\right) = 0 \qquad b_\ell = 0, \; a_\ell = 0.$$

Thus finally

.5 $$u = P_1(\cos\theta)\left(-\cancel{\tfrac{1}{3}}\tfrac{1}{7} r + \cancel{\tfrac{4}{3}}\tfrac{8}{7} r^{-2}\right).$$

4. [10 marks] Solve on $\{(r,\theta,\phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \left.\frac{\partial u}{\partial r}\right|_{r=1}, \quad u|_{r=2} = \begin{cases} -1, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

The following may be useful: $(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x)$. [Hint: do not confuse this problem with problem 2!]

As in problem 3, we have the series expansion

.5 $$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right);$$

the boundary conditions give (letting $h(x) = \begin{cases} 1, & -1 \le x < 0 \\ -1, & 0 < x \le 1 \end{cases}$, so that $u|_{r=2} = h(\cos\theta)$)

$$u|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)(a_\ell + b_\ell) = u_r|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\ell a_\ell - (\ell+1)b_\ell\right),$$ .5

so $(1-\ell)a_\ell + (\ell+2)b_\ell = 0$ for all ℓ, .5

.5 $$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + b_\ell 2^{-(\ell+1)}\right) = h(\cos\theta);$$

this gives

$$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = \frac{(h(\cos\theta), P_\ell(\cos\theta))}{(P_\ell(\cos\theta), P_\ell(\cos\theta))} = \frac{2\ell+1}{2}(h(\cos\theta), P_\ell(\cos\theta)).$$ 1 .5

Now (let us denote this quantity by c_ℓ)

$$(h(\cos\theta), P_\ell(\cos\theta)) = \int_0^\pi h(\cos\theta) P_\ell(\cos\theta)\sin\theta\, d\theta = \int_{-1}^1 h(x) P_\ell(x)\, dx$$ .5

$$= \int_{-1}^0 P_\ell(x)\,dx - \int_0^1 P_\ell(x)\,dx = -\int_0^1 (P_\ell(x) - P_\ell(-x))\,dx,$$ .5

so $c_\ell = 0$ if ℓ is even, since then $P_\ell(x) = P_\ell(-x)$, while if $\ell = 2k+1$ we have .5

$$c_{2k+1} = -2\int_0^1 P_{2k+1}(x)\,dx = -\frac{2}{4k+3}\int_0^1 P'_{2k+2}(x) - P'_{2k}(x)\,dx = -\frac{2}{4k+3}\Big[P_{2k+2}(1) - P_{2k}(1) - \left(P_{2k+2}(0) - P_{2k}(0)\right)\Big]$$ .5

$$= + \frac{2}{4k+3}\left(P_{2k+2}(0) - P_{2k}(0)\right),$$ .5

So if ℓ is even $a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0$, while if ℓ is odd

$$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = +\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right).$$

(See p. 10.)

Thus for ℓ even we have

1 $\quad (1-\ell)a_\ell + (\ell+2)b_\ell = 0$ $\qquad a_\ell = -2^{-(2\ell+1)}b_\ell \qquad b_\ell\left[(\ell+2)-(\ell-1)2^{-(2\ell+1)}\right] = 0$

$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0$

$b_\ell = 0,\ a_\ell = 0,$

while if ℓ is odd we have

$(1-\ell)a_\ell + (\ell+2)b_\ell = 0 \qquad b_\ell = \frac{\ell-1}{\ell+2}a_\ell$ .5

$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = +(P_{\ell+1}(0) - P_{\ell-1}(0))$

$a_\ell\left[2^\ell + 2^{-(\ell+1)}\frac{\ell-1}{\ell+2}\right] = -(P_{\ell-1}(0) - P_{\ell+1}(0))$ .5

2 $\quad a_\ell = \dfrac{P_{\ell+1}(0) - P_{\ell-1}(0)}{2^\ell + 2^{-(\ell+1)}\frac{\ell-1}{\ell+2}}$ .5 $, \quad b_\ell = \cancel{P_{\ell+1}(0)-} \; \dfrac{\ell-1}{\ell+2}\cdot\dfrac{P_{\ell+1}(0) - P_{\ell-1}(0)}{2^\ell + 2^{-(\ell+1)}\cdot\frac{\ell-1}{\ell+2}}.$ .5

✓ It looks rather like you made a copy error which cancelled the above sign error...

This gives finally for u

.5 $\quad u = \sum_{k=0}^{\infty} P_{2k+1}(\cos\theta)\left(\dfrac{P_{2k+2}(0) - P_{2k}(0)}{2^{2k+1} + 2^{-(2k+2)}\frac{2k}{2k+3}}\right)\left(r^{2k+1} + \dfrac{2k}{2k+3}r^{-(2k+2)}\right).$ ✓ok

5. [20 marks] Solve on $\{(r,\theta,\phi) | r<1\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \sin\theta\cos\phi, & 0 \le \theta < \frac{\pi}{2} \\ -\sin\theta\cos\phi, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

The following identities may be useful: $(1-x^2)P_\ell' = \ell P_{\ell-1} - \ell x P_\ell$, $xP_\ell = \frac{\ell+1}{2\ell+1}P_{\ell+1} + \frac{\ell}{2\ell+1}P_{\ell-1}$, $\int_0^1 P_\ell(x)\,dx = \frac{1}{2\ell+1}\left(P_{\ell-1}(0) - P_{\ell+1}(0)\right)$. You may leave your final answer in terms of $P_\ell(0)$ if it is nonzero. [Hint: what is the relationship between $P_{\ell m}$ and P_ℓ?] $\int_{-1}^{1} P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$

Since we no longer have azimuthal symmetry, but are solving on the whole sphere, we have the series expansion

$$u = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi\right) r^\ell.$$

The boundary condition gives (letting h be the function from 3)

$$u|_{r=1} = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi\right) = -\sin\theta\, h(\cos\theta)\cos\phi,$$

whence we see that $b_{\ell m} = 0$ for all ℓ, m, while $a_{\ell m} = 0$ unless $m=1$ and

$$\sum_{\ell=1}^{\infty} a_{\ell 1} P_{\ell 1}(\cos\theta) = -\sin\theta\, h(\cos\theta).$$

By the orthogonality of $\{P_{\ell 1}(x)\}$ on $[-1,1]$, this gives

$$a_{\ell 1} = \frac{\left(-\sin\theta\, h(\cos\theta), P_{\ell 1}(\cos\theta)\right)}{\left(P_{\ell 1}(\cos\theta), P_{\ell 1}(\cos\theta)\right)} = \frac{(\ell-1)!}{(\ell+1)!}\frac{2\ell+1}{2}\int_0^\pi -\sin^2\theta\, h(\cos\theta) P_{\ell 1}(\cos\theta)\,d\theta,$$

$x = \cos\theta$

Performing the usual change of variables on this integral gives

$$\int_{-1}^{1} -(1-x^2)^{1/2} h(x) P_{\ell 1}(x)\,dx = -\int_{-1}^{0}(1-x^2)^{1/2}P_{\ell 1}(x)\,dx + \int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx.$$

Now

$$P_{\ell m}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \qquad P_{\ell 1}(x) = (1-x^2)^{1/2}P_\ell'(x),$$

So

$$\int (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int (1-x^2)P_\ell'(x)\,dx = \int \ell P_{\ell-1}(x) - \ell x P_\ell(x)\,dx$$

$$= \int \ell P_{\ell-1}(x) - \left[\frac{\ell^2+\ell}{2\ell+1} P_{\ell+1}(x) + \frac{\ell^2}{2\ell+1}P_{\ell-1}(x)\right] dx$$

(see p. 12)

$$\int (1-x^2)^{1/2} P_\ell'(x)\,dx = \frac{\ell^2+\ell}{2\ell+1}\int P_{\ell-1}(x) - P_{\ell+1}(x)\,dx,$$

so (note that we have $\ell \geq 1$, so $\ell - 1 \geq 0$)

$$\int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int_0^1 (1-x^2) P_\ell'(x)\,dx = \frac{\ell^2+\ell}{2\ell+1}\int_0^1 P_{\ell-1}(x) - P_{\ell+1}(x)\,dx$$

1

$$= \frac{\ell^2+\ell}{2\ell+1}\cdot\left[\frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right)\right]$$

while

1

$$\int_{-1}^0 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int_{-1}^0 (1-x^2) P_\ell'(x)\,dx = \int_0^1 (1-x^2) P_\ell'(-x)\,dx;$$

now P_ℓ' is odd when ℓ is even, and vice versa, so that when ℓ is even we have

$$a_{\ell 1} = \frac{1}{\ell(\ell+1)}\cdot\frac{2\ell+1}{2}\left[2\int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx\right] = \frac{2\ell+1}{\ell(\ell+1)}\cdot\frac{\ell^2+\ell}{2\ell+1}\left[\frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right)\right]$$

2

$$= \frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right),$$

while if ℓ is odd we have $\int_0^\pi -\sin\theta\, h(\cos\theta) P_{\ell 1}(\cos\theta)\,d\theta = 0$, so $a_\ell = 0$. Thus we have for u (remember that $\ell \geq 1$!)

2

$$u = \sum_{k=1}^\infty P_{2k}(\cos\theta)\, r^{2k} \overset{\cos\varphi}{\cdot} \left[\frac{1}{4k-1}\left(P_{2k-2}(0) - P_{2k}(0)\right) - \frac{1}{4k+3}\left(P_{2k}(0) - P_{2k+2}(0)\right)\right]$$

6. [15 marks] Solve on $\{(\rho,\phi,z)|\rho<1,\ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\quad u|_{\rho=1}=0,\quad u|_{z=0}=0,\quad u|_{z=1}=\begin{cases}-\rho^3\cos 3\phi, & 0\le\rho<\frac{1}{2}\\ \rho^3\cos 3\phi, & \frac{1}{2}<\rho<1\end{cases}.$$

The following identity may be useful: $\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$. You may leave your final answer in terms of quantities of the form $J_{m+1}\left(\frac{1}{2}\lambda_{mi}\right)$ and $J_{m+1}\left(\lambda_{mi}\right)$ (as long as you say what the λ_{mi} are!).

We have the series expansion (since $u|_{\rho=1}=0$; here λ_{mi} is the ith positive zero of J_m) .5

1 $$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\left(c_{mi}\cosh\lambda_{mi}z + d_{mi}\sinh\lambda_{mi}z\right)$$

The condition $u|_{z=0}=0$ gives

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)c_{mi} = 0, \quad .5$$

whence we take $c_{mi}=0$ .5 for all m and all i. Redefining a_{mi} and b_{mi} by absorbing d_{mi}, we may write the third boundary condition as

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{mi} = g(\rho)\cos 3\phi, \quad .5$$

where $g(\rho)=\begin{cases}-\rho^3, & 0\le\rho<1/2\\ \rho^3, & 1/2<\rho<1\end{cases}$. As in problem 5, this allows us to conclude that 1 $b_{mi}=0$ for all m and all i, while $a_{mi}=0$ for $m\ne 3$ and all i 1 and finally

$$\sum_{i=1}^{\infty} J_3(\lambda_{3i}\rho)\, a_{3i}\sinh\lambda_{3i} = g(\rho). \quad .5$$

By the orthogonality of $\{J_3(\lambda_{3i}\rho)\}$ on $[0,1]$, we have

$$a_{3i} = \frac{(g(\rho), J_3(\lambda_{3i}\rho))}{\sinh\lambda_{3i}\,(J_3(\lambda_{3i}\rho), J_3(\lambda_{3i}\rho))} \;1\; = \frac{2}{J_4^2(\lambda_{3i})}\;1\;\int_0^1 g(\rho)J_3(\lambda_{3i}\rho)\rho\,d\rho \;1$$

$$= \frac{2}{J_4^2(\lambda_{3i})}\left(-\int_0^{1/2}\rho^4 J_3(\lambda_{3i}\rho)\,d\rho + \int_{1/2}^1 \rho^4 J_3(\lambda_{3i}\rho)\,d\rho\right). \quad .5$$

Now from the given identity,

$$\int x^4 J_3(x)\,dx = x^4 J_4(x) + C, \quad 0.5 \quad \int \rho^4 J_3(\lambda_{3i}\rho)\,d\rho = \frac{1}{\lambda_{3i}^5}\left[\lambda_{3i}^4\rho^4 J_4(\lambda_{3i}\rho) + C\right] \quad 1$$

$$= \frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho) + C, \quad .5$$

(see p. 14)

whence

$$a_{3i} = \frac{2}{J_4^2(\lambda_{3i})\sinh\lambda_{3i}}\left(-\frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho)\Big|_0^{1/2} + \frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho)\Big|_{1/2}^{1}\right) \quad .5$$

$$= \frac{2}{J_4^2(\lambda_{3i})\sinh\lambda_{3i}}\left(-\frac{1}{16\lambda_{3i}}J_4(\tfrac{1}{2}\lambda_{3i}) + \frac{1}{\lambda_{3i}}J_4(\lambda_{3i}) - \frac{1}{16\lambda_{3i}}J_4(\tfrac{1}{2}\lambda_{3i})\right) \quad 1$$

$$= \frac{2}{\lambda_{3i}J_4(\lambda_{3i})\sinh\lambda_{3i}}\left(1 - \frac{1}{8}\frac{J_4(\frac{1}{2}\lambda_{3i})}{J_4(\lambda_{3i})}\right). \quad .5$$

and we have finally for u

$$2 \quad u = \sum_{i=1}^{\infty} J_3(\lambda_{3i}\rho)\cos 3\varphi \sinh\lambda_{3i}z \cdot \frac{2}{\lambda_{3i}J_4(\lambda_{3i})\sinh\lambda_{3i}}\left(1 - \frac{1}{8}\frac{J_4(\frac{1}{2}\lambda_{3i})}{J_4(\lambda_{3i})}\right).$$

Scratch work