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University of Toronto, Faculty of Arts and Science

APM346 H1Y, August 2019 Final Examinations

Instructor: Nathan Carruth

Duration: 3 hours. **No aids allowed.**

Please read the following instructions:

- Please fill out the front of this exam booklet, but do not begin writing the actual exam until the announcements are over and the Exam Facilitator has started the exam.
- No aids of any form are allowed on this exam. Possessing an aid during this exam may result in your being charged with an academic offence.
- All cell phones, smart watches, electronic devices, toasters, etc., must be turned off and placed in your bag under your desk. All study materials must also be placed in your bag under your desk. Having such items on your person after the exam has started may be an academic offence.
- [Your instructor does not recommend carrying toasters in your pockets anyway.]
- When you are done with your exam, please raise your hand and wait for someone to come and collect it. Do not collect your bag and jacket while still in possession of the exam paper.
- If you are feeling ill and unable to finish your exam, please let an Exam Facilitator know this prior to leaving the exam hall so it can be properly noted.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special instructions:

- You *must* use the definition of the Fourier transform given in class. Use of a different definition (including that given in the textbook) may result in lost marks.
- Use of an incorrect orthogonal set on a problem may result in a very low score for the entire problem. Please check the sets you use. Sets which were derived in class, in the notes, or in the homework solutions on the course webpage may be used without derivation. Other sets, if needed, may be stated without derivation, but then no partial credit will be given for a partially correct set.
- This exam has eight questions, for a total of 125 marks. The weighting is indicated on each question. Note that weighting may not directly correspond either to difficulty or to amount of writing required, and that the ordering of the problems may not be the best order in which to write the exam. You must show all of your work for credit.
- You may use the back sides of the pages, as well as the last four pages, to continue your solutions, as long as this is clearly indicated.
- Unless otherwise stated, you must write out the full form of the final answer for full marks.

1. [8 marks] Solve the following boundary-value problem on the unit cube $Q = \{(x, y, z)|x, y, z \in [0, 1]\}$:

$$\nabla^2 u = 0, \quad u|_{\partial Q} = \begin{cases} 1, & z = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We have from class that the general solution to $\nabla^2 u = 0$ on Q with $u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0$ is

$$u = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \sinh \pi\sqrt{\ell^2+m^2}z + b_{\ell m}\cosh \pi\sqrt{\ell^2+m^2}z\right). \qquad \text{[2 marks]}$$

The boundary conditions then give

$$u|_{z=0} = 0 = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y\,(b_{\ell m}),\ \text{[1 mark]} \qquad \text{so} \qquad b_{\ell m} = 0\ \text{[1 mark]}$$

$$u|_{z=1} = 1 = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \sinh \pi\sqrt{\ell^2+m^2}\right) \qquad \text{[1 mark]}$$

$$\begin{aligned}
a_{\ell m} \sinh \pi\sqrt{\ell^2+m^2} &= 4\int_0^1\int_0^1 \sin \ell\pi x \sin m\pi y\, dy\, dx = 4\left(\int_0^1 \sin \ell\pi x\, dx\right)\left(\int_0^1 \sin m\pi y\, dy\right) \ \text{[1 mark]}\\
&= r\left(-\frac{1}{\ell\pi}\cos \ell\pi x\Big|_0^1\right)\left(-\frac{1}{m\pi}\cos m\pi y\Big|_0^1\right)\\
&= \frac{4}{\pi^2 \ell m}\left(1-(-1)^\ell\right)\left(1-(-1)^m\right),\ \text{[1 mark]}
\end{aligned}$$

so the solution is

$$u = \sum_{\ell=1,\ell\,\text{odd}}^{\infty}\ \sum_{m=1,m\,\text{odd}}^{\infty} \frac{16}{\pi^2 \ell m \sinh \pi\sqrt{\ell^2+m^2}} \sin \ell\pi x \sin m\pi y \sinh \pi\sqrt{\ell^2+m^2}z. \qquad \text{[1 mark]}$$

NOTES. 1 mark was given if the form for the expansion was not quite correct. Writing out a sum over only ℓ and m odd (as done here) was not required. Taking the initial value of ℓ and m to be 0 instead of 1 should typically result in a deduction of 0.5 marks, since in this case the final expression is meaningless.

2. [22 marks] Solve the following boundary-value problem on the spherical shell $\{(r, \theta, \phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \ u|_{r=1} = \begin{cases} 0, & 0 \leq \theta < \frac{\pi}{2} \\ \sin 2\phi, & \frac{\pi}{2} < \theta \leq \pi \end{cases}, \ u|_{r=2} = \begin{cases} \sin 2\phi, & 0 \leq \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta \leq \pi \end{cases}.$$

Recall *Legendre's equation:* $(1-x^2)P_\ell'' - 2xP_\ell' + \ell(\ell+1)P_\ell = 0$. [Can you see a certain $P_{\ell m}$ hiding here?] The following identities may be useful: $P_{\ell+1}' - xP_\ell' = (\ell+1)P_\ell$, $(2\ell+1)P_\ell = P_{\ell+1}' - P_{\ell-1}'$. [Hint: the algebra is probably easiest if you write everything in terms of derivatives of P_n for various n *before* integrating.] Your answer may include $P_n(0)$ for values of n for which this is nonzero. You may also use the normalisation integral for $P_{\ell m}$: $\int_{-1}^1 P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$.

We have the general solution

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left[\cos m\phi\left(\alpha_{\ell m}r^\ell + \beta_{\ell m}r^{-(\ell+1)}\right) + \sin m\phi\left(\gamma_{\ell m}r^\ell + \delta_{\ell m}r^{-(\ell+1)}\right)\right]. \quad \text{[1 mark]}$$

The first boundary condition [1 mark] then gives

$$\alpha_{\ell m} + \beta_{\ell m} = 0, \qquad \text{all } \ell, m \qquad \text{[1 mark]}$$
$$\gamma_{\ell m} + \delta_{\ell m} = 0, \qquad m \neq 2 \qquad \text{[0.5 marks]}$$
$$\sum_{\ell=2}^{\infty} P_{\ell 2}(\cos\theta)\,(\gamma_{\ell 2} + \delta_{\ell 2}) = \begin{cases} 0, & 0 \leq \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \leq \pi \end{cases} \qquad \text{[1 mark]}$$

Similarly, the second boundary condition [1 mark] gives

$$2^\ell\alpha_{\ell m} + 2^{-(\ell+1)}\beta_{\ell m} = 0, \qquad \text{all } \ell, m \qquad \text{[1 mark]}$$
$$2^\ell\gamma_{\ell m} + 2^{-(\ell+1)}\delta_{\ell m} = 0, \qquad m \neq 2 \qquad \text{[0.5 marks]}$$
$$\sum_{\ell=2}^{\infty} P_{\ell 2}(\cos\theta)\left(2^\ell\gamma_{\ell 2} + 2^{-(\ell+1)}\delta_{\ell 2}\right) = \begin{cases} 1, & 0 \leq \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta \leq \pi \end{cases} \qquad \text{[1 mark]}$$

Since the matrix

$$\begin{pmatrix} 1 & 1 \\ 2^\ell & 2^{-(\ell+1)} \end{pmatrix} \qquad \text{has inverse} \qquad \frac{1}{2^{-(\ell+1)} - 2^\ell}\begin{pmatrix} 2^{-(\ell+1)} & -1 \\ -2^\ell & 1 \end{pmatrix},$$

we see that $\alpha_{\ell m} = \beta_{\ell m} = 0$ for all ℓ, m [1 mark], while $\gamma_{\ell m} = \delta_{\ell m} = 0$ for all $m \neq 2$ [1 mark]. We now need to expand the two functions appearing in the remaining two conditions. To do this, we note that

$$\begin{aligned} P_{\ell 2}(x) &= (1-x^2)P_\ell'' \text{[0.5 marks]} = 2xP_\ell' - \ell(\ell+1)P_\ell \text{[0.5 marks]} = 2\left[P_{\ell+1}' - (\ell+1)P_\ell\right] - \ell(\ell+1)P_\ell \\ &= 2P_{\ell+1}' - (\ell+2)(\ell+1)P_\ell = 2P_{\ell+1}' - \frac{(\ell+2)(\ell+1)}{2\ell+1}\left(P_{\ell+1}' - P_{\ell-1}'\right) \\ &= \frac{4\ell+2-(\ell^2+3\ell+2)}{2\ell+1}P_{\ell+1}' + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}' = -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}' + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}', \end{aligned}$$

[3.5 marks]

so

$$\int P_{\ell 2}(x)\,dx = -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1} + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1} + C.$$

Thus (making the change of variables $x = \cos\theta$, as usual)

$$\begin{aligned}\left[\frac{(\ell+2)!}{(\ell-2)!}\frac{2}{2\ell+1}\right] \text{[0.5 marks]} \left(2^\ell\gamma_{\ell 2} + 2^{-(\ell+1)}\delta_{\ell 2}\right) &= -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1} + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}\Big|_0^1 \text{ [0.5 marks]}\\ &= \frac{\ell^2+3\ell+2-\ell^2+\ell}{2\ell+1} + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\\ &= 2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0), \quad \text{[1 mark]}\end{aligned}$$

while since $P'_{\ell+1}$, $P'_{\ell-1}$ are even or odd as ℓ is [1 mark],

$$\left[\frac{(\ell+2)!}{(\ell-2)!}\frac{2}{2\ell+1}\right](\gamma_{\ell 2}+\delta_{\ell 2}) = (-1)^\ell\left[2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\right].$$

[0.5 marks] [0.5 marks]

Thus finally

$$\begin{aligned}\begin{pmatrix}\gamma_{\ell 2}\\ \delta_{\ell 2}\end{pmatrix} &= \frac{(\ell-2)!}{(\ell+2)!}\frac{2\ell+1}{2}\frac{1}{2^{-(\ell+1)}-2^\ell}\begin{pmatrix}2^{-(\ell+1)} & -1\\ -2^\ell & 1\end{pmatrix}\begin{pmatrix}(-1)^\ell\\ 1\end{pmatrix}\\ &\qquad\cdot\left[2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\right]\\ &= \frac{(-1)^\ell}{2^{-(\ell+1)}-2^\ell}\left[\frac{2\ell+1}{(\ell+2)(\ell+1)\ell(\ell-1)} + \frac{1}{2(\ell+2)(\ell+1)}P_{\ell+1}(0) - \frac{1}{2\ell(\ell-1)}P_{\ell-1}(0)\right]\\ &\qquad\cdot\begin{pmatrix}2^{-(\ell+1)}-(-1)^\ell\\ -2^\ell+(-1)^\ell\end{pmatrix}\end{aligned}$$

[3.5 marks]

and we have the final answer

$$\begin{aligned}u &= \sum_{\ell=2}^\infty P_{\ell 2}(\cos\theta)\sin 2\phi\frac{2\ell+1}{(\ell+2)(\ell+1)\ell(\ell-1)\left(2^{-(\ell+1)}-2^\ell\right)}\\ &\qquad\cdot\left[r^\ell\left((-1)^\ell 2^{-(\ell+1)}-1\right) - r^{-(\ell+1)}\left((-1)^\ell 2^\ell - 1\right)\right]\\ &\quad+\sum_{\ell=2,\ell\,\mathrm{odd}}^\infty P_{\ell 2}(\cos\theta)\sin 2\phi\frac{1}{2^{-(\ell+1)}-2^\ell}\left[\frac{1}{2(\ell+2)(\ell+1)}P_{\ell+1}(0) - \frac{1}{2\ell(\ell-1)}P_{\ell-1}(0)\right]\\ &\qquad\cdot\left[r^\ell\left((-1)^\ell 2^{-(\ell+1)}-1\right) - r^{-(\ell+1)}\left((-1)^\ell 2^\ell - 1\right)\right]. \quad \text{[0.5 marks]}\end{aligned}$$

NOTES. One can also use the alternative (less general) form for the solution

$$u(r,\theta,\phi) = \sum_{\ell=0}^\infty\sum_{m=0}^\ell P_{\ell m}(\cos\theta)\left(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi\right)\left(c_{\ell m}r^\ell + d_{\ell m}r^{-(\ell+1)}\right).$$

However, in either case it is necessary to solve systems for all of the coordinates; and concluding too quickly that (for example) $a_{\ell m} = 0$ for all ℓ and m led to lost marks. (This is analogous to problem 3 on the midterm.) Additionally, the identity $(2\ell+1)P_\ell = P'_{\ell+1} - P'_{\ell-1}$ only applies to P_ℓ, *not* to the $P_{\ell 2}$ with which we need to work here: attempting to solve the problem that way probably led to little credit being given.

Beyond the foregoing, most lost marks on this problem were probably due to algebraic errors or simply not finishing.

The alert reader will note that the marks above add up to 22.5, not 22. This was an inadvertant slip on the part of the instructor which was felt not to be serious enough to attempt to correct once it was discovered. Thus this problem had effectively 0.5 bonus marks attached to it.

3. [9 marks] Solve the following boundary-value problem on the cylinder $\{(\rho,\phi,z)|\rho<1,\ 0<z<2\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=2} = 0, \quad u|_{\rho=1} = z\cos 2\phi.$$

We have the general expansion

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi}{2}\rho\right)\left[\cos m\phi\beta_{nm}\sin\frac{n\pi}{2}z + \sin m\phi\delta_{nm}\sin\frac{n\pi}{2}z\right]; \qquad \text{[1.5 marks]}$$

applying the boundary condition [1 mark] gives

$$\begin{aligned} \delta_{nm} &= 0 \qquad \text{for all}\, n, m && \text{[1 mark]}\\ \beta_{nm} &= 0 \qquad \text{for all}\, m \neq 2 && \text{[0.5 marks]}\\ \sum_{n=1}^{\infty} I_2\left(\frac{n\pi}{2}\right)&\left(\beta_{n2}\sin\frac{n\pi}{2}z\right) = z; && \text{[1 mark]} \end{aligned}$$

thus (since $\int_0^2 \sin^2\frac{n\pi}{2}z\,dz = 1$ [0.5 mark])

$$\begin{aligned} \beta_{n2} I_2\left(\frac{n\pi}{2}\right) &= \int_0^2 z\sin\frac{n\pi}{2}z\,dz \text{ [0.5 marks]} = \left[-\frac{2}{n\pi}z\cos\frac{n\pi}{2}z\Big|_0^2 + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}z\Big|_0^2\right] \text{ [1 mark]}\\ &= \frac{4}{n\pi}(-1)^{n+1}, \text{ [1 mark]} \end{aligned}$$

so $\beta_{n2} = \frac{4(-1)^{n+1}}{n\pi I_2\left(\frac{n\pi}{2}\right)}$ [0.5 marks], and the solution is

$$u(\rho,\phi,z) = \sum_{n=1}^{\infty} I_2\left(\frac{n\pi}{2}\rho\right)\cos 2\phi(-1)^{n+1}\frac{4}{n\pi I_2\left(\frac{n\pi}{2}\right)}\sin\frac{n\pi}{2}z. \qquad \text{[0.5 marks]}$$

NOTES. Probably the single most common mistake on this problem was forgetting the factor of $\frac{1}{2}$ in the z separation constant, i.e., using $n\pi$ instead of $\frac{n\pi}{2}$ in the foregoing. This fails to give a correct answer since $\{\sin n\pi z\}$ is not a complete set on the interval $[0,2]$. This generally resulted in the deduction of 0.5 marks. As with problem 1, beginning the sum for n at 0 instead of 1 should generally result in a deduction of 0.5 marks.

4. [12 marks] Suppose that $n \in \mathbf{Z}$, $n > 0$. Solve the following problem on $(0, +\infty) \times \mathbf{R}^3$, using Fourier transforms:

$$\frac{\partial u}{\partial t} = \nabla^2 u + (4\pi t)^{-\frac{3}{2}} e^{-\frac{x^2}{4t}}, \quad u|_{t=0} = \left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}} e^{-n^2|\mathbf{x}|^2}.$$

Find the limit of the solution as $n \to \infty$. What does the initial data behave like in this limit?

We have, upon Fourier transforming in space,

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &= -4\pi^2|\mathbf{k}|^2\hat{u} \,\text{[1 mark]} + (4\pi t)^{-\frac{3}{2}} \left(\frac{\pi}{\frac{1}{4t}}\right)^{\frac{3}{2}} e^{-4\pi^2|\mathbf{k}|^2 t} \\ &= -4\pi^2|\mathbf{k}|^2\hat{u} + e^{-4\pi^2|\mathbf{k}|^2 t} \,\text{[1 mark]} \end{aligned}$$

$$\begin{aligned} \hat{u}|_{t=0} &= \left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}} \left(\frac{\pi}{n^2}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{n^2}} \\ &= e^{-\frac{\pi^2|\mathbf{k}|^2}{n^2}} \,\text{[1 mark]} \end{aligned}$$

whence, using the integrating factor $e^{4\pi^2|\mathbf{k}|^2 t}$ [1 mark],

$$\begin{aligned} \frac{\partial}{\partial t}\left(e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}\right) &= 1, \\ \hat{u} &= [\hat{u}(0) \,\text{[1 mark]} + t \,\text{[1 mark]}]\, e^{-4\pi^2|\mathbf{k}|^2 t} \\ &= te^{-4\pi^2|\mathbf{k}|^2 t} + e^{-|\mathbf{k}|^2\pi^2\left(4t+\frac{1}{n^2}\right)}, \,\text{[0.5 marks]} \end{aligned}$$

whence we obtain upon inverse transforming

$$\begin{aligned} u &= t\left(\frac{\pi}{4\pi^2 t}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \,\text{[1 mark]} + \left(\frac{\pi}{\pi^2\left(4t+\frac{1}{n^2}\right)}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t+\frac{1}{n^2}}} \,\text{[1 mark]} \\ &= \frac{1}{8\pi^{\frac{3}{2}} t^{\frac{1}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}} \,\text{[0.5 marks]} + \frac{1}{\left(\pi\left(4t+\frac{1}{n^2}\right)\right)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t+\frac{1}{n^2}}} \,\text{[1 mark]}. \end{aligned}$$

In the limit as $n \to \infty$, the second term becomes simply $\frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}}$, and the whole solution is

$$u = \frac{1}{(4\pi t)^{\frac{3}{2}}}(1+t)e^{-\frac{|\mathbf{x}|^2}{4t}}. \qquad \text{[1 mark]}$$

Since

$$\int_{\mathbf{R}^3} \pi^{-\frac{3}{2}} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{-\frac{3}{2}} \left(\frac{\pi}{1}\right)^{\frac{3}{2}} = 1,$$

and

$$\left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}} e^{-n^2|\mathbf{x}|^2} = n^3 \left[\pi^{-\frac{3}{2}} e^{-|n\mathbf{x}|^2}\right],$$

we see that the initial data is an approximate identity and behaves like the delta function $\delta(\mathbf{x})$ in the limit $n \to \infty$.[1 mark]

NOTES. Probably the most common mistake here was incorrectly taking the forward or inverse Fourier transform of a Gaussian. I think almost nobody correctly found the indicated limit of the initial data (many people said it was zero, which is true only for $\mathbf{x} \neq 0$).

5. (a) [19 marks] Solve the following problem on $(0,+\infty)\times B$, where B is the unit ball $\{(r,\theta,\phi)|r<1\}$:

$$\frac{\partial u}{\partial t}=\nabla^2 u,\quad u|_{t=0}=r^2\sin^2\theta\sin 2\phi,\quad u|_{\partial B}=0.$$

[If you wish to use quantities like $\kappa_{\ell n}$, you must define them explicitly.] Find the limit of the solution as $t\to+\infty$.

(b) [4 marks] Suppose that the condition $u|_{\partial B}=0$ were replaced by the condition $u|_{\partial B}=\cos\theta$. Explain how you would solve the problem in this case (you need not actually calculate anything). What would you expect the limit of the solution to be in this case as $t\to+\infty$? [You need not give an explicit formula, but your answer must be a definite function, not just a description in words.]

(a) The Laplacian on B with Dirichlet boundary conditions has eigenfunctions

$$j_\ell(\kappa_{\ell n}r)\,P_{\ell m}(\cos\theta)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases}\qquad\text{[1 mark]}$$

(where $\kappa_{\ell n}$, $n=1,2,\ldots$, is the nth positive root of j_ℓ [0.5 marks]) with corresponding eigenvalues $\lambda_{\ell nm}=-\kappa_{\ell n}^2$ [1 mark]. Suppose that we expand u in this basis as

$$u=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell(\kappa_{\ell n}r)\,P_{\ell m}(\cos\theta)\,(a_{\ell nm}\cos m\phi+b_{\ell nm}\sin m\phi)\,.\qquad\text{[1 mark]}$$

Then substituting into the equation gives

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell(\kappa_{\ell n}r)\,P_{\ell m}(\cos\theta)\,(a'_{\ell nm}\cos m\phi+b'_{\ell nm}\sin m\phi)\quad\text{[1 mark]}$$
$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}-\kappa_{\ell n}^2 j_\ell(\kappa_{\ell n}r)\,P_{\ell m}(\cos\theta)\,(a_{\ell nm}\cos m\phi+b_{\ell nm}\sin m\phi)\,,$$

[1 mark]

so

$$a'_{\ell nm}=-\kappa_{\ell n}^2 a_{\ell nm},\ \text{[1 mark]}\qquad b'_{\ell nm}=-\kappa_{\ell n}^2 b_{\ell nm},\ \text{[1 mark]}$$

and

$$a_{\ell nm}(t)=a_{\ell nm}(0)e^{-\kappa_{\ell n}^2 t},\ \text{[1 mark]}\qquad b_{\ell nm}(t)=b_{\ell nm}(0)e^{-\kappa_{\ell n}^2 t}.\text{[1.5 marks]}$$

The initial values can be obtained from $u|_{t=0}$:

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell(\kappa_{\ell n}r)\,P_{\ell m}(\cos\theta)\,(a_{\ell nm}(0)\cos m\phi+b_{\ell nm}(0)\sin m\phi)=r^2\sin^2\theta\sin 2\phi.\qquad\text{[1 mark]}$$

Since $P_{22}(\cos\theta) = 3\sin^2\theta$, we see that $b_{\ell nm}(0) = 0$ unless $\ell = m = 2$ [0.5 marks], and $a_{\ell nm} = 0$ for all ℓ, n, m [1 mark]; finally

$$\sum_{n=1}^{\infty} b_{2n2}(0) j_2(\kappa_{2n} r) = \frac{r^2}{3}, \qquad \text{[1 mark]}$$

$$\begin{aligned} b_{2n2}(0) &= \frac{2}{j_3^2(\kappa_{2n})}\text{[1 mark]} \int_0^1 \frac{r^4}{3} j_2(\kappa_{2n} r)\, dr\text{[1 mark]} = \frac{2}{3j_3^2(\kappa_{2n})}\sqrt{\frac{\pi}{2\kappa_{2n}}} \int_0^1 r^{\frac{7}{2}} J_{\frac{5}{2}}\left(\lambda_{\frac{5}{2},n} r\right) \text{[0.5 marks]}\, dr \\ &= \frac{2}{3j_3^2(\kappa_{2n})}\sqrt{\frac{\pi}{2}} \frac{1}{\kappa_{2n}^{\frac{3}{2}}} J_{\frac{7}{2}}(\kappa_{2n}) \text{[1 mark]} = \frac{2}{3j_3^2(\kappa_{2n})\kappa_{2n}} j_3(\kappa_{2n}) \\ &= \frac{2}{3j_3(\kappa_{2n})\kappa_{2n}}, \text{[0.5 marks]} \end{aligned}$$

and the final solution is

$$u = \sum_{n=1}^{\infty} j_2(\kappa_{2n} r) P_{22}(\cos\theta)\sin 2\phi \frac{2}{3j_3(\kappa_{2n})\kappa_{2n}} e^{-\kappa_{2n}^2 t}. \qquad \text{[0.5 marks]}$$

Since $\kappa_{2n} > 0$ for all n, we see that $u \to 0$ as $t \to +\infty$. [1 mark]

(b) In this case we would first solve the problem on B

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = \cos\theta, \qquad \text{[1 mark]}$$

and then solve on $(0, +\infty) \times B$

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = r^2 \sin^2\theta \sin 2\phi - U_1, \quad u_2|_{\partial B} = 0; \qquad \text{[1 mark]}$$

the full solution would be $u = U_1 + u_2$ [1 mark]. We expect $\lim_{t\to+\infty} u = U_1$ [1 mark] in this case.

Notes. Probably the biggest single reason for deducted marks in (a) was not deriving the equations satisfied by the coefficients, but rather assuming the solutions from the outset. For (b), the single biggest quantitative error was probably taking $u_2|_{t=0} = r^2\sin^2\theta\sin 2\phi - \cos\theta$, or even dropping the subtracted term altogether.

Starting the n sum at 0 instead of 1 should not result in lost marks (since n is just a counter, which can just as well be started at 0 as at 1, though in class we always started it at 1).

The curious asymmetry in marking the expressions for $a_{\ell nm}(t)$ and $b_{\ell nm}(t)$ was not intended to create any asymmetry in practice, in that if only one appeared, it would be given the higher mark. (I probably had some reason in mind when I wrote 1 mark for a and 1.5 marks for b, but I have long since forgotten what it was.)

6. [24 marks] Solve the following problem on the unit disk $D = \{(\rho, \phi) | \rho < 1\}$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \rho^2 \sin 2\phi.$$

[As in problem 5, if you wish to use quantities like λ_{mi}, you must define them explicitly.] What is the lowest frequency occurring? [A symbolic answer is sufficient.]

In this case we have the eigenfunctions $J_m(\lambda_{mi}\rho)\begin{cases}\cos m\phi \\ \sin m\phi\end{cases}$ [1 mark] (where λ_{mi} is the ith positive root of $J_{\ell m}(x)$ [0.5 marks]) with eigenvalues $-\lambda_{mi}^2$ [1 mark]. Expanding u as

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi), \qquad \text{[1 mark]}$$

we have, upon substituting into the equation,

$$\sum_{n=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a''_{mi}\cos m\phi + b''_{mi}\sin m\phi) \text{ [1 mark]}$$
$$= \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(-\lambda_{mi}^2\right)(a_{mi}\cos m\phi + b_{mi}\sin m\phi), \text{ [1 mark]}$$

so that the a_{mi} and b_{mi} satisfy

$$a''_{mi} = -\lambda_{mi}^2 a_{mi}, \text{ [1 mark]} \qquad b''_{mi} = -\lambda_{mi}^2 b_m i, \text{ [1 mark]}$$

so

$$a_{mi}(t) = \alpha_{mi}\cos\lambda_{mi}t + \beta_{mi}\sin\lambda_{mi}t, \qquad b_{mi}(t) = \gamma_{mi}\cos\lambda_{mi}t + \delta_{mi}\sin\lambda_{mi}t. \qquad \text{[1 mark]}$$

Now we see that

$$a_{mi}(0) = \alpha_{mi}, \quad a'_{mi}(0) = \lambda_{mi}\beta_{mi}, \quad b_{mi}(0) = \gamma_{mi}, \quad b'_{mi}(0) = \lambda_{mi}\delta_{mi};$$

and these initial values can be determined from the initial conditions for u:

$$0 = u|_{t=0} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}(0)\cos m\phi + b_{mi}(0)\sin m\phi) \text{ [1 mark]}$$

so $\alpha_{mi} = \gamma_{mi} = 0$ for all m, i [1 mark];

$$\rho^2\sin 2\phi = u_t|_{t=0} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a'_{mi}(0)\cos m\phi + b'_{mi}(0)\sin m\phi), \text{ [1 mark]}$$

so $a'_{mi}(0) = 0$ for all m, i [1 mark], which gives $\beta_{mi} = 0$ and $a_{mi}(t) = 0$ for all t, all m, i [2 marks], while $b'_{mi}(0) = 0$ for all $m \neq 2$ [0.5 marks] , which gives $\delta_{mi} = 0$, hence $b_{mi}(t) = 0$ for all t [0.5 marks], for $m \neq 2$ [1 mark]; finally,

$$\rho^2 = \sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho)\, b'_{2i}(0), \qquad \text{[1 mark]}$$

so

$$b'_{2i}(0) = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i}\rho)\, d\rho = \frac{2}{J_3^2(\lambda_{2i})} \frac{1}{\lambda_{2i}} J_3(\lambda_{2i}) = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})},$$ [3.5 marks]

whence

$$\delta_{2i} = \frac{2}{\lambda_{2i}^2 J_3(\lambda_{2i})}$$ [1 mark]

and we have finally for u

$$u(t,\rho,\phi) = \sum_{i=1}^{\infty} \frac{2}{\lambda_{2i}^2 J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho) \sin 2\phi \sin \lambda_{2i} t.$$ [1 mark]

The lowest frequency is thus $\frac{\lambda_{21}}{2\pi}$. [1 mark]

NOTES. As with problem 5, probably the biggest reason for lost marks was starting directly with the solutions for the coefficients rather than deriving them as here. For the last part of the question, an answer λ_{21} was also acceptable (missing the factor of 2π did not result in lost marks): while technically only $\frac{\lambda_{21}}{2\pi}$ is the frequency, λ_{21} is the so-called *angular frequency*, and since we didn't spend much time on this point in class I didn't see a point in deducting marks for missing the 2π.

As with problem 5, starting the i sum at 0 should not result in lost marks.

7. [18 marks] Solve the following problem on the unit cube Q (defined in problem 1):

$$\nabla^2 u = \sin 4\pi x \sin 2\pi y \cos \pi z, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0, \qquad u(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0.$$

(Here $\frac{\partial}{\partial n}$ denotes the derivative in the normal direction to the surface ∂Q.)

We have the eigenfunctions $\cos \ell\pi x \cos m\pi y \cos n\pi z$ [2 marks], with eigenvalues $-\pi^2\left(\ell^2+m^2+n^2\right)$ [1 mark]. Expanding u as

$$u(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z, \qquad \text{[1 mark]}$$

we see that the equation gives

$$\sum_{\ell,m,n=0}^{\infty} -\pi^2\left(\ell^2+m^2+n^2\right) a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z = \sin 4\pi x \sin 2\pi y \cos \pi z, \qquad \text{[1 mark]}$$

whence we see that

$$-\pi^2\left(\ell^2+m^2+n^2\right) a_{\ell mn} = n_\ell n_m n_n \int_Q \sin 4\pi x \sin 2\pi y \cos \pi z \cos \ell\pi x \cos m\pi y \cos n\pi z \, dV, \tag{1}$$

where $n_\ell = \begin{cases} 2, & \ell \neq 0 \\ 1, & \ell = 0 \end{cases}$ is the appropriate normalisation constant. Now we see that the integral above vanishes for $n \neq 1$, while

$$\begin{aligned}
\int_0^1 \sin 2k\pi x \cos \ell\pi x \, dx\text{[0.5 marks]} &= \frac{1}{2}\int_0^1 \sin\left[(2k\pi+\ell\pi)x\right] + \sin\left[(2k\pi-\ell\pi)x\right] \, dx\text{[1 mark]} \\
&\left[= 0 \text{ if } \ell = 2k\right]\text{[1 mark]} \\
&= -\frac{1}{2}\left[\frac{1}{(2k+\ell)\pi}\cos\left[(2k+\ell)\pi x\right]\Big|_0^1 + \frac{1}{(2k-\ell)\pi}\cos\left[(2k-\ell)\pi x\right]\Big|_0^1\right], \\
&\qquad \text{[0.5 marks]} \quad \ell \neq 2k \\
&= \frac{1}{2\pi}\left(1-(-1)^\ell\right)\left(\frac{1}{2k+\ell}+\frac{1}{2k-\ell}\right) = \frac{2k}{\pi}\left(1-(-1)^\ell\right)\frac{1}{4k^2-\ell^2}, \\
&\qquad \text{[1 mark]}
\end{aligned}$$

so for $(\ell,m,n) \neq (0,0,0)$ we have

$$\begin{aligned}
a_{\ell mn} &= \begin{cases} 0, & n \neq 1, \text{ or } m = 2,\text{[0.5 marks] or } \ell = 4 \text{ [0.5 marks]} \\ n_\ell n_m \frac{8}{\pi^2}\left(1-(-1)^\ell\right)\left(1-(-1)^m\right)\frac{1}{16-\ell^2} \\ \quad \cdot\frac{1}{4-m^2}\text{[1 mark]}\cdot\frac{-1}{\pi^2(\ell^2+m^2+1)}, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0\text{[0.5 marks]}, & n \neq 1, \text{ or } m \text{ or } \ell \text{ even[0.5 marks]} \\ -\frac{128}{\pi^2}\frac{1}{16-\ell^2}\frac{1}{4-m^2}\text{[0.5 marks]}\frac{1}{\pi^2(\ell^2+m^2+1)}\text{[1 mark]}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now if $\ell = m = n = 0$, then the integral in (1) is zero, as is the left-hand side. Thus this equation is consistent but tells us nothing about a_{000} [1 mark]. However, since our series for u only has nonzero coefficients for ℓ, m, n all odd, and

$$\cos\frac{\ell\pi}{2}\cos\frac{m\pi}{2}\cos\frac{n\pi}{2} = 0 \qquad \text{[1 mark]}$$

in such a case, the final condition gives $a_{000} = 0$ [1 mark]. Thus

$$u = \sum_{\ell,m=1,\ell,m\text{ odd}}^{\infty} -\frac{128}{\pi^2}\frac{1}{16-\ell^2}\frac{1}{4-m^2}\frac{1}{\pi^2\left(\ell^2+m^2+1\right)}\cos\ell\pi x\cos m\pi y\cos\pi z. \qquad \text{[1.5 marks]}$$

NOTES. Again, some marks were lost by simply assuming the general form of the solution to Poisson's equation rather than deriving it as here (though this is less of an issue than with 5 and especially 6). Marks were also lost for being insufficiently careful with the term a_{000}.

8. [9 marks] Solve the following problem on $\mathbf{R}^3$ (here x is the first coordinate of $\mathbf{x} = (x, y, z)$):

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{\partial u}{\partial x}, \quad u|_{t=0} = e^{-|\mathbf{x}|^2}.$$

We have the Fourier transform:

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u}\text{[1 mark]} + 2\pi i k_1 \hat{u}\text{[1 mark]}, \quad \hat{u}|_{t=0} = \pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2}.\text{[1 mark]}$$

The equation gives

$$\frac{\partial}{\partial t}\left(e^{\left(4\pi^2|\mathbf{k}|^2 - 2\pi i k_1\right)t}\hat{u}\text{[1 mark]}\right) = 0,$$

$$\begin{aligned} \hat{u} &= \hat{u}(0)e^{-\left(4\pi^2|\mathbf{k}|^2 - 2\pi i k_1\right)t} \\ &= \pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2(4t+1)} e^{2\pi i k_1 t}.\text{[1 mark]} \end{aligned}$$

Since

$$\mathcal{F}^{-1}\left[\pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2(4t+1)}\right] = \pi^{\frac{3}{2}}\left(\frac{1}{\pi(4t+1)}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t+1}} = \frac{1}{(4t+1)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t+1}}, \qquad \text{[2 marks]}$$

we see by properties of Fourier transforms that

$$u = \frac{1}{(4t+1)^{\frac{3}{2}}} e^{-\frac{1}{4t+1}\left(y^2+z^2+(x+t)^2\right)}. \qquad \text{[2 marks]}$$

NOTES. Again, probably the biggest issue with this problem was the mishandling of the relevant Fourier transforms. Another issue which came up was failure to use the property

$$\mathcal{F}[f(\mathbf{x} - \mathbf{x}_0)](\mathbf{k}) = e^{-2\pi i \mathbf{k}\cdot\mathbf{x}_0}\hat{f}(\mathbf{k}).$$

Some solutions wrote effectively $\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial \hat{u}}{\partial x}$, probably by analogy with a similar (though correct) formula for $\frac{\partial u}{\partial t}$: unfortunately this formula is not only wrong in actuality but meaningless even in principle, since $\hat{u}$ is a function of $\mathbf{k}$ and t and hence does not depend on x. The point behind the analogous formula for $\frac{\partial u}{\partial t}$ is that we are taking a function of $(t, \mathbf{x})$ and transforming *only in* $\mathbf{x}$, meaning that t is essentially a parameter with respect to which we can differentiate either before or after transforming (assuming, as always, that our functions are sufficiently well-behaved that we are allowed to take the derivative inside the integral representing the Fourier transform). x, however, is one of the variables with respect to which we are transforming; i.e., it will be one of the variables over which we integrate, and hence it does not appear in the transformed function and it makes no sense to speak of the derivative of the transform with respect to it. More explicitly:

$$\begin{aligned} \mathcal{F}\left[\frac{\partial u}{\partial t}\right] &= \int_{\mathbf{R}^3} \frac{\partial u}{\partial t} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \frac{\partial}{\partial t}\int_{\mathbf{R}^3} u e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \frac{\partial}{\partial t}\mathcal{F}[u] = \frac{\partial \hat{u}}{\partial t}, \end{aligned}$$

while attempting to do the same thing with $\frac{\partial u}{\partial x}$ leads to

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \int_{\mathbf{R}^3} \frac{\partial u}{\partial x} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x},$$

and now there is no way to take the derivative outside of the integral since the integral over $\mathbf{x}$ includes an integral over x: one needs instead to do an integration by parts, which leads to the formula

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = 2\pi i k_1 \hat{u}$$

used here, as derived in class. (Here k_1 represents the component of $\mathbf{k}$ corresponding to x.)

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