APM346 (Summer 2019), Quiz 5, late tutorial.

Solve the following boundary-value problem on $\{(r,\theta,\phi)|r<1\}$:

$$\nabla^2 u = 0, \qquad u|_{r=1} = \begin{cases} 2\cos\theta, & 0 \le \theta < \frac{\pi}{2} \\ -2\cos\theta, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

The following formulas may be useful:

$$(\ell+1)P_{\ell+1}(x) - (2\ell+1)xP_\ell(x) + \ell P_{\ell-1}(x) = 0,$$

$$\int_0^1 P_\ell(x)\,dx = \frac{1}{2\ell+1}\left(P_{\ell-1}(0) - P_{\ell+1}(0)\right),$$

$$\int_{-1}^1 P_\ell^2(x)\,dx = \frac{2}{2\ell+1}.$$

Your answer can contain quantities like $P_\ell(0)$, if these are nonzero.

Since the set on which we are solving Laplace's equation and the boundary data are both azimuthally symmetric, and since the set on which we are solving includes the origin, the solution can be written in the form

$$u = \sum_{\ell=0}^\infty a_\ell P_\ell(\cos\theta) r^\ell.$$

The boundary condition then gives

$$u|_{r=1} = \sum_{\ell=0}^\infty a_\ell P_\ell(\cos\theta) = \begin{cases} 2\cos\theta, & 0 \le \theta < \frac{\pi}{2} \\ -2\cos\theta, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

From the orthogonality properties of the Legendre polynomials, this gives

$$a_\ell = \frac{2\ell+1}{2}\left(\int_0^1 2xP_\ell(x)\,dx - \int_{-1}^0 2xP_\ell(x)\,dx\right).$$

Now from the given identity, we have

$$xP_\ell(x) = \frac{1}{2\ell+1}\left((\ell+1)P_{\ell+1}(x) + \ell P_{\ell-1}(x)\right),$$

whence by the integral formula

$$\begin{aligned}
\int_0^1 xP_\ell(x) &= \frac{\ell+1}{2\ell+1}\int_0^1 P_{\ell+1}(x)\,dx + \frac{\ell}{2\ell+1}\int_0^1 P_{\ell-1}(x)\,dx \\
&= \frac{\ell+1}{2\ell+1}\left(\frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right)\right) + \frac{\ell}{2\ell+1}\left(\frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right)\right) \\
&= \frac{1}{(2\ell-1)(2\ell+1)(2\ell+3)}\left((\ell+1)(2\ell-1)(P_\ell(0) - P_{\ell+2}(0)) + \ell(2\ell+3)(P_{\ell-2}(0) - P_\ell(0))\right) \\
&= \frac{1}{(2\ell-1)(2\ell+1)(2\ell+3)}\left(\ell(2\ell+3)P_{\ell-2}(0) - (2\ell+1)P_\ell(0) - (\ell+1)(2\ell-1)P_{\ell+2}(0)\right).
\end{aligned}$$

We see that this quantity is zero if ℓ is odd. Now we have also

$$\int_{-1}^0 xP_\ell(x)\,dx = -\int_0^1 xP_\ell(-x)\,dx,$$

so that if ℓ is odd this second integral also vanishes. Thus $a_\ell = 0$ for ℓ odd, while if ℓ is even we have

$$a_\ell = \frac{4\ell+2}{(2\ell-1)(2\ell+1)(2\ell+3)}\left(\ell(2\ell+3)P_{\ell-2}(0) - (2\ell+1)P_\ell(0) - (\ell+1)(2\ell-1)P_{\ell+2}(0)\right),$$

so that our final solution for u is

$$u = \sum_{k=0}^\infty \frac{8k+2}{(4k-1)(4k+1)(4k+3)}\left(2k(4k+3)P_{2k-2}(0) - (4k+1)P_{2k}(0) - (2k+1)(4k-1)P_{2k+2}(0)\right) \cdot P_{2k}(\cos\theta)r^{2k}.$$