APM 346, final exam review practice problems, solutions and sketches.

1. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = x, \quad u|_{y=1} = 1-x.$$

The general solution to Laplace's equation on the unit square satisfying the first two boundary conditions is

$$u(x,y) = \sum_{n=1}^{\infty} \sin n\pi x (a_n \cosh n\pi y + b_n \sinh n\pi y).$$

Substituting this into the second set of boundary conditions gives

$$u(x,0) = \sum_{n=1}^{\infty} \sin n\pi x a_n = x,$$

whence

$$\begin{aligned} a_n &= 2\int_0^1 x \sin n\pi x\, dx = 2\left(-\frac{x}{n\pi}\cos n\pi x\bigg|_0^1 + \int_0^1 \frac{1}{n\pi}\cos n\pi\, dx\right) \\ &= 2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x\bigg|_0^1\right) = 2\frac{(-1)^{n+1}}{n\pi}, \end{aligned}$$

and

$$u(x,1) = \sum_{n=1}^{\infty} \sin n\pi x(a_n \cosh n\pi + b_n \sinh n\pi) = 1-x,$$

whence

$$\begin{aligned} a_n \cosh n\pi + b_n \sinh n\pi &= 2\int_0^1 (1-x)\sin n\pi x\, dx = 2\left(-(1-x)\frac{1}{n\pi}\cos n\pi x\bigg|_0^1 - \int_0^1 \frac{1}{n\pi}\cos n\pi x\, dx\right) \\ &= 2\left(\frac{1}{n\pi} - \frac{1}{n^2\pi^2}\sin n\pi x\bigg|_0^1\right) = \frac{2}{n\pi}, \end{aligned}$$

so

$$b_n = \frac{2}{n\pi}\left((-1)^n \coth n\pi + \operatorname{csch} n\pi\right),$$

and finally

$$u(x,y) = \sum_{n=1}^{\infty} \sin n\pi x \left(\frac{2}{n\pi}\right)\left((-1)^{n+1}\cosh n\pi y + ((-1)^n \coth n\pi + \operatorname{csch} n\pi)\sinh n\pi y\right).$$

2. Solve on $[0,2]\times[0,3]$:

$$\nabla^2 u = 0, \quad u|_{x=0} = 1 - |y-1|, \quad u|_{x=2} = 0, \quad u|_{y=0} = u|_{y=3} = 0.$$

This is quite similar to the previous problem, in principle. Here the general solution satisfying the last two boundary conditions[1] is

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{3} y(a_n \cosh\frac{n\pi}{3}x + b_n \sinh\frac{n\pi}{3}y),$$

[1] In both cases, the key is that the solution has to satisfy the *homogeneous* boundary conditions

and the boundary conditions are used to determine a_n and b_n as before.

3. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 1, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0.$$

While not explicitly derived in class, it should be evident from our treatment of the corresponding problem on the unit *cube* that the eigenfunctions and eigenvalues of the Laplacian satisfying Dirichlet boundary conditions on the unit square (the set given here) are

$$\mathbf{e}_{\ell m} = \sin \ell\pi x \sin m\pi y, \quad \lambda_{\ell m} = -\pi^2\left(\ell^2 + m^2\right), \quad \ell, m \in \mathbf{Z},\ \ell, m \geq 1.$$

Expanding u in this basis, we have

$$u(x,y) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell m} \sin \ell\pi x \sin m\pi y;$$

substituting this into the equation $\nabla^2 u = 1$ and using the fact that the functions $\sin \ell\pi x \sin m\pi y$ are eigenfunctions of ∇^2 gives

$$\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} -\pi^2(\ell^2+m^2) a_{\ell m} \sin \ell\pi x \sin m\pi y = 1.$$

Now since

$$\int_0^1 \sin \ell\pi x\, dx = \frac{1}{\ell\pi}(1-(-1)^\ell),$$

we see that

$$\begin{aligned} a_{\ell m} &= -\frac{4}{\pi^2(\ell^2+m^2)}\int_0^1\int_0^1 \sin \ell\pi x \sin m\pi y\, dx\, dy \\ &= -\frac{4}{\pi^2(\ell^2+m^2)}\frac{1}{\ell\pi}(1-(-1)^\ell)\frac{1}{m\pi}(1-(-1)^m) \\ &= -\frac{4}{\pi^4 \ell m(\ell^2+m^2)}(1-(-1)^\ell)(1-(-1)^m), \end{aligned}$$

and thus

$$u(x,y) = -\sum_{\ell=1, \ell\,\mathrm{odd}}^{\infty}\ \sum_{m=1, m\,\mathrm{odd}}^{\infty} \frac{16}{\pi^4 \ell m(\ell^2+m^2)} \sin \ell\pi x \sin m\pi y.$$

4. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 1, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = x, \quad u|_{y=1} = 1-x.$$

Do this twice: once by splitting up into two separate problems, and once by using a Green's function (expressed as a series in the eigenfunctions of the Laplacian on $[0,1]\times[0,1]$).

The first method gives simply the sum of the solution to 1 and the solution to 3.

5. Solve on the ball $\{(r,\theta,\phi)|r<1\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} 1-\cos\theta, & \theta \in [0, \frac{\pi}{2}] \\ 1+\cos\theta, & \theta \in [\frac{\pi}{2}, \pi] \end{cases}.$$

6. Solve on the ball $\{(r,\theta,\phi)|r<2\}$:

$$\nabla^2 u = 0, \quad \left.\frac{\partial u}{\partial r}\right|_{r=2} = \begin{cases} \cos^2\theta, & \theta \in [0, \frac{\pi}{2}] \\ -\cos^2\theta, & \theta \in [\frac{\pi}{2}, \pi] \end{cases}, \quad u|_{r=0} = 0.$$

The main idea here is to use the identity $(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$ twice in order to reduce $x^2 P_\ell$ to a linear combination of $P_{\ell+2}$, P_ℓ, and $P_{\ell-2}$, which may be integrated as usual using the identity $(2\ell+1)P_\ell = P'_{\ell+1} - P'_{\ell-1}$.

7. Solve on the shell $\{(r,\theta,\phi)|1 < r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \cos\theta, & \theta \in [0, \frac{\pi}{2}] \\ -\cos\theta, & \theta \in [\frac{\pi}{2}, \pi] \end{cases}, \quad u|_{r=2} = \begin{cases} \cos\theta \sin 2\phi, & \theta \in [0, \frac{\pi}{2}] \\ -\cos\theta\cos 2\phi, & \theta \in [\frac{\pi}{2}, \pi] \end{cases}.$$

[Hint: Use Legendre's equation!] This problem requires heavy use of Legendre polynomial identities. It is nevertheless good preparation for the exam.

We provide a rough sketch of the solution. We have the general expansion (using the most general, unfactored version)

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(\alpha_{\ell m} r^\ell \cos m\phi + \beta_{\ell m} r^\ell \sin m\phi + \gamma_{\ell m} r^{-(\ell+1)}\cos m\phi + \delta_{\ell m} r^{-(\ell+1)}\sin m\phi\right).$$

The first boundary condition gives

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left((\alpha_{\ell m}+\gamma_{\ell m})\cos m\phi + (\beta_{\ell m}+\delta_{\ell m})\sin m\phi\right) = \begin{cases} \cos\theta, & \theta \in [0, \frac{\pi}{2}] \\ -\cos\theta, & \theta \in [\frac{\pi}{2}, \pi] \end{cases},$$

from which we see that $\alpha_{\ell m} + \gamma_{\ell m} = 0$ for all $m \neq 0$, while $\beta_{\ell m} + \delta_{\ell m} = 0$ for all m (both $\beta_{\ell m}$ and $\delta_{\ell m}$ are zero for $m = 0$ by convention). For $m = 0$, we have

$$\begin{aligned} \alpha_{\ell 0} + \delta_{\ell 0} &= \frac{2\ell+1}{2}\left(\int_0^{\frac{\pi}{2}} \cos\theta P_\ell(\cos\theta)\sin\theta\, d\theta - \int_{\frac{\pi}{2}}^{\pi} \cos\theta P_\ell(\cos\theta)\sin\theta\, d\theta\right) \\ &= \frac{2\ell+1}{2}\left(\int_0^1 xP_\ell(x)\,dx - \int_{-1}^0 xP_\ell(x)\,dx\right), \end{aligned}$$

which we see is equal to 0 if ℓ is odd (i.e., if $xP_\ell(x)$ is even). If ℓ is even, we may use Legendre function identities to write

$$\begin{aligned} xP_\ell(x) &= \frac{1}{2\ell+1}\left((\ell+1)P_{\ell+1} + \ell P_{\ell-1}\right) \\ &= \frac{1}{2\ell+1}\left(\frac{\ell+1}{2\ell+3}(P'_{\ell+2} - P'_\ell) + \frac{\ell}{2\ell-1}(P'_\ell - P'_{\ell-2})\right) \\ &= \frac{\ell+1}{(2\ell+3)(2\ell+1)}P'_{\ell+2} + \frac{1}{(2\ell+3)(2\ell-1)}P'_\ell - \frac{\ell}{(2\ell-1)(2\ell+1)}P'_{\ell-2}, \end{aligned}$$

whence

$$\int_0^1 xP_\ell(x)\,dx = \frac{\ell+1}{(2\ell+3)(2\ell+1)}(1-P_{\ell+2}(0)) + \frac{1}{(2\ell+3)(2\ell-1)}(1-P_\ell(0)) - \frac{\ell}{(2\ell-1)(2\ell+1)}(1-P_{\ell-2}(0)),$$

so that

$$\alpha_{\ell 0} + \delta_{\ell 0} = \frac{\ell+1}{2\ell+3}(1-P_{\ell+2}(0)) + \frac{2\ell+1}{(2\ell+3)(2\ell-1)}(1-P_\ell(0)) - \frac{\ell}{2\ell-1}(1-P_{\ell-2}(0)).$$

Let us call this quantity C_ℓ. Now the second boundary condition gives similarly

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(\left(\alpha_{\ell m}2^\ell + \gamma_{\ell m}2^{-(\ell+1)}\right)\cos m\phi + \left(\beta_{\ell m}2^\ell + \delta_{\ell m}2^{-(\ell+1)}\right)\sin m\phi\right)$$
$$= \begin{cases} \cos\theta\sin 2\phi, & \theta \in [0, \frac{\pi}{2}] \\ -\cos\theta\cos 2\phi, & \theta \in [\frac{\pi}{2}, \pi] \end{cases}; \quad (1)$$

from this we see as before that $\alpha_{\ell m}2^\ell + \gamma_{\ell m}2^{-(\ell+1)} = 0$ and $\beta_{\ell m}2^\ell + \delta_{\ell m}2^{-(\ell+1)} = 0$ for all $m \neq 2$. Since the matrix

$$\begin{pmatrix} 1 & 1 \\ 2^\ell & 2^{-(\ell+1)} \end{pmatrix}$$

has nonzero determinant for $\ell \in \mathbf{Z}$, $\ell \geq 0$, this implies that $\alpha_{\ell m} = \beta_{\ell m} = \gamma_{\ell m} = \delta_{\ell m} = 0$ unless $m = 0$ or $m = 2$. Equation (1) above then gives

$$\begin{aligned} \alpha_{\ell 2}2^\ell + \gamma_{\ell 2}2^{-(\ell+1)} &= \frac{(\ell-2)!}{(\ell+2)!}\frac{2\ell+1}{2}\int_0^{\frac{\pi}{2}} \cos\theta P_{\ell 2}(\cos\theta)\sin\theta\, d\theta \\ &= \frac{(\ell-2)!}{(\ell+2)!}\frac{2\ell+1}{2}\int_0^1 xP_{\ell 2}(x)\, dx, \end{aligned}$$

and similarly

$$\beta_{\ell 2}2^\ell + \delta_{\ell 2}2^{-(\ell+1)} = -\frac{(\ell-2)!}{(\ell+2)!}\frac{2\ell+1}{2}\int_{-1}^0 xP_{\ell 2}(x)\, dx.$$

Now

$$P_{\ell 2}(x) = (1-x^2)\frac{d^2}{dx^2}P_\ell(x),$$

so by Legendre's equation $(1-x^2)P_\ell'' - 2xP_\ell' + \ell(\ell+1)P_\ell = 0$ we have

$$\begin{aligned} xP_{\ell 2}(x) &= 2x^2P_\ell' - \ell(\ell+1)xP_\ell \\ &= 2\left(P_\ell' - \ell P_{\ell-1} + \ell xP_\ell\right) - \ell(\ell+1)xP_\ell \\ &= 2P_\ell' - 2\ell P_{\ell-1} - \ell(\ell-1)xP_\ell, \end{aligned} \tag{2}$$

whence we have, from our work above,

$$\begin{aligned} \int_0^1 xP_{\ell 2}(x)\, dx &= 2(1-P_\ell(0)) - \frac{2\ell}{2\ell+1}\left(P_{\ell-1}(0) - P_{\ell+1}(0)\right) \\ &- \ell(\ell-1)\left[\frac{\ell+1}{(2\ell+3)(2\ell+1)}(1-P_{\ell+2}(0)) + \frac{1}{(2\ell+3)(2\ell-1)}(1-P_\ell(0)) - \frac{\ell}{(2\ell-1)(2\ell+1)}(1-P_{\ell-2}(0))\right] \\ &= \frac{2\ell}{2\ell+1}(P_{\ell+1}(0) - P_{\ell-1}(0)) \\ &- \left[\frac{(\ell+1)\ell(\ell-1)}{(2\ell+3)(2\ell+1)}(1-P_{\ell+2}(0)) + \frac{3\ell^2+5\ell-3}{(2\ell+3)(2\ell-1)}(1-P_\ell(0)) - \frac{\ell^2(\ell-1)}{(2\ell-1)(2\ell+1)}(1-P_{\ell-2}(0))\right] \end{aligned}$$

This gives

$$\begin{aligned} \alpha_{\ell 2}2^\ell + \gamma_{\ell 2}2^{-(\ell+1)} &= \frac{1}{(\ell+2)(\ell+1)(\ell-1)}(P_{\ell+1}(0) - P_{\ell-1}(0)) \\ &- \left[\frac{1}{(2\ell+4)(2\ell+3)}(1-P_{\ell+2}(0)) - \frac{\ell}{(2\ell+4)(2\ell-1)(\ell+1)}(1-P_{\ell-2}(0))\right. \\ &\left. + \frac{(2\ell+1)(3\ell^2+5\ell-3)}{(2\ell+4)(\ell+1)\ell(\ell-1)(2\ell+3)(2\ell-1)}(1-P_\ell(0))\right]. \end{aligned}$$

Let us call this quantity on the right D_ℓ. Since relation (2) above implies that $xP_{\ell 2}$ is odd or even as ℓ is even or odd, i.e., that its parity is the opposite of that of ℓ, we see that we have also

$$\beta_{\ell 2}2^\ell + \delta_{\ell 2}2^{-(\ell+1)} = (-1)^\ell D_\ell.$$

We thus have the systems

$$\begin{aligned} \alpha_{\ell 0} + \gamma_{\ell 0} &= C_\ell & \alpha_{\ell 2} + \gamma_{\ell 2} &= 0 & \beta_{\ell 2} + \delta_{\ell 2} &= 0 \\ \alpha_{\ell 0} 2^\ell + \gamma_{\ell 0} 2^{-(\ell+1)} &= 0 & \alpha_{\ell 2} 2^\ell + \gamma_{\ell 2} 2^{-(\ell+1)} &= D_\ell & \beta_{\ell 2} 2^\ell + \delta_{\ell 2} 2^{-(\ell+1)} &= (-1)^\ell D_\ell \end{aligned}$$

Since the coefficient matrix has inverse

$$\begin{pmatrix} 1 & 1 \\ 2^\ell & 2^{-(\ell+1)} \end{pmatrix}^{-1} = \frac{1}{2^{-(\ell+1)} - 2^\ell} \begin{pmatrix} 2^{-(\ell+1)} & -1 \\ -2^\ell & 1 \end{pmatrix},$$

these systems have solutions, letting $\Delta = \frac{1}{2^{-(\ell+1)} - 2^\ell}$,

$$\begin{aligned} \alpha_{\ell 0} &= \Delta 2^{-(\ell+1)} C_\ell, & \alpha_{\ell 2} &= -\Delta D_\ell & \beta_{\ell 2} &= (-1)^{\ell+1} \Delta D_\ell \\ \gamma_{\ell 0} &= -\Delta 2^\ell C_\ell & \gamma_{\ell 2} &= \Delta D_\ell & \delta_{\ell 2} &= (-1)^\ell \Delta D_\ell \end{aligned}$$

whence we have finally for u the most imposing expression

$$\begin{aligned} u(r,\theta,\phi) &= \sum_{\ell=0}^\infty \frac{1}{2} \Delta C_\ell \left(\left(\frac{r}{2}\right)^e ll - \left(\frac{2}{r}\right)^{\ell+1} \right) P_\ell(\cos\theta) \\ &\quad + \sum_{\ell=2}^\infty \Delta D_\ell (-r^\ell + r^{-(\ell+1)})(\cos 2\phi + (-1)^\ell \sin 2\phi) P_{\ell 2}(\cos\theta) \\ &= \sum_{\ell=0}^\infty \frac{1}{2^{-\ell} - 2^{\ell+1}} \left[\frac{\ell+1}{2\ell+3}(1 - P_{\ell+2}(0)) + \frac{2\ell+1}{(2\ell+3)(2\ell-1)}(1 - P_\ell(0)) - \frac{\ell}{2\ell-1}(1 - P_{\ell-2}(0)) \right] \\ &\qquad \cdot \left(\left(\frac{r}{2}\right)^\ell - \left(\frac{2}{r}\right)^{\ell+1} \right) P_\ell(\cos\theta) \\ &\quad + \sum_{\ell=2}^\infty \frac{1}{2^{-(ell+1)} - 2^\ell} \Bigg[\frac{1}{(\ell+2)(\ell+1)(\ell-1)} (P_{\ell+1}(0) - P_{\ell-1}(0)) \\ &- \Bigg[\frac{1}{(2\ell+4)(2\ell+3)}(1 - P_{\ell+2}(0)) - \frac{\ell}{(2\ell+4)(2\ell-1)(\ell+1)}(1 - P_{\ell-2}(0)) \\ &+ \frac{(2\ell+1)(3\ell^2 + 5\ell - 3)}{(2\ell+4)(\ell+1)\ell(\ell-1)(2\ell+3)(2\ell-1)}(1 - P_\ell(0)) \Bigg] \Bigg] \left(-r^\ell + r^{-(\ell+1)} \right) (\cos 2\phi + (-1)^\ell \sin 2\phi) P_{\ell 2}(\cos\theta). \end{aligned}$$

8. Solve on the cylinder $\{(\rho,\phi,z) | \rho < 1, 0 < z < 2\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0, \quad u|_{z=0} = 0, \quad u|_{z=2} = 1.$$

This can just about be written down without any work; the answer is

$$u(\rho,\phi,z) = \sum_{n=1}^\infty \frac{2}{\lambda_{0i} J_1(\lambda_{0i})} J_0(\lambda_{0i}\rho) \frac{\sinh \lambda_{0i} z}{\sinh 2\lambda_{0i}}.$$

(On the exam, of course, I need to see all of the work behind this!)

9. Solve on the cylinder $\{(\rho,\phi,z) | \rho < 2, 0 < z < 3\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=2} = 0, \quad u|_{z=0} = \rho^3 \cos 3\phi, \quad u|_{z=3} = \rho^2 \sin 2\phi.$$

This one is similar; the answer should be something like (this is not guaranteed to be exactly correct! – in particular I am not entirely sure I have the overall factor correct)

$$u(\rho, \phi, z) = \sum_{n=1}^{\infty} \frac{1}{\lambda_{3i} J_4(\lambda_{3i})} J_3\left(\frac{1}{2}\lambda_{3i}\rho\right)\left(\cosh\frac{1}{2}\lambda_{3i}z - \coth\frac{3}{2}\lambda_{3i}\sinh\frac{1}{2}\lambda_{3i}z\right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\lambda_{2i} J_3(\lambda_{2i})} J_2\left(\frac{1}{2}\lambda_{2i}\rho\right)\frac{\sinh\frac{1}{2}\lambda_{2i}z}{\sinh\frac{3}{2}\lambda_{2i}}.$$

(Again, of course, on the exam I need to see all of the work behind this!)

10. Solve on the cylinder $\{(\rho, \phi, z)|\rho < 4, 0 < z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u|_{\rho=4} = \phi(2\pi - \phi)z(1 - z).$$

11. Solve on the cylinder $\{(\rho, \phi, z)|\rho < 2, 0 < z < 4\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = 1, \quad u|_{z=4} = \rho^3 \sin 3\phi, \quad u|_{\rho=2} = \sin 2\phi \sin 16\pi z.$$

12. Solve on the cube $\{(x, y, z)|0 < x, y, z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0, \quad u|_{z=0} = 0, \quad u|_{z=1} = \sin x \sin y.$$

13. Solve on the cube $\{(x, y, z)|0 < x, y, z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = \sin\pi x \sin\pi z, \quad u|_{y=1} = \sin 3\pi x \sin 3\pi z,$$
$$u|_{z=0} = \sin 2\pi x \sin 2\pi y, \quad u|_{z=1} = x(1 - x)y(1 - y).$$

14. The same as 13, except that the conditions on $x = 0$ and $x = 1$ are replaced by

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0.$$

15. Solve on the cube $Q = \{(x, y, z)|0 < x, y, z < 1\}$:

$$\nabla^2 u = x(1 - x)y(1 - y)z(1 - z), \quad u|_{\partial Q} = 0.$$

16. Solve on the cube $Q = \{(x, y, z)|0 < x, y, z < 1\}$:

$$\nabla^2 u = xy(1 - y)z(1 - z), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0,$$

where $\frac{\partial}{\partial n}$ denotes the outwards normal derivative at the boundary of ∂Q.

17. Solve on the unit ball $B = \{(r, \theta, \phi)|r < 1\}$:

$$\nabla^2 u = r\cos\theta, \quad u|_{r=1} = 0.$$

18. Solve on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = r^2 \sin\theta \cos\theta \sin\phi, \quad u|_{r=1} = 0.$$

19. Solve on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = r \sin\theta \cos\phi, \quad u|_{r=1} = \cos\theta.$$

Try doing this problem two ways, one by splitting it up into the sum of two separate problems, and the other by using an appropriate Green's function.

20. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \rho^3 \sin 3\phi (1 - z), \quad u|_{\partial C} = 0.$$

21. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \left\{ \begin{array}{ll} \rho^3 \sin 3\phi, & \rho \in [0, \frac{1}{2}] \\ \rho^4 \cos 4\phi, & \rho \in [\frac{1}{2}, 1] \end{array} \right\} (1 - z), \quad u|_{\partial C} = 0.$$

22. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \left\{ \begin{array}{ll} \rho^2 \cos 2\phi, & \rho \in [0, \frac{1}{2}] \\ 0, & \rho \in [\frac{1}{2}, 1] \end{array} \right\} \sin z, \quad u|_{z=0} = 1, \quad u|_{z=1} = \rho^2 \cos 2\phi, \quad u|_{\rho=1} = 0.$$

Again, try doing this two ways, one by directly writing out an orthogonal expansion and the other by using an appropriate Green's function.

23. Repeat the previous eight problems, but instead of solving Poisson's equation $\nabla^2 u = f$ solve the problem on $(0, +\infty) \times X$ (where X is either Q, B, or C as appropriate)

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f,$$

with the boundary conditions on ∂X unchanged. What is the behaviour of the solutions in the limit $t \to \infty$?

24. Again repeat the same eight problems, but now instead of solving the heat equation as in 23, solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

with f taken alternatively as the initial data for u and $\frac{\partial u}{\partial t}$ at $t = 0$, with the other one set to zero there.

25. Repeat problems 20 − 22, dropping the z dependence, on the unit disk $D = \{(\rho, \phi) | \rho < 1\}$, and then solve the corresponding heat and wave equation problems as in 23 and 24. For the wave equations, comment on the lowest frequency appearing. Can you say anything about the which frequency will be the loudest (i.e., have the largest coefficient in the orthogonal expansion)?

26. Solve on $\mathbf{R}^1$:

$$\nabla^2 u = \left(4x^2 - 2\right) e^{-x^2}, \quad \lim_{|x| \to \infty} u = 0.$$

Try using Fourier transforms in space. (There is actually a much easier way of solving this problem which doesn't require anything more than elementary calculus; can you see it? Even if you can, try doing this using Fourier transforms anyway as it is good practice.)

27. Solve on $\mathbf{R}^3$:

$$\nabla^2 u = e^{-|\mathbf{x}|^2}, \quad \lim_{|\mathbf{x}|\to\infty} u = 0.$$

You can do this either using Fourier transforms or the Green's function on $\mathbf{R}^3$ which we derived in class.

28. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = e^{-|\mathbf{x}|^2} \sin x.$$

(Here x is the first component of $\mathbf{x} = (x, y, z)$.) Hint: write $\sin x$ in terms of complex exponentials and use properties of the Fourier transform.

29. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = \begin{cases} 1, & x, y, z \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Express your answer in terms of the function (related to the error function)

$$E(x) = \int_0^x e^{-u^2}\, du.$$

30. Repeat the previous two problems, but with the initial data taken as the inhomogeneous term f for the equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + f$$

and with the initial data $u|_{t=0} = 0$. What is the behaviour of the solutions as $t \to \infty$?

31. Do problems 7 and 8 from the week 12 practice problem sheet, if you have not already done so. Then redo them, changing which of $u|_{t=0}$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0}$ is set to zero.

32. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + bu, \quad u|_{t=0} = \frac{\sin 2\pi x \sin 2\pi y \sin 2\pi z}{xyz}.$$

[Hint: what is the inverse Fourier transform of the function $\chi(x)\chi(y)\chi(z)$ (where $\chi(x) = \begin{cases} 1, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$)? What does this say about the Fourier transform of the inhomogeneous function above (assuming that it exists)?] Consider both $b > 0$ and $b < 0$. What is the behaviour of the solution in the limit $t \to +\infty$? How does it depend on b?

33. [This problem is interesting but less relevant than the others for exam preparation.] Redo 32, but with the initial data multiplied by $\sin 200\pi x$. Consider the dependence of the behaviour as $t \to \infty$ on b. Is there a critical value for b at which the behaviour changes drastically?

34. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n} \cdot \nabla u, \quad u|_{t=0} = e^{-|\mathbf{x}|^2}.$$

Here $\mathbf{n}$ is some fixed unit vector. How does this solution compare to the solution for the same problem without the $\mathbf{n} \cdot \nabla u$ term?