APM 346, final exam review practice problems.

1. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = x, \quad u|_{y=1} = 1-x.$$

2. Solve on $[0,2]\times[0,3]$:

$$\nabla^2 u = 0, \quad u|_{x=0} = 1-|y-1|, \quad u|_{x=2} = 0, \quad u|_{y=0} = u|_{y=3} = 0.$$

3. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 1, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0.$$

4. Solve on $[0,1]\times[0,1]$:

$$\nabla^2 u = 1, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = x, \quad u|_{y=1} = 1-x.$$

Do this twice: once by splitting up into two separate problems, and once by using a Green's function (expressed as a series in the eigenfunctions of the Laplacian on $[0,1]\times[0,1]$).

5. Solve on the ball $\{(r,\theta,\phi)|r<1\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} 1-\cos\theta, & \theta\in[0,\frac{\pi}{2}] \\ 1+\cos\theta, & \theta\in[\frac{\pi}{2},\pi] \end{cases}.$$

6. Solve on the ball $\{(r,\theta,\phi)|r<2\}$:

$$\nabla^2 u = 0, \quad \left.\frac{\partial u}{\partial r}\right|_{r=2} = \begin{cases} \cos^2\theta, & \theta\in[0,\frac{\pi}{2}] \\ -\cos^2\theta, & \theta\in[\frac{\pi}{2},\pi] \end{cases}, \quad u|_{r=0} = 0.$$

7. Solve on the shell $\{(r,\theta,\phi)|1<r<2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \cos\theta, & \theta\in[0,\frac{\pi}{2}] \\ -\cos\theta, & \theta\in[\frac{\pi}{2},\pi] \end{cases}, \quad u|_{r=2} = \begin{cases} \cos\theta\sin 2\phi, & \theta\in[0,\frac{\pi}{2}] \\ -\cos\theta\cos 2\phi, & \theta\in[\frac{\pi}{2},\pi] \end{cases}.$$

[Hint: Use Legendre's equation!] This problem requires heavy use of Legendre polynomial identities. It is nevertheless good preparation for the exam.

8. Solve on the cylinder $\{(\rho,\phi,z)|\rho<1, 0<z<2\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0, \quad u|_{z=0} = 0, \quad u|_{z=2} = 1.$$

9. Solve on the cylinder $\{(\rho,\phi,z)|\rho<2, 0<z<3\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=2} = 0, \quad u|_{z=0} = \rho^3\cos 3\phi, \quad u|_{z=3} = \rho^2\sin 2\phi.$$

10. Solve on the cylinder $\{(\rho,\phi,z)|\rho<4, 0<z<1\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u|_{\rho=4} = \phi(2\pi-\phi)z(1-z).$$

11. Solve on the cylinder $\{(\rho, \phi, z) | \rho < 2, 0 < z < 4\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = 1, \quad u|_{z=4} = \rho^3 \sin 3\phi, \quad u|_{\rho=2} = \sin 2\phi \sin 16\pi z.$$

12. Solve on the cube $\{(x, y, z) | 0 < x, y, z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0, \quad u|_{z=0} = 0, \quad u|_{z=1} = \sin x \sin y.$$

13. Solve on the cube $\{(x, y, z) | 0 < x, y, z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \quad u|_{y=0} = \sin \pi x \sin \pi z, \quad u|_{y=1} = \sin 3\pi x \sin 3\pi z,$$
$$u|_{z=0} = \sin 2\pi x \sin 2\pi y, \quad u|_{z=1} = x(1-x)y(1-y).$$

14. The same as 13, except that the conditions on $x = 0$ and $x = 1$ are replaced by

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0.$$

15. Solve on the cube $Q = \{(x, y, z) | 0 < x, y, z < 1\}$:

$$\nabla^2 u = x(1-x)y(1-y)z(1-z), \quad u|_{\partial Q} = 0.$$

16. Solve on the cube $Q = \{(x, y, z) | 0 < x, y, z < 1\}$:

$$\nabla^2 u = xy(1-y)z(1-z), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0,$$

where $\frac{\partial}{\partial n}$ denotes the outwards normal derivative at the boundary of ∂Q.

17. Solve on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = r \cos \theta, \quad u|_{r=1} = 0.$$

18. Solve on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = r^2 \sin \theta \cos \theta \sin \phi, \quad u|_{r=1} = 0.$$

19. Solve on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = r \sin \theta \cos \phi, \quad u|_{r=1} = \cos \theta.$$

Try doing this problem two ways, one by splitting it up into the sum of two separate problems, and the other by using an appropriate Green's function.

20. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \rho^3 \sin 3\phi (1-z), \quad u|_{\partial C} = 0.$$

21. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \left\{ \begin{array}{ll} \rho^3 \sin 3\phi, & \rho \in [0, \frac{1}{2}] \\ \rho^4 \cos 4\phi, & \rho \in [\frac{1}{2}, 1] \end{array} \right\} (1 - z), \quad u|_{\partial C} = 0.$$

22. Solve on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = \left\{ \begin{array}{ll} \rho^2 \cos 2\phi, & \rho \in [0, \frac{1}{2}] \\ 0, & \rho \in [\frac{1}{2}, 1] \end{array} \right\} \sin z, \quad u|_{z=0} = 1, \quad u|_{z=1} = \rho^2 \cos 2\phi, \quad u|_{\rho=1} = 0.$$

Again, try doing this two ways, one by directly writing out an orthogonal expansion and the other by using an appropriate Green's function.

23. Repeat the previous eight problems, but instead of solving Poisson's equation $\nabla^2 u = f$ solve the problem on $(0, +\infty) \times X$ (where X is either Q, B, or C as appropriate)

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f,$$

with the boundary conditions on ∂X unchanged. What is the behaviour of the solutions in the limit $t \to \infty$?

24. Again repeat the same eight problems, but now instead of solving the heat equation as in 23, solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

with f taken alternatively as the initial data for u and $\frac{\partial u}{\partial t}$ at $t = 0$, with the other one set to zero there.

25. Repeat problems 20 – 22, dropping the z dependence, on the unit disk $D = \{(\rho, \phi) | \rho < 1\}$, and then solve the corresponding heat and wave equation problems as in 23 and 24. For the wave equations, comment on the lowest frequency appearing. Can you say anything about the which frequency will be the loudest (i.e., have the largest coefficient in the orthogonal expansion)?

26. Solve on $\mathbf{R}^1$:

$$\nabla^2 u = \left(4x^2 - 2\right) e^{-x^2}, \quad \lim_{|x| \to \infty} u = 0.$$

Try using Fourier transforms in space. (There is actually a much easier way of solving this problem which doesn't require anything more than elementary calculus; can you see it? Even if you can, try doing this using Fourier transforms anyway as it is good practice.)

27. Solve on $\mathbf{R}^3$:

$$\nabla^2 u = e^{-|\mathbf{x}|^2}, \quad \lim_{|\mathbf{x}| \to \infty} u = 0.$$

You can do this either using Fourier transforms or the Green's function on $\mathbf{R}^3$ which we derived in class.

28. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = e^{-|\mathbf{x}|^2} \sin x.$$

(Here x is the first component of $\mathbf{x} = (x, y, z)$.) Hint: write $\sin x$ in terms of complex exponentials and use properties of the Fourier transform.

29. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = \left\{ \begin{array}{ll} 1, & x, y, z \in [-1, 1] \\ 0, & \text{otherwise} \end{array} \right. .$$

Express your answer in terms of the function (related to the error function)

$$E(x) = \int_0^x e^{-u^2}\, du.$$

30. Repeat the previous two problems, but with the initial data taken as the inhomogeneous term f for the equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + f$$

and with the initial data $u|_{t=0} = 0$. What is the behaviour of the solutions as $t \to \infty$?

31. Do problems 7 and 8 from the week 12 practice problem sheet, if you have not already done so. Then redo them, changing which of $u|_{t=0}$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0}$ is set to zero.

32. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + bu, \quad u|_{t=0} = \frac{\sin 2\pi x \sin 2\pi y \sin 2\pi z}{xyz}.$$

[Hint: what is the inverse Fourier transform of the function $\chi(x)\chi(y)\chi(z)$ (where $\chi(x) = \begin{cases} 1, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$)? What does this say about the Fourier transform of the inhomogeneous function above (assuming that it exists)?] Consider both $b > 0$ and $b < 0$. What is the behaviour of the solution in the limit $t \to +\infty$? How does it depend on b?

33. [This problem is interesting but less relevant than the others for exam preparation.] Redo 32, but with the initial data multiplied by $\sin 200\pi x$. Consider the dependence of the behaviour as $t \to \infty$ on b. Is there a critical value for b at which the behaviour changes drastically?

34. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n} \cdot \nabla u, \quad u|_{t=0} = e^{-|\mathbf{x}|^2}.$$

Here $\mathbf{n}$ is some fixed unit vector. How does this solution compare to the solution for the same problem without the $\mathbf{n} \cdot \nabla u$ term?