APM 346, practice problems for week 12.

1. Solve the following problem on D, using the eigenfunctions for the Laplacian on D which we found in class:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 1, \quad u|_{\partial D} = 0.$$

2. Same as 1, but with the initial condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \rho^4 \cos 4\phi$.

3. Solve on D:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = \rho^2 \sin 2\phi, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \rho^3 \cos 3\phi, \quad u|_{\partial D} = 0.$$

4. Solve on D:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + u, \quad u|_{t=0} = 1, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{\partial D} = 0.$$

What is the behaviour of the solution in the limit $t \to \infty$?

5. Same as 4, but with the initial conditions

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 1.$$

6. Repeat the previous two problems on the unit ball B instead of D.

7. Solve the following problem on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u(0, x, y, z) = \begin{cases} 1, & \sqrt{x^2 + y^2 + z^2} \in [\frac{1}{2}, 1] \\ 0, & \text{otherwise} \end{cases}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

8. Solve on $(0, +\infty) \times \mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = e^{-|\mathbf{x}|^2}.$$

[Hint: it may be easier to solve this problem by working directly with Fourier transforms rather than with the representation formula given in class.]

[The following two problems develop concepts related to the wave equation which are useful and important but are not considered examinable material for this course.]

9. Suppose that $u = e^{2\pi i f t - 2\pi i \mathbf{k} \cdot \mathbf{x}}$ is a solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$ on $\mathbf{R}^3$. What relation must f and $\mathbf{k}$ satisfy? (This is known as a *dispersion relation.*) Solutions of this form are known as *plane waves* since, for fixed t, the surfaces of constant u are planes $\mathbf{k} \cdot \mathbf{x} = \text{constant}$.

10. Similarly, suppose that $\mathbf{e} = j_\ell(\lambda r) P_{\ell m}(\cos\theta) \cos m\phi$ is an eigenfunction of the Laplacian ∇^2 on $\mathbf{R}^3$ (so that now λ may be any real number). Find a relation between f and λ such that $e^{2\pi i f t}\mathbf{e}$ is a solution of the wave equation on $\mathbf{R}^3$. (These expressions are known as *spherical waves.*)