APM 346, practice problems for week 11.

1. Redo problem 1 on Homework 11, but with your own functions for the nonhomogeneous term and the boundary data. Also try using the Dirichlet Green's function and solving with Dirichlet boundary data instead of Neumann. (Examples of nonhomogeneous terms might be the following: x, y, z, xy^2, xz^2, xyz, ye^x, $y\cos x$, $y\cos x\cos 2z$. Examples of boundary data might be the same functions with perhaps one coordinate set to 1, or anything similar to that.)

2. Using the heat kernel we found in class, solve the following problem on $\mathbf{R}^m$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f,$$

where f is respectively the following functions: $e^{-|\mathbf{x}|^2}$, $x_j e^{-|\mathbf{x}|^2}$, $\mathbf{n}\cdot\mathbf{x}e^{-4|\mathbf{x}|^2}$, $e^{-|\mathbf{x}|^2}\sin(\mathbf{n}\cdot\mathbf{x})$ [hint: complex exponentials!], $|\mathbf{x}|^2 e^{-|\mathbf{x}|^2}$. Here $\mathbf{n}$ is any fixed vector in $\mathbf{R}^m$. [Hint: work at the level of Fourier transforms, and simplify until you start to recognise the quantity at hand as the Fourier transform of a known function.]

3. Work out the solution to the optional problem on Homework 11, if you have not already done so. (Note that this can be done just using the information in the problem; one does not need to know anything else about the wave equation – and in fact most of what we shall do in class will be related to the wave equation on $\mathbf{R}^3$, not on a bounded region.)

4. Starting from separation of variables, work out the eigenvalues and eigenfunctions for the Laplacian on the cylinder $\{(\rho,\phi,z)|\rho < a, 0 \le z \le 1\}$ with homogeneous Dirichlet conditions, where $a > 0$ is any positive real number. (If you prefer, work this problem with a set equal to some constant number, say 2.)

5. Working along the lines of 4, find the eigenvalues and eigenfunctions for the Laplacian on the disk $\{(\rho,\phi)|\rho < a\}$ (where again you may take a to be some fixed real number, if you like).

6. Using the results of 5, work as in 3 to determine the possible frequencies f for which there is a solution of the form $u(t,\mathbf{x}) = e^{2\pi i f t} g(\mathbf{x})$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0$$

with homogeneous Dirichlet boundary conditions on the disk $D = \{(\rho,\theta)|\rho < a\}$. (Again, take a to be some specific real number, if you like.)

7. Using 6, provide a partial explanation as to why a bigger drum has a lower pitch. [Hint: low pitch corresponds to small f.]

8. Redo question 6, but now look instead for solutions of the form $u(t,\mathbf{x}) = e^{2\pi i f t} e^{\nu t} g(\mathbf{x})$, where ν is real, to the equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + bu, \quad u|_{\partial D} = 0.$$

(You may work on the unit disk – setting $a = 1$ – for simplicity.) What condition must b satisfy in order for such solutions to exist? How does this relate to the conditions which b must satisfy for an equation of the form

$$\frac{d^2 x}{dt^2} = bx$$

to have solutions of the form $x = e^{\nu t}$, ν real? [Terms of the form bu, where b satisfies such a condition, act to damp the solutions in time.] Are there values of b for which there is a solution of the above form with $f = \nu = 0$? How does this relate to our previous work (specifically with eigenfunctions)?

9. Suppose that you measure the temperature of a unit cylinder and find that it obeys the relation

$$u(t, \rho, \phi, z) = 1 - \rho^2,$$

where t is some particular time. How do you expect the quantity $u(t, 0, 0, 0)$ to change as t increases from this point? In particular, is it possible that the above u represents a steady-state solution (i.e., one where $\frac{\partial u}{\partial t} = 0$)? [Hint: remember that the temperature satisfies the heat equation!]

10. Using the definition of the Fourier transform and the properties of the Dirac delta function, work formally to calculate $\mathcal{F}[\delta(\mathbf{x})](\mathbf{k})$ on $\mathbf{R}^m$. Proceeding in the same way, calculate $\mathcal{F}[\delta(\mathbf{x} - \mathbf{x}')](\mathbf{k})$ for some fixed $\mathbf{x}' \in \mathbf{R}^m$.

11. Use Fourier transforms in space to solve the problem on $\mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u + u, \quad u|_{t=0} = f.$$

(Here and in the next problem, by 'solve' we mean find an expression for u as an integral in f, similar to what we did with the homogeneous heat equation in class and in the lecture notes.) What can you say about the limiting behaviour of the solution as $t \to \infty$?

12. Use Fourier transforms in space to solve the problem on $\mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \partial_j u, \quad u|_{t=0} = f,$$

and more generally

$$\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n} \cdot \nabla u, \quad u|_{t=0} = f,$$

where $\mathbf{n}$ is some vector. What can you say in these cases about the limiting behaviour of u as $t \to \infty$?

13. Repeat the previous two problems, but instead of working on $\mathbf{R}^m$ work on one of the bounded sets Q, C, B for which we have calculated the eigenfunctions and eigenvalues of the Laplacian subject to homogeneous Dirichlet boundary conditions, and assume that the solutions to the equations satisfy homogeneous Dirichlet boundary conditions, as well, for all time. In problem 11, consider the more general equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + au, \quad u|_{t=0} = f,$$

where a is some real number. Consider the limiting behaviour of the solutions to these equations as $t \to \infty$, and in particular how this behaviour depends on a. Is there a 'critical' value for a at which the behaviour changes? [In this problem, by 'solve' we mean find an expression for u as a series expansion whose coefficients depend on inner products involving f – in other words, the solution will still be at a fairly abstract level, since we really don't know anything about f.]