APM 346, practice problems for week 10.

1. Consider problem 1 from Homework 10. If the Poisson equation were replaced by

$$\nabla^2 u = \begin{cases} 2, & 0 \le z < \frac{1}{2}, \\ -1, & \frac{1}{2} < z \le 1 \end{cases},$$

would the problem still have a solution? Also, what role does the requirement $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$ play in the problem? [Hint: does the Laplacian have a zero eigenvalue in this case? What does that mean about the possibility of solving Poisson's equation?]

2. Find the eigenfunctions and eigenvalues for the Laplacian on the unit ball with homogeneous Neumann conditions; i.e., solve the problem on B

$$\nabla^2 u = \lambda u, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial B} = 0.$$

[It turns out that these eigenfunctions also give a complete orthogonal set on the unit ball; in fact, the same is true with various Robin conditions as well.]

3. Using the expression for the Green's function on $\mathbf{R}^3$ derived in class (and the lecture notes), solve the following problem on $\mathbf{R}^3$:

$$\nabla^2 u = \begin{cases} 1, & x^2+y^2+z^2 < 1 \\ 0, & x^2+y^2+z^2 > 1 \end{cases}, \quad u \to 0 \,\text{as}\, \mathbf{x} \to \infty.$$

4. Similarly, solve the following problem on $\mathbf{R}^3$ for $a > 0$:

$$\nabla^2 u = e^{-a|\mathbf{x}|}, \quad u \to 0 \,\text{as}\, \mathbf{x} \to \infty.$$

5. We provide a simple introduction to the method of images. Consider the following problem on the halfspace $H = \{(x, y, z)|z > 0\}$:

$$\nabla^2 u = f, \quad u|_{\partial H} = 0.$$

By our work in class (and in the lecture notes), an appropriate Green's function for this problem would solve the problem

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}'), \quad G|_{\mathbf{x} \in \partial H} = 0.$$

The idea behind the method of images is to write the Green's function as a sum of *two* free-space Green's functions, which exactly cancel on the boundary. (Note that it is not clear *a priori* that such a thing is actually possible; but it turns out that it is for a plane – as here – and also for a sphere – see Example 8.2.1 in the text.) The second one will correspond to a point outside of the region H, which means that it will satisfy Laplace's equation in H and hence won't disturb the first equation for G above. Putting all of this together, we seek a point $\mathbf{x}^* = (x^*, y^*, z^*)$ with $z^* < 0$ such that

$$\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi|\mathbf{x} - \mathbf{x}^*|} = 0$$

for any $\mathbf{x} \in \partial H$. ($\mathbf{x}^*$ may depend on $\mathbf{x}'$.) Show that such an $\mathbf{x}^*$ does indeed exist and find a formula for it in terms of $\mathbf{x}'$; then use the resulting Green's function to solve the problem on H (i.e., $z > 0$)

$$\nabla^2 u = \begin{cases} 1, & x^2+y^2+z^2 < 1 \\ 0, & x^2+y^2+z^2 > 1 \end{cases}, \quad u|_{\partial H} = 0.$$

6. Repeat 5, but with homogeneous Neumann conditions instead of Dirichlet conditions.

7. Consider the following sequence of functions:

$$f_n(x) = \begin{cases} 1, & x \in [-n, n] \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathcal{F}[f_n](k)$. What can you say about the limit of $\mathcal{F}[f_n](k)$ as $n \to \infty$? [Hint: does it behave like a delta function, possibly after a rescaling?]

8. Same as 7, but with the sequence

$$f_n(x) = e^{-\frac{1}{n}x^2}.$$

9. Can you extend the previous two problems to $\mathbf{R}^3$?