APM 346, practice problems for week 9.

1. Solve the following problem on $C = \{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = f, \quad u|_{\partial C} = 0$$

with (a) $f = \sin z$, (b) $f = \chi_{[\frac{1}{3}, \frac{2}{3}]}(z)$ (i.e., the function which is 1 on $[\frac{1}{3}, \frac{2}{3}]$ and zero elsewhere), (c) $f = \rho^2 \sin 2\phi$, (d) $f = \begin{cases} \rho^2 \sin 2\phi, & 0 < \rho < \frac{1}{2} \\ \rho^2 \cos 2\phi, & 0 < \rho < \frac{1}{2} \end{cases}$, (e) Your own f! [Try mixing the z and ρ, ϕ dependence of various parts of this question if you like.]

2. Solve the following problem on C:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial C} = 0,$$

taking for f a couple of the functions from problem 1.

3. Solve the following problem on $B = \{(r, \theta, \phi) | r < 1, 0 < z < 1\}$:

$$\nabla^2 u = f, \quad u|_{\partial B} = 0$$

with (a) $f = 1$, (b) $f = 3z^2 - 1$ ($z = r\cos\theta$), (c) $f = \begin{cases} 0, & 0 < r < \frac{1}{2} \\ r\cos\theta, & \frac{1}{2} < r < 1 \end{cases}$, (d) Look up $P_{\ell m}$ for some higher values of ℓ and m and make up your own f.

4. Solve the following problem on B:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial B} = 0,$$

taking for f a couple of the functions from problem 3.

5. Solve the following problem on C:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = 0, \quad u|_{(0,+\infty)\times\partial C} = \begin{cases} 0, & z \neq 1 \\ \rho^3 \cos 3\phi, & z = 1 \end{cases}.$$

[Hint: this still requires solving two problems, even though the initial data is homogeneous.]

6. Same as 5, but with $u|_{t=0} = f$, where f is one of the functions in problem 1 above.

7. Solve the following problem on B:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = 0, \quad u|_{(0,+\infty)\times\partial B} = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < \theta \le 1 \end{cases}.$$

[Hint: we have solved the ancillary boundary-value problem for Laplace's equation before.]

8. Same as 7, but with $u|_{t=0} = f$, where f is one of the functions in problem 3 above.

9. Formulate and solve additional problems similar to the previous two, by putting together problems we have solved (or know how to solve, at any rate) for Laplace's equation with various kinds of initial data we have used for solving the heat equation. (Or, just take $u|_{t=0} = 0$ as in the first two problems above!)