APM 346, practice problems for week 7.

1. Solve on $\{(\rho, \phi, z) | \rho < 3, 0 < z < 4\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=4} = 0, \quad u|_{\rho=3} = \begin{cases} \sin\phi, & 0 < \phi < \pi \\ 0, & \pi < \phi < 2\pi \end{cases}$$

2. Same as 1, but with $u|_{z=0} = u|_{z=4} = 1$. [Hint: try subtracting off a trivial solution of Laplace's equation to turn this into a homogeneous condition.]

3. Solve on $\{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = 0, \quad u|_{z=1} = -\left.\frac{\partial u}{\partial z}\right|_{z=1}, \quad u|_{\rho=3} = \sin 2\phi \cos z.$$

4. Solve on $\{(\rho, \phi, z) | \rho < 1, 0 < z < 1, 0 < \phi < \frac{\pi}{4}\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0, \quad u|_{\phi=0} = u|_{\phi=\frac{\pi}{4}} = 0, \quad u|_{z=0} = 1, \quad u|_{z=1} = \rho^4 \sin 4\phi.$$

[Hint: Start directly from separation of variables. Also, recall that $\{J_m(\lambda_{mi}\rho)\}_{i=1}^{\infty}$ is a complete orthogonal set on $[0, 1]$ for any value of m.]

5. Solve on $\{(\rho, \phi, z) | \rho < 1, 0 < z < 1, 0 < \phi < \frac{\pi}{3}\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u|_{\phi=0} = u|_{\phi=\frac{\pi}{3}} = 0, \quad u|_{\rho=1} = (1 - \cos 6\phi)e^z.$$

6. And finally, everything rolled all together into one big problem! Solve on $\{(\rho, \phi, z) | \rho < 2, 0 < z < 4, 0 < \phi < \frac{\pi}{6}\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = \rho^{12} \sin 12\phi, \quad u|_{z=4} = \rho^{24} \cos 24\phi, \quad u|_{\phi=0} = u|_{\phi=\frac{\pi}{6}} = 0, \quad u|_{\rho=2} = \phi\left(\phi - \frac{\pi}{6}\right)e^z.$$