APM 346, practice problems.

1. Solve on $\{(x,y)|0 \le x \le 1,\ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0,\ u|_{y=0} = \sin \pi x,\ u|_{y=1} = \sin 2\pi x.$$

2. Solve on $\{(x,y)|0 \le x \le 2,\ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{y=0} = u|_{y=1} = 0, \quad u|_{x=0} = 1,\ u|_{x=2} = \sin \pi y.$$

3. Same as 2, but with $u|_{x=2} = y$.
4. Redo the previous problems with u in one of the inhomogeneous boundary conditions replaced by its normal derivative (e.g., for problem 1, take $u_y|_{y=0} = \sin \pi x$).
5. Same as 3 (or, if you like, 1 or 2), with $u|_{y=0} = u|_{y=1} = 0$ replaced by $u|_{y=0} = 0$, $u|_{y=1} = -u_y|_{y=1}$ (or, if you are doing 1, replace y by x in the foregoing). [Hint: Homework 4!]
6. Solve on $\{(r,\theta,\phi)|r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=2} = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{3} \\ 1, & \frac{\pi}{3} < \theta < \frac{2\pi}{3} \\ 0, & \frac{2\pi}{3} < \theta < \pi \end{cases}.$$

[Hint: Homework 5!]
7. Same as 6, but with the boundary data multiplied by $\cos\theta$. [Hint: Quiz 5!]
8. Solve on $\{(r,\theta,\phi)|r < 1\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \sin\theta\cos\phi, & 0 \le \theta < \frac{\pi}{2}, \\ -\sin\theta\cos\phi, & \frac{\pi}{2} < \theta < \pi \end{cases}.$$

[Hint: Use the identity $\left(1-x^2\right)P_\ell' = \ell P_{\ell-1} - \ell x P_\ell$.]
9. Same as 8, but with the boundary condition applied to $u_r|_{r=1}$, and with the additional condition that $u|_{r=0} = 0$.
10. Solve on $\{(r,\theta,\phi)|1 < r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \cos 2\phi, & 0 \le \theta < \frac{\pi}{2} \\ \sin 2\phi, & \frac{\pi}{2} < \theta < \pi \end{cases},\ u|_{r=2} = \begin{cases} \sin 2\phi, & 0 \le \theta < \frac{\pi}{2} \\ \cos 2\phi, & \frac{\pi}{2} < \theta < \pi \end{cases}.$$

[Hint: Can you find $P_{\ell,2}$ sitting somewhere inside the equation satisfied by P_ℓ? If you can, it is just a matter of manipulating the identities for the P_ℓ and P_ℓ' to integrate the resulting expressions. But you need multiple identities and probably multiple applications of them, so this is a long problem. Beyond that, the suggestion after the last problem below may be helpful for this problem also.]
11. Make up additional problems of the foregoing type by replacing u by u_r and mixing up the different kinds of boundary data – and then solve them!
12. Solve on $\{(\rho,\phi,z)|\rho < 1, 0 \le z \le 2\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0, u|_{z=0} = 0,\ u|_{z=2} = \rho^2 \sin 2\phi.$$

13. Solve on $\{(\rho,\phi,z)|\rho < 1, 0 \le z \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0,\ u|_{z=0} = \rho^3 \sin 3\phi,\ u|_{z=1} = \rho^2 \cos 2\phi.$$

14. Solve on $\{(\rho,\phi,z)|\rho < 1, 0 \le z \le 2\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0,\ u|_{z=0} = \begin{cases} -\rho^3 \sin 3\phi, & 0 \le \rho < \frac{1}{2} \\ \rho^3 \sin 3\phi, & \frac{1}{2} < \rho \le 1 \end{cases},\ u|_{z=1} = \begin{cases} 2\rho^4 \cos 4\phi, & 0 \le \rho < \frac{1}{2} \\ -\rho^4 \cos 4\phi, & \frac{1}{2} < \rho \le 1 \end{cases}.$$

15. Solve on $\{(\rho,\phi,z)|\rho < 1, 0 \le z \le 3\}$:

$$\nabla^2 u = 0, \quad u|_{\rho=1} = 0,\ u|_{z=0} = \begin{cases} -\rho^3 \sin 3\phi, & 0 \le \rho < \frac{1}{2} \\ \rho^2 \cos 2\phi, & \frac{1}{2} < \rho \le 1 \end{cases},\ u|_{z=1} = \begin{cases} \rho \sin\phi, & 0 \le \rho < \frac{1}{2} \\ \rho^4 \cos 4\phi, & \frac{1}{2} < \rho \le 1 \end{cases}.$$

[This is a very hard (not to mention long) problem. One method is to break it up into four problems, where each one has only one of the four pieces of boundary data above, with all other boundary data set to zero. Each one of these can be solved reasonably straightforwardly, and then their sum gives the solution to the above problem.]