APM 346, practice problems for week 4.

1. Recall the decomposition of x^4 in terms of Legendre polynomials. Use this to solve the following boundary-value problem on the unit ball:

$$\nabla^2 u = 0, \qquad u|_{r=1} = \cos^4 \theta.$$

2. Now solve problem 1, but on the complement of the unit ball (i.e., the set $\{(r, \theta, \phi) | r \geq 1\}$). [Hint: this solution can be written down almost immediately given that to 1, after a little thought.]

3. Solve on $\{(r, \theta, \phi) | r \leq 2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = \cos \theta.$$

4. Solve on $\{(r, \theta, \phi) | \frac{1}{2} \leq r \leq 1\}$:

$$\nabla^2 u = 0, \qquad u_r|_{r=1} = 1, \quad u|_{r=\frac{1}{2}} = 0.$$

5. Same as 4, but on the exterior of the unit ball and without the second boundary condition. Is this solution unique?

6. Solve on $\{(r, \theta, \phi) | 1 \leq r \leq 2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = \cos \theta, \quad u|_{r=1} = 0.$$

7. Solve on $\{(r, \theta, \phi) | \frac{1}{2} \leq r \leq 1\}$:

$$\nabla^2 u = 0, \qquad u|_{r=\frac{1}{2}} = 1, \quad u|_{r=1} = 0.$$

[Can you see how to put these last two problems together – with a slight modification of 7, of course – to solve problem 5 on Homework 4?]

8. Make up your own linear combination of Legendre polynomials, call it $f(x)$, and solve Laplace's equation on the unit ball subject to $u|_{r=1} = f(\cos \theta)$.

9. Same as 8, but instead of solving on the unit ball solve on the shell $\{(r, \theta, \phi) | 2 \leq r \leq 3\}$, with the conditions $u|_{r=2} = f(\cos \theta)$, $u|_{r=3} = 0$.