We review the definition and an elementary property of the Wronskian. We recall that the notation $f^{(n)}(x)$ denotes the nth derivative of the function f.

DEFINITION. Let $f_1, \ldots, f_n : (a, b) \to \mathbf{R}$, $a, b \in \mathbf{R} \cup \{-\infty, +\infty\}$, and suppose that the first $n-1$ derivatives of all n functions exist on (a, b). Then the Wronskian of $f_1, \ldots, f_n$ is the function $W : (a, b) \to \mathbf{R}$ defined by

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}.$$

EXAMPLES.

(a) Let $(a, b) = \mathbf{R}$, $f_1(x) = x$, $f_2(x) = x^2$. Then the Wronskian of f_1 and f_2 is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} \\ &= 2x^2 - x^2 = x^2. \end{aligned}$$

(b) Let $(a, b) = \mathbf{R}$, $f_1(x) = e^x$, $f_2(x) = e^{-x}$. Then the Wronskian of f_1 and f_2 is

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= -1 - 1 = -2. \end{aligned}$$

The importance of the Wronskian can be seen from the following proposition.

PROPOSITION. Suppose that the functions $f_1, f_2, \ldots, f_n : (a, b) \to \mathbf{R}$ possess derivatives of up to order $n-1$ and are linearly dependent on (a, b). Then their Wronskian is zero everywhere on (a, b).

Proof. Since the functions $f_1, f_2, \ldots, f_n$ are linearly dependent on (a, b), there must exist constants $c_1, c_2, \ldots, c_n$ such that for all $x \in (a, b)$ we have

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0.$$

Since $c_1, c_2, \ldots, c_n$ are all constants, we may differentiate this equation k times to obtain

$$c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \cdots + c_n f_n^{(k)}(x) = 0,$$

where $k = 1, \ldots, n-1$. Thus we see that for each $x \in (a, b)$, the vectors

$$\begin{pmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(n-1)}(x) \end{pmatrix}, \quad \begin{pmatrix} f_2(x) \\ f_2'(x) \\ \vdots \\ f_2^{(n-1)}(x) \end{pmatrix}, \quad \ldots \quad \begin{pmatrix} f_n(x) \\ f_n'(x) \\ \vdots \\ f_n^{(n-1)}(x) \end{pmatrix}$$

are linearly dependent. Thus the matrix

$$D = \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}$$

is not full-rank, so its determinant $|D|$ must be zero. But $|D|$ is exactly the Wronskian of $f_1, f_2, \ldots, f_n$, so this completes the proof of the proposition. QED.

From this it follows that if the Wronskian is not identically zero on (a, b), then $f_1, f_2, \ldots, f_n$ must be linearly independent on (a, b).

EXAMPLES.

(c) From examples (a) and (b) above, we see that x and x^2 are linearly independent on $\mathbf{R}$, as are e^x and e^{-x}.