Generalities; Laplace's equation

If $\{e_\alpha\}$ is a complete, orthogonal set with respect to an inner product $(\cdot,\cdot)$, then any f can be written $f = \sum_\alpha a_\alpha e_\alpha$, where $a_\alpha = \frac{(f,e_\alpha)}{(e_\alpha,e_\alpha)}$.

Laplace's equation $\nabla^2 u = 0$ has the following general series expansions as its solutions when solved in the indicated regions and with the indicated boundary conditions:

Region and boundary conditions	Series expansion, related complete orthogonal set, and inner product
$\{(x,y)\|0 \le x \le 1,\ \ 0 \le y \le 1\}$ $u\|_{x=0} = u\|_{x=1} = 0$	$u = \sum_{n=0}^{\infty} \sin n\pi x \,(a_n \sinh n\pi y + b_n \cosh n\pi y)$ $\{\sin n\pi x\}_{n=1}^{\infty}, (f(x), g(x)) = \int_0^1 f(x)\overline{g(x)}\,dx$
$\{(r,\theta,\phi)\|r \le a\}$ azimuthally symmetric u finite and single-valued	$u = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta) r^\ell$ $\{P_\ell(\cos\theta)\}_{\ell=0}^{\infty}, (f(x), g(x)) = \int_{-1}^1 f(x)\overline{g(x)}\,dx, (f(\theta), g(\theta)) = \int_0^\pi f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$
$\{(r,\theta,\phi)\|a \le r \le b\}$ azimuthally symmetric u finite and single-valued	$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right)$ $\{P_\ell(\cos\theta)\}_{\ell=0}^{\infty}, (f(x), g(x)) = \int_{-1}^1 f(x)\overline{g(x)}\,dx, (f(\theta), g(\theta)) = \int_0^\pi f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$
$\{(\rho,\phi,z)\|\rho \le a,\ \ 0 \le z \le z_0\}$ azimuthally symmetric $u\|_{\rho=a} = 0$, u finite	$u = \sum_{i=1}^{\infty} J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\left(a_i \cosh\frac{\lambda_{0i}}{a}z + b_i \sinh\frac{\lambda_{0i}}{a}z\right)$ $\{J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\}_{i=1}^{\infty},\ (f(\rho), g(\rho)) = \int_0^a f(\rho)\overline{g(\rho)}\rho\,d\rho$ λ_{mi}, $m \in \mathbf{Z}$, $m \ge 0$, $i \in \mathbf{Z}$, $i \ge 1$ denotes the ith *positive* zero of $J_m(x)$
$\{(r,\theta,\phi)\|a \le r \le b\}$ u finite and single-valued	$u = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\,(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi)\left(c_{\ell m} r^\ell + d_{\ell m} r^{-(\ell+1)}\right)$ $\{P_{\ell m}(\cos\theta)\cos m\phi, P_{\ell m}(\cos\theta)\sin m\phi \| \ell \in \mathbf{Z}, \ell \ge 0, m \in \mathbf{Z}, 0 \le m \le \ell\}$ $(f(\theta,\phi), g(\theta,\phi)) = \int_0^\pi\int_0^{2\pi} f(\theta,\phi)\overline{g(\theta,\phi)}\sin\theta\,d\phi\,d\theta$
$\{(\rho,\phi,z)\|\rho \le a,\ \ 0 \le z \le z_0\}$ $u\|_{\rho=a} = 0$, u finite	$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{\lambda_{mi}}{a}\rho\right)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\left(c_{mi}\cosh\frac{\lambda_{mi}}{a}z + d_{mi}\sinh\frac{\lambda_{mi}}{a}z\right)$ $\{J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi, J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi \| m \in \mathbf{Z}, m \ge 0, i \in \mathbf{Z}, i \ge 1\}$ $(f(\rho,\phi), g(\rho,\phi)) = \int_0^a\int_0^{2\pi} f(\rho,\phi)\overline{g(\rho,\phi)}\rho\,d\phi\,d\rho$ λ_{mi}, $m \in \mathbf{Z}$, $m \ge 0$, $i \in \mathbf{Z}$, $i \ge 1$ denotes the ith *positive* zero of $J_m(x)$

In all cases, solving Laplace's equation proceeds as follows:

1. Determine the correct coordinate system and boundary conditions (including azimuthal symmetry or lack thereof).
2. Assuming this corresponds to an entry in the above table, write down the corresponding general series expansion.
3. Apply the remaining boundary conditions to this series and equate the result to the given boundary data to determine the expansion coefficients.

For the first four examples above, the boundary data is essentially one-dimensional, so that only one set of integrals occurs in step 3. In the last two examples, the expansion part of step 3 can be split into two steps, as follows:

3.1. Expand in ϕ for fixed θ (resp. ρ) to obtain θ- (resp. ρ-) dependent coefficients a_m, b_m.

3.2. Expand a_m and b_m in the basis $\{P_{\ell m}(\cos\theta)\}_{\ell=m}^{\infty}$ (resp. $\{J_m\left(\lambda_{mi}\frac{\rho}{a}\right)\}_{i=1}^{\infty}$; both of these are complete orthogonal sets) to obtain the final expansion coefficients $a_{\ell m}$, $b_{\ell m}$ (resp. a_{mi}, b_{mi}).

Special functions: equations and properties

Associated Legendre functions. These are solutions $P_{\ell m}(x)$, $\ell \in \mathbf{Z}$, $\ell \geq 0$, $m \in \mathbf{Z}$, $0 \leq m \leq \ell$ to the equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$

which are finite at $x = 0$. For fixed m, the set $\{P_{\ell m}(x)\}_{\ell=m}^{\infty}$ is complete and orthogonal on the interval $[-1,1]$ with respect to the inner product $(f(x), g(x)) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$; equivalently, $\{P_{\ell m}(\cos\theta)\}_{\ell=m}^{\infty}$ is complete and orthogonal (in θ) on the interval $[0,\pi]$ with respect to the inner product $(f(\theta), g(\theta)) = \int_0^{\pi} f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$. They have normalisation

$$\int_{-1}^{1} P_{\ell m}^2(x)\,dx = \int_0^{\pi} P_{\ell m}^2(\cos\theta)\sin\theta\,d\theta = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}.$$

The first few for $m > 0$ are as follows. (For $m = 0$, see the Legendre polynomials below.)

$$P_{1,1}(\cos\theta) = \sin\theta, \quad P_{2,1}(\cos\theta) = 3\sin\theta\cos\theta, \quad P_{2,2}(\cos\theta) = 3\sin^2\theta.$$

The associated Legendre functions satisfy the following relation:

$$P_{\ell m}(x) = \left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_{\ell,0}(x).$$

Legendre polynomials. When $m = 0$, the associated Legendre functions $P_{\ell,0}(x)$ are polynomials and denoted by $P_\ell(x)$. By the foregoing, they satisfy the equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

and form a complete orthogonal set on $[-1,1]$ with respect to the above-given inner product, with normalisation

$$\int_{-1}^{1} P_\ell^2(x)\,dx = \int_0^{\pi} P_\ell^2(\cos\theta)\sin\theta\,d\theta = \frac{2}{2\ell+1}.$$

The first few are as follows:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

They satisfy the following recursion and differentiation relations:

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0, \quad P'_{n+1} - 2xP'_n + P'_{n-1} = P_n, \quad xP'_n - P'_{n-1} = nP_n,$$

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n, \quad (1-x^2)P'_n = nP_{n-1} - nxP_n.$$

$P_\ell(x)$ is an odd or even function as ℓ is odd or even. Thus $P_\ell(0) = 0$ if ℓ is odd.

Bessel functions. These are solutions $J_m(x)$, $m \in \mathbf{Z}$, $m \geq 0$ to the equation

$$\frac{d^2J}{dx^2} + \frac{1}{x}\frac{dJ}{dx} + \left(1 - \frac{m^2}{x^2}\right)J = 0$$

which are finite at $x = 0$. It can be shewn that each $J_m(x)$ has infinitely many zeroes, and we denote the ith *positive* zero of J_m by λ_{mi}, $i = 1, 2, \ldots$. It can be shewn that the spacing between zeroes approaches a constant value when $i \to +\infty$, but there is no closed-form formula for them. $J_m(x)$ has the Taylor series expansion

$$J_m(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}.$$

For any positive number a and any $m \geq 0$, the set $\{J_m\left(\lambda_{mi}\frac{\rho}{a}\right)\}_{i=1}^{\infty}$ is complete orthogonal on the interval $[0,a]$ with respect to the inner product $(f(\rho), g(\rho)) = \int_0^a f(\rho)\overline{g(\rho)}\rho\,d\rho$. They have normalisation

$$\int_0^a J_m^2\left(\lambda_{mi}\frac{\rho}{a}\right)\rho\,d\rho = \frac{1}{2}a^2J_{m+1}^2\left(\lambda_{mi}\right).$$

The Bessel functions cannot be expressed in any simple way in terms of elementary functions. They satisfy the relations $(m > 0)$

$$J'_0(x) = -J_1(x),$$

$$J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x), \quad J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x}J_m(x), \quad J_{m-1}(x) = J'_m(x) + \frac{m}{x}J_m(x),$$

$$J_{m+1}(x) = -J'_m(x) + \frac{m}{x}J_m(x), \quad \frac{d}{dx}\left(x^mJ_m(x)\right) = x^mJ_{m-1}(x), \quad \frac{d}{dx}\left(x^{-m}J_m(x)\right) = -x^{-m}J_{m+1}(x).$$