We review some concepts and methodology from linear algebra.

DEFINITION. Let V and W be two vector spaces[1]. A map $T : V \to W$ is called a *linear transformation* if it satisfies $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ for all $v, w \in V$ and all scalars α, β.

DEFINITION. Let V be a vector space, and let $S \subset V$. We say that S *spans* V if for all $v \in V$ there are $w_1, \ldots, w_n \in S$ and scalars $\alpha_1, \ldots, \alpha_n$ such that $v = \alpha_1 w_1 + \cdots + \alpha_n w_n$. We say that S is *linearly independent* if for any $w_1, \ldots, w_n \in S$ the equation $\alpha_1 w_1 + \cdots + \alpha_n w_n = 0$ has only $\alpha_1 = \cdots = \alpha_n = 0$ as a solution. If S both spans V and is linearly independent then it is called a *basis* for V. In this case, the number of elements of S is called the *dimension* of V. It could be finite or infinite[2].

EXAMPLE. If $V = \mathbf{R}^n$ or $V = \mathbf{C}^n$, then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is a basis for V.

DEFINITION. Let V, W be vector spaces with bases $B = \{v_1, \ldots, v_n\}$, $D = \{w_1, \ldots, w_m\}$, and let $T : V \to W$ be a linear transformation. Then the *basis representation of T with respect to B and D*, $[T]_B^D$, is defined as follows. For each $v_k \in V$, $T(v_k) \in W$ can be expressed in a unique way as a linear combination of elements of D, say

$$T(v_k) = a_{1k} w_1 + \cdots + a_{mk} w_m.$$

We define

$$[T]_B^D = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

In linear algebra courses, one learns about the properties of these matrices, and how to transform from one basis to another, but we do not need all this at the moment.

DEFINITION. Let V be a vector space, and let $T : V \to V$. If $v \in V$, $v \neq 0$, is such that $T(v) = \lambda v$ for some scalar λ, then v is said to be an *eigenvector* of T with *eigenvalue* λ. If there is a basis $B = \{v_1, \ldots, v_n\}$ of V, each element of which is an eigenvector of T, then T is said to be *diagonalisable*.

In this case, it is not hard to see that $[T]_B^B$ is a diagonal matrix, with the kth element being the eigenvalue corresponding to v_k.

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[1] I am not going to give the formal definition of a vector space here. Roughly, a vector space is a collection of objects (which can be vectors in $\mathbf{R}^n$ but can also be other things, such as functions) which can be added and multiplied by scalars (real or complex numbers) in such a way that vector addition and scalar multiplication interact as one would expect. Those of you who have never seen abstract vector spaces can think of $\mathbf{R}^n$ or $\mathbf{C}^n$ for the time being.

[2] For the benefit of those who know a little set theory, we note that in the case of an infinite-dimensional vector space V, by 'the number of elements of S' we mean the cardinality of S. Much of the numerology of finite-dimensional linear algebra can be carried over to the infinite-dimensional case in this way. However, in this case an (algebraic) basis as defined here is not particularly useful and one prefers to use something like an orthogonal basis, as we shall see later, where one is able (essentially) to represent elements of V as infinite linear combinations of elements of S.

DEFINITION. Let V be a complex vector space. An *inner product* on V is a map $(\cdot,\cdot) : V \times V$ to $\mathbf{C}$ satisfying the following properties:

1. $(av + bw, u) = a(v, u) + b(w, u)$ for all $v, w, u \in V$ and all $a, b \in \mathbf{C}$;
2. $(v, u) = \overline{(u, v)}$ for all $v, u \in V$;
3. $(v, v) \geq 0$ for all $v \in V$, and $(v, v) = 0$ if and only if $v = 0$.

The first and second properties imply that $(\cdot,\cdot)$ is *conjugate linear* in the second argument, i.e., $(v, aw + bu) = \overline{a}(v, w) + \overline{b}(v, u)$. This is sometimes combined with property 1 above to say that $(\cdot,\cdot)$ is a *sesquilinear*[3] map. (It would be *bilinear*, i.e., linear in each argument separately, if it weren't for the conjugate on the a and b.)

The text has an introduction to inner products in section 0.3, and we shall go over similar material from a slightly different perspective in class.

[3]While I have never checked this, 'sesqui' apparently means 'one-and-a-half', as in sesquicentennial, or 150th anniversary.