Additional solutions to Laplace's equation

Laplace's equation $\nabla^2 u = 0$ has the following general series expansions as its solutions when solved in the indicated regions and with the indicated boundary conditions:

Region and boundary conditions, and dates for notes	Series expansion, related complete orthogonal set, and inner product
$\{(\rho,\phi,z)\mid\rho\le a, 0\le z\le b\}$ $u\|_{z=0} = u\|_{z=b} = 0$ July 2 – 4	$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m\left(\frac{n\pi}{b}\rho\right)(a_{nm}\cos m\phi + b_{nm}\sin m\phi)\sin\frac{n\pi}{b}z$ $\{\cos m\phi\sin\frac{n\pi}{b}z, \sin m\phi\sin\frac{n\pi}{b}z\mid n,m\in\mathbf{Z}, n\ge 1, m\ge 0\}$ $(f(\phi,z),g(\phi,z)) = \int_0^{2\pi}\int_0^b f(\phi,z)\overline{g(\phi,z)}\,dz\,d\phi$
$\{(x,y,z)\mid 0\le x,y,z\le 1\}$ $u\|_{x=0} = u\|_{x=1} = u\|_{y=0} = u\|_{y=1} = 0$ July 9 – 11	$\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sin\ell\pi x\sin m\pi y\left(a_{\ell m}\cosh\sqrt{\ell^2+m^2}\pi z + b_{\ell m}\sinh\sqrt{\ell^2+m^2}\pi z\right)$ $\{\sin\ell\pi x\sin m\pi y\mid \ell,m\in\mathbf{Z}, \ell,m\ge 1\}, (f(x,y),g(x,y)) = \int_0^1\int_0^1 f(x,y)\overline{g(x,y)}\,dx\,dy$

We may interchange x, y, and z in the last example to obtain additional solutions on the cube.

In cases where more than one set of boundary conditions is inhomogeneous, we may express the solution as a sum of two or three separate ones, each of which satisfies a problem with one set of inhomogeneous boundary conditions. See notes of July 2 – 4, pp. 3 – 6 for an example.

Eigenfunctions and eigenvalues for the Laplacian: $\nabla^2 u = \lambda u$

Region and boundary conditions, and dates for notes	Eigenfunctions, eigenvalues, and parameter ranges
$Q = \{(x,y,z)\mid 0\le x,y,z\le 1\},\ u\|_{\partial Q} = 0$ July 9 – 11	$\sin\ell\pi x\sin m\pi y\sin n\pi z,\ -\pi^2\left(\ell^2+m^2+n^2\right),\ \ell,m,n\in\mathbf{Z},\ \ell,m,n\ge 1$
$Q = \{(x,y,z)\mid 0\le x,y,z\le 1\},\ \left.\frac{\partial u}{\partial n}\right\|_{\partial Q} = 0$ [Homeworks 10 and 11]	$\cos\ell\pi x\cos m\pi y\cos n\pi z,\ -\pi^2\left(\ell^2+m^2+n^2\right),\ \ell,m,n\in\mathbf{Z},\ \ell,m,n\ge 0$
$C = \{(\rho,\phi,z)\mid\rho\le 1, 0\le z\le 1\},\ u\|_{\partial C} = 0$ July 9 – 11, 16 – 18	$J_m(\lambda_{mi}\rho)\sin n\pi z\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\lambda_{mi}^2 - n^2\pi^2,$ $m,n,i\in\mathbf{Z},\ m\ge 0,\ n,i\ge 1,\ \lambda_{mi}$ the ith positive zero of $J_m(x)$
$B = \{(r,\theta,\phi)\mid r<1\},\ u\|_{\partial B} = 0$ July 16 – 18	$j_\ell(\kappa_{\ell i}r)P_{\ell m}(\cos\theta)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\kappa_{\ell i}^2,\ \ell,m,i\in\mathbf{Z},\ \ell\ge 0,\ 0\le m\le\ell,\ i\ge 1,$ $\kappa_{\ell i} = \lambda_{\ell+\frac{1}{2},i}$ the ith positive zero of $j_\ell(x)$
$D = \{(\rho,\phi)\mid\rho<a\},\ u\|_{\partial D} = 0$ August 6 – 8	$J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\frac{1}{a^2}\lambda_{mi}^2,\ m,i\in\mathbf{Z}, m\ge 0, i\ge 1,$ λ_{mi} the ith positive zero of $J_m(x)$

The inner product used is $(f,x) = \int_X f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}$, where X is the region and $d\mathbf{x}$ is the volume or area element.

All of the above sets are complete and orthogonal with respect to their respective inner product.

For general concepts relating to eigenfunctions and eigenvalues, see notes of July 2 – 4.

Additional special functions: equations and properties

Modified Bessel functions. These are solutions $I_m(x)$, $m\in\mathbf{Z}$, $m\ge 0$ to the equation

$$\frac{d^2 I}{dx^2} + \frac{1}{x}\frac{dI}{dx} - \left(1 + \frac{m^2}{x^2}\right)I = 0$$

(compare the equation satisfied by Bessel functions $J_m(x)$). They are exponential rather than oscillatory in nature and hence do not form an orthogonal basis. They satisfy many similar identities to the unmodified Bessel functions but we do not need these identities in this course. *(continued)*

Spherical Bessel functions. These are solutions $j_\ell(x)$, $\ell \in \mathbf{Z}$, $\ell \geq 0$ to the equation

$$\frac{d^2 j}{dx^2} + \frac{2}{x}\frac{dj}{dx} + \left(1 - \frac{\ell(\ell+1)}{x^2}\right) j = 0$$

and can be expressed as $j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x)$. They can be expressed in terms of elementary functions (though we don't use that here). If $\kappa_{\ell i}$ denotes the ith positive zero of $j_\ell(x)$, then for each ℓ the set $\{j_\ell(\kappa_{\ell i} r)\}_{i=1}^\infty$ forms a complete orthogonal set on $[0,1]$ with respect to the inner product

$$(f(r), g(r)) = \int_0^1 f(r)\overline{g(r)}\, r^2\, dr.$$

Their normalisation with respect to this inner product is

$$(j_\ell(\kappa_{\ell i} r), j_\ell(\kappa_{\ell i} r)) = \frac{1}{2} j_{\ell+1}^2(\kappa_{\ell i}).$$

The j_ℓ satisfy many identities similar to those satisfied by the ordinary Bessel functions, but everything we shall need to calculate can be obtained by reducing to the ordinary Bessel functions so we do not give them.

Poisson's equation on a bounded domain. Let X denote one of Q, C, and B. The problem on X

$$\nabla^2 u = f, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I \mathbf{e}_I$, where $a_I = \frac{(f, \mathbf{e}_I)}{(\mathbf{e}_I, \mathbf{e}_I)}$, and $u = \sum_I b_I \mathbf{e}_I$; $\nabla^2 u = f$ then gives

$$\lambda_I^2 b_I = a_I.$$

Here $\mathbf{e}_I$ is the eigenfunction of the Laplacian satisfying

$$\nabla^2 \mathbf{e}_I = \lambda_I \mathbf{e}_I, \quad \mathbf{e}_I|_{\partial X} = 0.$$

See the notes of July 9 – 11 and 16 – 18 for examples. The more general problem

$$\nabla^2 u = f, \quad u|_{\partial X} = g$$

may be solved as the sum $u = u_1 + u_2$ of the two problems

$$\nabla^2 u_1 = f, \quad u_1|_{\partial X} = 0, \qquad \nabla^2 u_2 = 0, \quad u_2|_{\partial X} = g.$$

See the notes of July 16 – 18 for examples of this type of problem. The related problem

$$\nabla^2 u = f, \quad \frac{\partial u}{\partial n}\Big|_{\partial X} = 0 \qquad [\tfrac{\partial}{\partial n} \text{ the outward normal derivative}]$$

may be solved in the same way, using the eigenfunctions satisfying

$$\nabla^2 \mathbf{e}_I = \lambda_I \mathbf{e}_I, \quad \frac{\partial \mathbf{e}_I}{\partial n}\Big|_{\partial} X = 0,$$

except when one or more of the eigenvalues vanish: in that case f must be orthogonal to all corresponding eigenfunctions, and additional conditions must be imposed on u to get a unique solution. See the Appendix to the solutions for Homework 11, and the notes for July 2 – 4. The inhomogeneous problem may then be treated as above.

Green's functions for Poisson's equation. Suppose that $G(\mathbf{x}, \mathbf{x}')$ is a function satisfying

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

where δ is the Dirac delta function (see the next page for a review of this function). Then for u sufficiently differentiable on a domain D we have

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}') \nabla_{\mathbf{x}'}^2 u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x}, \mathbf{x}') \frac{\partial u}{\partial n'} - u(\mathbf{x}') \frac{\partial G}{\partial n'}\, dS'.$$

We may use Green's functions satisfying certain boundary conditions to solve boundary-value problems.

$$G(\mathbf{x}, \mathbf{x}')|_{\mathbf{x} \in \partial D} = 0: \qquad u = -\int_D G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' - \int_{\partial D} \frac{\partial G}{\partial n'} g(\mathbf{x}')\, dS' \text{ solves } \nabla^2 u = f,\ u|_{\partial D} = g$$

$$\frac{\partial G}{\partial n}\bigg|_{\mathbf{x} \in \partial D} = 0: \qquad u = -\int_D G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x}, \mathbf{x}') g(\mathbf{x}')\, dS' \text{ solves } \nabla^2 u = f,\ \frac{\partial u}{\partial n}\bigg|_{\partial D} = g$$

On $\mathbf{R}^3$, the solution vanishing at infinity to

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \quad \text{is} \quad G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

Thus on $\mathbf{R}^3$ the solution vanishing at infinity to Poisson's equation

$$\nabla^2 u = f \quad \text{is} \quad u(\mathbf{x}) = -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'.$$

Fourier transforms. These are covered in the notes for July 23 – 25, July 30 – August 1, and August 6 – August 8. If $f(\mathbf{x})$ is a function on $\mathbf{R}^m$ which satisfies $\int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x} < \infty$, then we say that f *is in* L^1 and define its *Fourier transform*

$$\hat{f}(\mathbf{k}) = \mathcal{F}[f(\mathbf{x})](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

If f is continuous and bounded and such that $\hat{f}(\mathbf{k})$ is in L^1, then we have the *Fourier inversion theorem*

$$f(\mathbf{x}) = \mathcal{F}^{-1}[\hat{f}(\mathbf{k})](\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

This may be shewn by making use of so-called *approximate identities*, which are sequences $\{\phi_n(\mathbf{x})\}$ of functions in L^1 satisfying

$$\int_{\mathbf{R}^m} \phi_n(\mathbf{x})\, d\mathbf{x} = 1, \quad \int_{\mathbf{R}^m} \phi_n(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \to f(0) \text{ as } n \to \infty$$

for all suitable (e.g., continuous and bounded) functions $f(\mathbf{x})$. If ϕ is any individual function in L^1 satisfying $\int_{\mathbf{R}^m} \phi(\mathbf{x})\, d\mathbf{x} = 1$, then the sequence $\{n^m \phi(n\mathbf{x})\}$ is an approximate identity.

If $f(\mathbf{x})$ and $g(\mathbf{x})$ are two functions in L^1 on $\mathbf{R}^m$, we define their *convolution* $f * g$ by

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}'.$$

The Fourier transform maps convolution to multiplication in the following sense:

$$\mathcal{F}[(f * g)(\mathbf{x})](\mathbf{k}) = \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}), \quad \mathcal{F}^{-1}[f(\mathbf{k})g(\mathbf{k})](\mathbf{x}) = (\mathcal{F}^{-1}[f] * \mathcal{F}^{-1}[g])(\mathbf{x}).$$

The Fourier transform possesses the following properties (see notes for July 23 – 25, p. 15):

$$\mathcal{F}[af + bg](\mathbf{k}) = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}), \quad \mathcal{F}[\partial_j f](\mathbf{k}) = 2\pi i k_j \mathcal{F}[f](\mathbf{k}), \quad \mathcal{F}[2\pi i x_j f](\mathbf{k}) = -\frac{\partial}{\partial k_j}\mathcal{F}[f](\mathbf{k})$$

$$\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) = e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\alpha}}\hat{f}(\mathbf{k}), \quad \mathcal{F}[e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x})](\mathbf{k}) = \mathcal{F}[f](\mathbf{k} - \boldsymbol{\alpha}).$$

The Fourier transform of a Gaussian is

$$\mathcal{F}\left[e^{-a|\mathbf{x}|^2}\right](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}}, \quad \mathcal{F}^{-1}\left[e^{-a|\mathbf{k}|^2}\right](\mathbf{x}) = \left(\frac{\pi}{a}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|\mathbf{x}|^2}{a}}.$$

Heat equation: bounded domains. Let X denote one of Q, C, and B. The problem on $(0, +\infty) \times X$

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I \mathbf{e}_I$, where $a_I = \frac{(f, \mathbf{e}_I)}{(\mathbf{e}_I, \mathbf{e}_I)}$, and $u = \sum_I b_I(t)\mathbf{e}_I$; the equation and initial condition then give

$$b_I'(t) = \lambda_I b_I, \quad b_I(0) = a_I, \quad \text{whence} \quad b_I(t) = a_I e^{\lambda_I t}.$$

Here $\mathbf{e}_I$ and λ_I denote the appropriate eigenfunctions and eigenvalues, as in our discussion of the Poisson equation. See the notes of July 2 – 4, 9 – 11, and 16 – 18 for details and examples. The more general problem

$$\tfrac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial X} = g,$$

where g is a function of $\mathbf{x}$ alone, can be solved as the sum $u = u_1 + u_2$ of the two problems

$$\nabla^2 u_1 = 0, \quad u_1|_{\partial X} = g, \quad \tfrac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = f - g, \quad u_2|_{\partial X} = 0.$$

See the notes of July 16 – 18, pp. 7 – 8, for discussion and an example.

Heat equation and generalisations on $\mathbf{R}^m$. The problem on $(0, +\infty) \times \mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0$$

can be solved using Fourier transforms, obtaining $\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u}$, $\hat{u}|_{t=0} = \hat{f}$, whence $\hat{u} = \hat{f}e^{-4\pi^2 t|\mathbf{k}|^2}$, and

$$u(t, \mathbf{x}) = (K_t * f)(\mathbf{x}), \quad \text{where the } \textit{heat kernel } K_t(\mathbf{x}) = \tfrac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}}.$$

Note that the heat kernel is an approximate identity in the limit $t \to 0^+$. The more general problem on $(0, +\infty) \times \mathbf{R}^m$

$$\tfrac{\partial u}{\partial t} = \nabla^2 u + g(\mathbf{x}, t), \quad u|_{t=0} = f$$

has solution

$$u(t, \mathbf{x}) = K_t(\mathbf{x}) * f(\mathbf{x}) + \int_0^t K_{t-s}(\mathbf{x}) * g(s, \mathbf{x})\, ds.$$

In practice it may be simpler to solve both of these problems by working directly with Fourier transforms. More general equations such as $\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n} \cdot \nabla u$ can be solved in this way. See notes for July 30 – August 1 and Homework 12, and the practice problems for week 12 and the final.

Wave equation: bounded domains. Again, let X denote one of Q, C, and B. The problem on $(0,+\infty)\times X$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I \mathbf{e}_I$, $g = \sum_I b_I \mathbf{e}_I$, where $a_I = \frac{(f,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}$ and $b_I = \frac{(g,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}$, and also $u = \sum_I c_I(t)\mathbf{e}_I$; the equation and initial conditions then give

$$c_I''(t) = \lambda_I c_I(t), \quad c_I(0) = a_I, \quad c_I'(0) = b_I.$$

This is a simple second-order constant-coefficient ordinary differential equation and can be solved easily. Here $\mathbf{e}_I$ and λ_I denote the appropriate eigenfunctions and eigenvalues, as above. The *frequencies* are $\sqrt{-\lambda_I}$. See the notes for August 6 – August 8 for an example on the disk.

Wave equation on $\mathbf{R}^3$. The problem on $(0,+\infty)\times\mathbf{R}^3$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \left.\frac{\partial u}{\partial t}\right|_{t=0} = g$$

can be solved using Fourier transforms, obtaining $\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u}$, $\hat{u}|_{t=0} = \hat{f}$, $\left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}$. Ultimately,

$$u(t,\mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,dS'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\,dS',$$

where $S_t(\mathbf{x})$ is the sphere of radius t centred at $\mathbf{x}$.

More general equations on $\mathbf{R}^m$. The problem on $(0,+\infty)\times\mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n}\cdot\nabla u + bu, \quad u|_{t=0} = f$$

can be solved by taking Fourier transforms, obtaining

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u} + 2\pi i\mathbf{n}\cdot\mathbf{k}\hat{u} + b\hat{u}, \quad \hat{u}|_{t=0} = \hat{f},$$

whence

$$\hat{u} = e^{-4\pi^2 t|\mathbf{k}|^2 + 2\pi i t\mathbf{n}\cdot\mathbf{k} + bt}\hat{f},$$

which can be inverted using properties of the Fourier transform to obtain u.