We give a brief review of complex numbers, and some results which we shall need in this course.

DEFINITION. A *complex number* is a number of the form $a+ib$, where $a, b \in \mathbf{R}$ and i satisfies $i^2=-1$. If $a+ib$ and $c+id$ are two complex numbers, we define their sum, difference, product, and quotient as follows:

$$\begin{aligned}
(a+ib)+(c+id) &= (a+c)+i(b+d) \\
(a+ib)-(c+id) &= (a-c)+i(b-d) \\
(a+ib)\cdot(c+id) &= (ac-bd)+i(bc+ad) \\
\frac{1}{c+id} &= \frac{c}{c^2+d^2}+i\frac{-d}{c^2+d^2}=\frac{c-id}{c^2+d^2}, \qquad c^2+d^2 \neq 0 \\
\frac{a+ib}{c+id} &= (a+ib)\cdot\frac{1}{c+id}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}, \qquad c^2+d^2 \neq 0.
\end{aligned}$$

If $z=a+ib$ is a complex number, we call a its *real part* and b its *imaginary part*, and write $a=\operatorname{Re} z$, $b=\operatorname{Im} z$. The *conjugate* of z is the number $\overline{z}=a-ib$. The quantity $\sqrt{z\overline{z}}=\sqrt{a^2+b^2}$ is called the *norm* of z, and is denoted $|z|$; note that it is equal to the norm of the vector $a\mathbf{i}+b\mathbf{j}$ in $\mathbf{R}^2$.[1] (We note in passing that $c^2+d^2=0$ if and only if both c and d are zero; thus requiring $c^2+d^2 \neq 0$ is equivalent to saying that at least one of c and d is nonzero.)

COMMENTARY. Essentially, the above definitions say that complex numbers obey all of the usual rules of algebra, supplemented by the condition $i^2=-1$. It turns out to be convenient to require them to also behave in a natural way with respect to the operations of the calculus, as in the following.[2]

DEFINITION. Let $f:[a,b]\to\mathbf{C}$, and suppose that $f_1=\operatorname{Re} f$ and $f_2=\operatorname{Im} f$ are differentiable. Then we define

$$f'(t)=f_1'(t)+if_2'(t)$$

(compare to the definition of a tangent vector to a plane parametric curve). If f_1 and f_2 are integrable, then we define

$$\int_a^b f(t)dt=\int_a^b f_1(t)dt+i\int_a^b f_2(t)dt.$$

DEFINITION. A sequence $\{z_n\}$ of complex numbers is said to *converge* to the complex number z if the sequence $\{|z_n-z|\}$ of real numbers converges to 0. (It can be shewn that this is equivalent to saying that $\operatorname{Re} z_n$ converges to $\operatorname{Re} z$ and $\operatorname{Im} z_n$ converges to $\operatorname{Im} z$.) Convergence of a series as convergence of its partial sums is defined as for real series.

COMMENTARY. Suppose that $\sum\limits_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R>0$ (we include the case $R=\infty$). If $z=a+ib$ is such that $|z|<R$, then it can be shewn that $\sum\limits_{n=0}^{\infty} a_n z^n$ converges also. This allows us to extend functions with convergent power series representations (such as e^x, $\sin x$, $\cos x$, etc.) to the complex plane. In particular, if we define e^z in this way for $z\in\mathbf{C}$, then it can be shewn that (exercise)

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

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[1]There are, in fact, some deep connections here, and various parts of two-dimensional calculus have some analogue in complex analysis. Three-dimensional vector calculus is related to a still higher kind of number, called a *quaternion*. Quaternions are very interesting and useful for some purposes (and have nice connections to the concept of *spin* in quantum mechanics) but we do not need them here (and unfortunately shall probably not have occasion to use them anywhere in this course).

[2]Note that we are developing here only a very small part of the theory of complex analysis – in particular, all of our functions are functions of a real variable, and we differentiate only with respect to real variables. As those of you who have had complex analysis are aware, the true power and depth of complex analysis only comes out when one considers derivatives with respect to a *complex* variable.