Summary:
- We provide some analogies between our work with Fourier transforms and our previous work with orthogonal expansions.
- We then derive the eigenfunctions for the Laplacian on a disk with Dirichlet boundary conditions and use them together with Fourier transforms to study the wave equation on a disk.
- Finally, we derive the solution to the initial value problem for the wave equation on $\mathbf{R}^3$ (and indicate how to derive a similar formula in dimensions 1 and 2). We indicate its qualitative content and sketch an example of its use. In an appendix, we show how to use similar methods to solve the inhomogeneous wave equation.

ANALOGIES BETWEEN FOURIER TRANSFORM METHODS AND ORTHOGONAL EXPANSIONS. Suppose that $D$ is some bounded region in $\mathbf{R}^m$, and that $\{\mathbf{e}_I\}_{I\in\mathcal{I}}$ is a complete orthonormal set of eigenfunctions for the Laplacian on $D$ with homogeneous Dirichlet boundary conditions, for some set $\mathcal{I}$. Then we know that any 'reasonable' function can be expanded as a series $u = \sum_{I\in\mathcal{I}} u_I\mathbf{e}_I$, where $u_I = (u, \mathbf{e}_I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x}$. Now we may view the coefficients $u_I$ as giving a function from the set of indices $\mathcal{I}$ to the complex numbers $\mathbf{C}$ (for the problems we have dealt with, the coefficients have generally been real; this is because we have used real functions $u$ and real eigenfunctions $\mathbf{e}_I$); we shall write such a function as $\tilde{u} : \mathcal{I} \to \mathbf{C}$, so that $\tilde{u}(I) = (u, \mathbf{e}_I)$. Let us denote the set of all such sequences by $\mathbf{C}^{\mathcal{I}}$ (there is a nice sense in which this set is a Cartesian product of $\mathcal{I}$ copies of $\mathbf{C}$, but it veers off into set theory and we shall not treat it here). Then the foregoing shows that we may define a *transform* $\mathcal{O} : L^1(D) \to \mathbf{C}^{\mathcal{I}}$[1] by

$$\mathcal{O}[u](I) = \tilde{u}(I) = (u, \mathbf{e}_I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x};$$

in other words, $\mathcal{O}[u]$ is the function from $\mathcal{I}$ to $\mathbf{C}$ which, for every $I \in \mathcal{I}$, gives the coefficient $(u, \mathbf{e}_I)$. (If the set $\{\mathbf{e}_I\}_{I\in\mathcal{I}}$ were not assumed to be normalised, then of course we would use $\frac{(u,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}$ instead here.) We then have the expansion

$$u(\mathbf{x}) = \sum_{I\in\mathcal{I}} \mathcal{O}[u](I)\mathbf{e}_I(\mathbf{x}).$$

Suppose that we let $\mathbf{O} \subset \mathbf{C}^{\mathcal{I}}$ denote the set of maps $v : \mathcal{I} \to \mathbf{C}$ such that the series

$$\sum_{I\in\mathcal{I}} v(I)\mathbf{e}_I(\mathbf{x})$$

converges in some appropriate sense, and such that this sum is in $L^1(D)$;[2] then we expect that $\mathcal{O}$ actually maps into $\mathbf{O}$ (much as we were able to show that $\mathcal{F}$ actually maps into $C_b(\mathbf{R}^m)$). If we now define the map

$$\mathcal{O}^{-1} : \mathbf{O} \to L^1(D)$$

by

$$\mathcal{O}^{-1}[v](\mathbf{x}) = \sum_{I\in\mathcal{I}} v(I)\mathbf{e}_I(\mathbf{x}),$$

then we see that (as our notation indicates) $\mathcal{O}^{-1}[\mathcal{O}[u]](\mathbf{x}) = u(\mathbf{x})$, $\mathcal{O}[\mathcal{O}^{-1}[v]](I) = v(I)$, i.e., that $\mathcal{O}^{-1}$ is actually an inverse to $\mathcal{O}$.

We may make the following comparison between the foregoing and the Fourier transform:

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x} = \hat{f}(\mathbf{k}) \qquad \mathcal{O}[u](I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \tilde{u}(I)$$

$$\mathcal{F}^{-1}[\hat{f}](\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k} = f(\mathbf{x}) \qquad \mathcal{O}^{-1}[\tilde{u}](I) = \sum_{I\in\mathcal{I}} \tilde{u}(I)\mathbf{e}_I(\mathbf{x}) = u(\mathbf{x})$$

[1] As with the Fourier transform, which, we recall, we showed in the previous set of lecture notes could be viewed as a map $\mathcal{F} : L^1(\mathbf{R}^m) \to C_b(\mathbf{R}^m)$, one can define $\mathcal{O}$ for various different function spaces. We use $L^1(D)$ here for convenience; doing so requires only that the eigenfunctions of the Laplacian are bounded on $D$, which is true for all of the cases we have studied in this course.

[2] Again, the exact spaces used here are not as important as the general idea.

The transform $\mathcal{O}$ possesses some (though certainly not all) of the properties of the transform $\mathcal{F}$. As an example, we compute the transform of the Laplacian of a function. (As usual, we assume that all relevant transforms exist.) This can be done two ways. The way most closely related to our derivation of a similar property for the Fourier transform is as follows (note that this is the first place we use the fact that the eigenfunctions satisfy homogeneous Dirichlet boundary conditions):

$$\begin{aligned}\mathcal{O}[\nabla^2 u](I) &= \int_D \nabla^2 u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \int_D \nabla\cdot\left(\nabla u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\right) - \nabla u(\mathbf{x})\cdot\nabla\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x}\\ &= \int_{\partial D}\mathbf{n}\cdot\nabla u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,dS - \int_D \nabla\cdot\left(u(\mathbf{x})\nabla\overline{\mathbf{e}_I(\mathbf{x})}\right) - u(\mathbf{x})\nabla^2\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x}\\ &= -\int_{\partial D} u(\mathbf{x})\mathbf{n}\cdot\nabla\overline{\mathbf{e}_I(\mathbf{x})}\,dS + \int_D u(\mathbf{x})\nabla^2\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \int_D u(\mathbf{x})\lambda_I\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x}\\ &= \lambda_I\int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \lambda_I\mathcal{O}[u](I);\end{aligned}$$

here we assume that, since $u$ is a series of functions satisfying homogeneous Dirichlet boundary conditions, it satisfies them itself. This result should be compared to the corresponding result for the Fourier transform:

$$\mathcal{F}[\nabla^2 u](\mathbf{k}) = -4\pi^2|\mathbf{k}|^2\mathcal{F}[u](\mathbf{k}).$$

Another way of deriving the above result for $\mathcal{O}[\nabla^2 u](I)$ which is much closer to our usual methods for manipulating orthogonal expansions (and also more general) is as follows. Writing $\tilde{u}(I) = \mathcal{O}[u](I)$, we have

$$u(\mathbf{x}) = \sum_{I\in\mathcal{I}}\tilde{u}(I)\mathbf{e}_I(\mathbf{x});$$

assuming that we may differentiate term-by-term, we have

$$\nabla^2 u(\mathbf{x}) = \sum_{I\in\mathcal{I}}\tilde{u}(I)\nabla^2\mathbf{e}_I(\mathbf{x}) = \sum_{I\in\mathcal{I}}\lambda_I\tilde{u}(I)\mathbf{e}_I(\mathbf{x}).$$

But this shows that

$$\begin{aligned}\mathcal{O}[\nabla^2 u](I) = (\nabla^2 u, \mathbf{e}_I) &= \left(\sum_{J\in\mathcal{I}}\lambda_J\tilde{u}(J)\mathbf{e}_J, \mathbf{e}_I\right)\\ &= \lambda_I\tilde{u}(I) = \lambda_I\mathcal{O}[u](I),\end{aligned}$$

by our usual manipulations with orthogonal expansions. This is our desired result.

The other major property of the Fourier transform, that of turning convolution into multiplication, does not have so happy a fate with $\mathcal{O}$; the details are quite beyond the scope of this course, but we provide an outline in Appendix I at the end for those who are interested. (This Appendix can be skipped entirely, though it does give another perspective on where convolution comes from.)

Given this property, we may proceed to solve the heat equation using $\mathcal{O}$ in a fashion exactly analogous to that by which we solved the heat equation using $\mathcal{F}$. To this end, consider the problem

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial D} = 0.$$

If we apply $\mathcal{O}$ to the entire problem, we obtain the transformed problem

$$\frac{\partial\tilde{u}(t,I)}{\partial t} = \lambda_I\tilde{u}(t,I), \quad \tilde{u}|_{t=0} = \tilde{f}(I);$$

from this we easily obtain

$$\tilde{u}(t,I) = \tilde{f}(I)e^{\lambda_I t},$$

whence

$$u = \mathcal{O}^{-1}[\tilde{u}](\mathbf{x}) = \sum_{I \in \mathcal{I}} \tilde{f}(I) e^{\lambda_I t} \mathbf{e}_I(\mathbf{x}),$$

where

$$\tilde{f}(I) = (f, \mathbf{e}_I).$$

This is identical to the result we obtained by our usual methods (see, for example, our treatment of the heat equation on the cube in the lecture notes for July 9 – 11).

The point of this is to try to make the Fourier method a little bit more understandable, rather than to suggest that we ought to use this method with orthogonal expansions! (Though we certainly can if we like.)

EIGENFUNCTIONS AND EIGENVALUES FOR THE LAPLACIAN ON A DISK. Let $D = \{(\rho, \phi) | \rho < a\}$, for some positive number $a$, and consider the problem

$$\nabla^2 u = \lambda u, \quad u|_{\partial D} = 0.$$

Now the Laplacian in polar coordinates can be obtained from the Laplacian in cylindrical coordinates by dropping the final $\frac{\partial^2}{\partial z^2}$; thus this equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = \lambda u.$$

We proceed as usual by looking for separated solutions to this equation. Thus suppose that $u = P(\rho)\Phi(\phi)$; substituting in and dividing through by $u$ then gives

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = \lambda.$$

As usual, since only the term $\frac{\Phi''}{\Phi}$ depends on $\phi$, it must be constant; and since $\phi$ is an angular variable which is only defined up at an additive term of a multiple of $2\pi$, our usual logic shows that this constant must be the negative square of an integer, i.e., that there must be an $m \in \mathbf{Z}$ such that $\frac{\Phi''}{\Phi} = -m^2$. From this we obtain readily the two solutions $\Phi_1(\phi) = \cos m\phi$, $\Phi_2(\phi) = \sin m\phi$. Substituting this back in, we obtain for $P$

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} - \frac{m^2}{\rho^2} = \lambda,$$

or

$$P'' + \frac{1}{\rho}P' + \left(-\lambda - \frac{m^2}{\rho^2}\right)P = 0.$$

This is seen, after scaling by $\sqrt{-\lambda}$, to be simply Bessel's equation; in other words, we must have

$$P(\rho) = J_m(\sqrt{-\lambda}\rho).$$

Somewhat more carefully: if $-\lambda \geq 0$ then we obtain the above formula; if $-\lambda < 0$ then we would obtain $I_m(\sqrt{\lambda}\rho)$. Since we require homogeneous Dirichlet boundary conditions on the boundary, i.e., at $\rho = a$, we must choose $J_m$ and not $I_m$. This forces $\lambda \leq 0$, say $\lambda = -\mu^2$. Now the boundary condition gives

$$P(a) = J_m(\mu a) = 0,$$

whence we see that $\mu = \frac{\lambda_{mi}}{a}$ for some $i$, where $\lambda_{mi}$ denotes as usual the $i$th positive zero of $J_m$. Thus we have the eigenfunctions

$$J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi, \quad J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi,$$

with the eigenvalues

$$\lambda = -\frac{\lambda_{mi}^2}{a^2}.$$

This set of eigenfunctions is seen to be complete, since the Bessel function factors are in $\rho$.

THE WAVE EQUATION ON A DISK. The *wave equation*,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

describes the motion of waves on elastic membranes, in gasses and fluids, and in various other circumstances (at least as long as the quantity $u$ is not large so that nonlinear effects can be neglected). Here $c$ is a parameter called the *wave speed* (we shall see the reason for this terminology later, when we discuss the wave equation on $\mathbf{R}^m$); we shall occasionally set it equal to 1 for convenience – any formula with $c = 1$ can be turned into a formula for general $c$ by multiplying $t$ by $c$ at each occurrence. Now consider the following problem on $(0, +\infty) \times D$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0;$$

this problem could describe the vibrations of a circular drumhead (in that case, $u$ would represent the vertical deflection from the equilibrium plane of the drumhead, so the Dirichlet condition $u|_{\partial D} = 0$ means physically that the edge of the drumhead is fixed and immobile). Here we have specified no initial conditions. If we Fourier transform in $t$, we obtain, using $f$ as our Fourier variable,

$$-4\pi^2 f^2 \hat{u} = \nabla^2 \hat{u}, \quad \hat{u}|_{\partial D} = 0;$$

from this we see that $-4\pi^2 f^2$ must be an eigenvalue of the Laplacian on $D$, which means that we must have

$$f = \pm\frac{\lambda_{mi}}{2\pi a}$$

for some $m$ and $i$; more specifically, for $f$ not of this form we must have $\hat{u}(f, \mathbf{x}) = 0$ for all $\mathbf{x}$. While we shall not pause to give a precise derivation of the following, this means that any solution $u$ must be simply a *sum* (rather than an integral) over frequencies; specifically, since $2\pi i \frac{\lambda_{mi}}{2\pi a} t = \frac{i\lambda_{mi} t}{a}$, we have

$$u(t, \mathbf{x}) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\left[e^{\frac{i\lambda_{mi}t}{a}}\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right) + e^{-\frac{i\lambda_{mi}t}{a}}\left(c_{mi}\cos m\phi + d_{mi}\sin m\phi\right)\right].$$

Here the coefficients can be complex to make $u$ real.

Let us now consider the slightly different problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g, \quad u|_{\partial D} = 0.$$

For this problem we shall begin by expanding $u$ in a series in terms of the eigenfunctions found above. We could proceed in the usual fashion; for the sake of illustration, we shall use the transform $\mathcal{O}$ introduced above. Transforming with $\mathcal{O}$, the above problem becomes

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = -\frac{\lambda_{mi}^2}{a^2}\tilde{u}, \quad \tilde{u}|_{t=0} = \tilde{f}, \quad \left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = \tilde{g}.$$

From the equation, we see that the general solution is of the form (writing $I = (m, i, \sigma)$, where $\sigma = 1$ for the eigenfunction with $\cos m\phi$ and $\sigma = -1$ for the eigenfunction with $\sin m\phi$)

$$\tilde{u}(t, I) = a(I)\cos\frac{\lambda_{mi}}{a}t + b(I)\sin\frac{\lambda_{mi}}{a}t;$$

applying the initial conditions gives

$$\tilde{u}|_{t=0} = a(I) = \tilde{f}(I),$$

$$\left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = \frac{\lambda_{mi}}{a}b(I) = \tilde{g}(I),$$

$$b(I) = \frac{a}{\lambda_{mi}}\tilde{g}(I),$$

so that

$$\tilde{u}(t,I) = \tilde{f}(I)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t,$$

and the solution is

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\left[\left[\tilde{f}(m,i,1)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}(m,i,1)\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t\right] J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi \right.$$
$$\left. + \left[\tilde{f}(m,i,-1)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}(m,i,-1)\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t\right] J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi\right].$$

The same result could of course be obtained by our usual methods. We now give a specific example.
EXAMPLE. Solve the following problem on $D$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 1, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{\partial D} = 0.$$

We first determine the transform $\mathcal{O}[1]$:

$$\mathcal{O}[1](m,i,\pm 1) = \frac{2}{(\cos m\phi, \cos m\phi)a^2 J_{m+1}^2(\lambda_{mi})}\int_D J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix} d\mathbf{x},$$

whence we see that $\mathcal{O}[1](m,i,\pm 1) = 0$ unless $m = 0$ and we take the $+1$ in the third slot, and that in that case

$$\begin{aligned}\mathcal{O}[1](0,i,1) &= \frac{1}{\pi a^2 J_1^2(\lambda_{0i})}\int_0^{2\pi}\int_0^a J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\rho\, d\rho\, d\phi\\ &= \frac{2}{a^2 J_1^2(\lambda_{0i})}\frac{a^2}{\lambda_{0i}^2}\left(xJ_0(x)\right)|_0^{\lambda_{mi}} = \frac{2}{\lambda_{0i}J_1(\lambda_{0i})};\end{aligned}$$

we note that the factors of $a$ cancel only because of the value of $m$ involved. Clearly $\mathcal{O}[0] = 0$, so substituting back into the general formula above, we have the solution

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty}\frac{2}{\lambda_{0i}J_1(\lambda_{0i})}\cos\frac{\lambda_{0i}}{a}tJ_0\left(\frac{\lambda_{0i}}{a}\rho\right).$$

This is, of course, what we would expect to obtain had we started by writing out the general series expansion for $u$ and then substituted it into the equation.

THE WAVE EQUATION ON $\mathbf{R}^m$. We now come to the last major topic of the course, namely the treatment of the initial value problem for the wave equation on $\mathbf{R}^m$. Thus we seek solutions to the following problem:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g.$$

(The treatment of the nonhomogeneous version, where there is a term $F$ added to the right-hand side, is beyond the scope of the course proper but will be sketched in Appendix II.) We approach this problem in a fashion analogous to that in which we approached the corresponding version on $D$. We begin by Fourier transforming:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

Now the first equation above clearly has the general solution

$$\hat{u}(t,\mathbf{k}) = a(\mathbf{k})\cos 2\pi|\mathbf{k}|t + b(\mathbf{k})\sin 2\pi|\mathbf{k}|t,$$

where $a(\mathbf{k})$ and $b(\mathbf{k})$ are two arbitrary functions. Applying the initial conditions, we obtain

$$\hat{u}|_{t=0} = a(\mathbf{k}) = \hat{f}(\mathbf{k}),$$
$$\left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = 2\pi|\mathbf{k}|b(\mathbf{k}) = \hat{g}(\mathbf{k}),$$
$$b(\mathbf{k}) = \frac{1}{2\pi|\mathbf{k}|}\hat{g}(\mathbf{k}),$$

so that

$$\hat{u}(t,\mathbf{k}) = \hat{f}(\mathbf{k})\cos 2\pi|\mathbf{k}|t + \hat{g}(\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|},$$

exactly analogous to the result we obtained above on $D$. (We note also that the above function is defined for all $\mathbf{k}$, even though $b(\mathbf{k})$ as given above is undefined for $\mathbf{k} = 0$.) We note that the result here is valid for all $m$; thus the Fourier transform of the solution does not depend in any way on the dimension of the space involved. (This is analogous to the situation for Poisson's equation: if one solves $\nabla^2 u = f$ by Fourier transform, one finds $u = -\frac{1}{4\pi^2|\mathbf{k}|^2}f(\mathbf{k})$, regardless of the dimension.)

We would now like to take the inverse Fourier transform of the above expression. Now the properties of the Fourier transform show that

$$\mathcal{F}^{-1}[\hat{f}\hat{g}](\mathbf{x}) = (f * g)(\mathbf{x})$$

for any appropriate functions $f$ and $g$; thus if we could recognise the two functions $\cos 2\pi|\mathbf{k}|t$ and $\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}$ as Fourier transforms, we would be able to write $u$ as a sum of two convolution integrals. We note that the former is the time derivative of the latter, which suggests that we start with the latter function. This is where the dimension of the space comes into play. The main case for us here will be $m = 3$ (and this is the only case we covered systematically in class), but we shall indicate what happens when $m = 1$ or $m = 2$.[3]

Let us denote the inverse transform we seek by $M(t,\mathbf{x})$; then

$$M(t,\mathbf{x}) = \mathcal{F}^{-1}\left[\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}\right](\mathbf{x})$$
$$= \int_{\mathbf{R}^m}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k}.$$

Before proceeding, we note that this function is real: its conjugate is just

$$\int_{\mathbf{R}^m}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k},$$

which can be turned back into the original integral by using the substitution $\mathbf{k}' = -\mathbf{k}$. This will be important below. Now it can be shown that for any $m > 1$ (we shall say more about the case $m = 1$ below), the $m$-dimensional volume element $d\mathbf{k}$ can be decomposed into the following (for simplicity, we shall write $k = |\mathbf{k}|$ where convenient):

$$d\mathbf{k} = k^{m-1}\,dk\,d\Omega,$$

where $d\Omega$ is an angular element; when $m = 2$ it is simply $d\theta$, while when $m = 3$ it is $\sin\theta d\theta d\phi$ (this is called an element of *solid angle*, in analogy with the element of angle $d\theta$ which one obtains in the case $m = 2$); when

[3]We shall not, however, treat higher values of $m$ since these involve progressively more pathological 'functions': we shall see in a moment that when $m = 3$ we get a Dirac delta function; for $m = 5$ we would get a second derivative of a Dirac delta function, and so on. (Even dimensions turn out to be somewhat more complicated than odd dimensions.) While these derivatives can be defined in a rigorous sense, doing so is beyond the scope of this course.

$m > 3$ it is a similar angular measure in $m-1$ angular variables obtained by parametrising the $m-1$-sphere. (For example, we may parametrise the 4-sphere thus (letting $\psi$ represent the normal polar angle in 3-space):

$$\begin{aligned} w &= \cos\theta \\ z &= \sin\theta\cos\psi \\ x &= \sin\theta\sin\psi\cos\phi \\ y &= \sin\theta\sin\psi\sin\phi, \end{aligned}$$

and for higher dimensions we may proceed by induction.) This general parametrisation is not important, beyond knowing that for all $m$ we can parametrise it in such a way that, for fixed $\mathbf{x}$, we have (writing $r = |\mathbf{x}|$)

$$\mathbf{k}\cdot\mathbf{x} = |\mathbf{k}| r\cos\theta$$

where $\theta$ is one of the angles parametrising the $m-1$-sphere, and which runs from 0 to $\pi$. (This is clearly true for $m = 2$ in polar coordinates – taking the $x$ axis along $\mathbf{x}$ – and for $m = 3$ in spherical coordinates – take the $z$ axis along $\mathbf{x}$ – and these are the only situations we are really concerned with here.) Thus we may rewrite the above integral as, letting $S_1$ denote the unit $m-1$-sphere (the unit circle if $m = 2$, the unit sphere if $m = 3$)

$$\int_{S_1}\int_0^\infty \frac{\sin 2\pi kt}{2\pi k} e^{2\pi i k r\cos\theta} k^{m-1}\, dk\, d\Omega.$$

Now as noted above, this integral is always a real number; thus we may replace the complex exponential with its real part, obtaining

$$\int_{S_1}\int_0^\infty \frac{\sin 2\pi kt}{2\pi k}\cos\left(2\pi kr\cos\theta\right) k^{m-1}\, dk\, d\Omega.$$

Now if $m$ is odd (for example, if $m = 3$), the integrand is an even function of $k$, so this integral equals

$$\frac{1}{2}\int_{S_1}\int_{-\infty}^\infty \frac{\sin 2\pi kt}{2\pi k}\cos\left(2\pi kr\cos\theta\right) k^{m-1}\, dk\, d\Omega = \int_{S_1}\int_{-\infty}^\infty \frac{\sin 2\pi kt}{2\pi k} e^{2\pi i kr\cos\theta} k^{m-1}\, dk\, d\Omega.$$

The point behind all of these manipulations is that the integral over $k$ here is now quite clearly the inverse Fourier transform of the function $k^{m-1}\frac{\sin 2\pi kt}{2\pi k}$ on $\mathbf{R}^1$, evaluated at the point $r\cos\theta$ – in other words, we have reduced a three-dimensional inverse Fourier transform to a one-dimensional one. The factor of $k^{m-1}$ indicates that the inverse transform of this function will be the $m-1$th derivative of the inverse transform of $\frac{\sin 2\pi kt}{2\pi k}$, which we now derive. (This is the reason why the function $M$ becomes increasingly less well-behaved in higher dimensions.)

Directly calculating the inverse Fourier transform of $\frac{\sin 2\pi kt}{2\pi k}$ is not easy, so we shall proceed as we did in class by finding a function whose Fourier transform it is. Let

$$\chi(x) = \chi_{[-t,t]}(x) = \begin{cases} 1, & x \in [-t,t] \\ 0, & x \notin [-t,t] \end{cases};$$

$\chi$ is just a rectangular bump function. The Fourier transform of $\chi$ is

$$\begin{aligned} \mathcal{F}[\chi](k) &= \int_{-\infty}^\infty \chi(x) e^{-2\pi i kx}\, dx = \int_{-t}^t e^{-2\pi i kx}\, dx = \int_{-t}^t \cos 2\pi kx\, dx \\ &= \left.\frac{\sin 2\pi kx}{2\pi k}\right|_{-t}^t = \frac{\sin 2\pi kt}{\pi k}, \end{aligned}$$

where we have made use of the fact that cos is an odd function and sin an even function. Thus we see that

$$\mathcal{F}^{-1}\left[\frac{\sin 2\pi kt}{2\pi k}\right](x) = \frac{1}{2}\chi(x).$$

It is worth noting that, were we working in dimension $m = 1$, this would be the only inverse Fourier transform we would need, i.e., this would be our function $M$. We shall not give the details here.

From this we obtain (pretending for the moment that $\chi$ is a twice-differentiable function, even though it is not even continuous at $x = \pm t$)

$$\begin{aligned}\mathcal{F}^{-1}\left[k^2\frac{\sin 2\pi kt}{2\pi k}\right](x) &= -\frac{1}{4\pi^2}\mathcal{F}^{-1}\left[-4\pi^2k^2\frac{\sin 2\pi kt}{2\pi k}\right](x)\\ &= -\frac{1}{8\pi^2}\chi''(x),\end{aligned}$$

and we see that our function $M$ is

$$\begin{aligned}M &= -\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\chi''(r\cos\theta)\sin\theta\,d\theta\,d\phi\\ &= -\frac{1}{8\pi}\left[-\frac{1}{r}\chi'(r\cos\theta)\right]\Bigg|_{\theta=0}^{\theta=\pi} = -\frac{1}{8\pi r}\cdot 2\chi'(r) = -\frac{1}{4\pi r}\chi'(r),\end{aligned}$$

where we have used the fact that $\chi'$ is odd since $\chi$ is even (again, pretending that $\chi'$ were a normal function!). We are, now, thus faced with the task of computing $\chi'(r)$, for $r > 0$ (remember that $r = |x|$). Clearly $\chi'(r) = 0$ for $r \neq t$. We claim that in fact $\chi'(r) = -\delta(r-t)$. The simplest way to see this is as follows. Let $H$ denote the *Heaviside function*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}.$$

Now suppose that $H'$ could be defined in such a way that integration by parts were still valid[4], if $f$ were any function vanishing as $x \to \infty$, we would have

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x)f(x)\,dx &= H(x)f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)f'(x)\,dx\\ &= -\int_0^{\infty} f'(x)\,dx = -f(x)|_0^{\infty} = f(0),\end{aligned}$$

so that $H'(x)$ does indeed behave as a delta function. Now on $r > 0$, we have $\chi(r) = H(t-r)$, so (proceeding formally) we have $\chi'(r) = -H'(t-r) = -\delta(t-r) = -\delta(r-t)$, as claimed. [Another, perhaps more rigorous, way of seeing this is as follows. Let $\{\phi_n\}$ be the approximate identity given by

$$\phi_n(x) = n\pi^{-\frac{1}{2}}e^{-n^2x^2},$$

and define

$$\Phi_n(x) = \int_0^x \phi_n(u)\,du;$$

then we have, doing a change of variables to $v = nu$,

$$\Phi_n(x) = \int_0^{nx} \phi(v)\,dv,$$

whence it is evident that for $x > 0$ we have $\Phi_n(x) \to \frac{1}{2}$ as $n \to \infty$, while $\Phi_n(x) \to -\frac{1}{2}$ as $n \to -\infty$; in other words, we have for all $x \neq 0$ the limit

$$\lim_{n\to\infty}\Phi_n(x) = H(x) - \frac{1}{2}.$$

[4] This is in fact the way in which differentiation of functions such as $H$ and 'functions' (distributions) such as $\delta$ may be defined rigorously: one requires that the normal integration-by-parts formulas hold and proceeds formally.

Thus, assuming that we can interchange differentiation with the limit, we obtain

$$H'(x) = \lim_{n\to\infty} \Phi_n'(x) = \lim_{n\to\infty} \phi_n(x),$$

and this latter limit 'is' just the delta function $\delta(x)$ since $\{\phi_n\}$ is an approximate identity.] Thus, finally, we have for $M$

$$M(t,\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\delta(|\mathbf{x}|-t) = \frac{1}{4\pi t}\delta(|\mathbf{x}|-t).$$

The inverse transform of $\hat{g}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|k|}$, which we shall denote $u_2(t,\mathbf{x})$, is thus equal to the convolution integral

$$\frac{1}{4\pi t}\int_{\mathbf{R}^3} g(\mathbf{x}-\mathbf{x}')\delta(|\mathbf{x}'|-t)\,d\mathbf{x}'.$$

Let us now set up a spherical coordinate system in $\mathbf{x}'$; then the above integral becomes

$$\begin{aligned}\frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} g(\mathbf{x}-\mathbf{x}')\delta(r'-t)r'^2\,dr'\sin\theta'\,d\theta'\,d\phi' &= \frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} g(\mathbf{x}-\mathbf{x}')\delta(r'-t)\,dr't^2\sin\theta'\,d\theta'\,d\phi' \\ &= \frac{1}{4\pi t}\int_{S_t(0)} g(\mathbf{x}-\mathbf{x}')\,dS' = \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}'')\,dS'',\end{aligned}$$

where in the last equation we have made the substitution $\mathbf{x}'' = \mathbf{x}-\mathbf{x}'$, which translates the sphere $S_t(0)$ to the sphere $S_t(\mathbf{x})$. (Here $S_t(\mathbf{x}) = \{\mathbf{x}' | |\mathbf{x}-\mathbf{x}'| = t\}$ is the sphere – not ball! – of radius $t$ centred at $\mathbf{x}$.) The second-to-last equality holds for the following reasons: first of all, the delta function forces the point $\mathbf{x}'$ in $g(\mathbf{x}-\mathbf{x}')$ to lie on the sphere; second, the remaining parts of the volume element, $t^2\sin\theta'\,d\theta'\,d\phi'$, give exactly the surface area element on a sphere of radius $t$.

This is thus the desired formula for the inverse Fourier transform of the second part of our expression for $\hat{u}$ obtained above.

To work out the first part, we proceed rather formally as follows, assuming that we can interchange $\mathcal{F}^{-1}$ and $\frac{\partial}{\partial t}$:

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}(\mathbf{k})\cos 2\pi|\mathbf{k}|t\right](\mathbf{x}) &= \frac{\partial}{\partial t}\mathcal{F}^{-1}\left[\hat{f}(\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}\right](\mathbf{x}) \\ &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,dS'\right].\end{aligned}$$

Thus finally we have the following formula for $u$:

$$u(t,\mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,dS'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\,dS';$$

or, putting back in the speed $c$,

$$u(t,\mathbf{x}) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{1}{4\pi ct}\int_{S_{ct}(\mathbf{x})} f(\mathbf{x}')\,dS'\right] + \frac{1}{4\pi ct}\int_{S_{ct}(\mathbf{x})} g(\mathbf{x}')\,dS'. \tag{1}$$

We note a qualitative result which follows from this: the solution $u$ at a point $\mathbf{x}$ and a time $t$ only depends on the initial data on (or, at any rate, in the case of $f$, infinitesimally close to) the sphere (not the ball!) of radius $ct$ centred at $\mathbf{x}$ – in other words, on the initial data on the set of points exactly a distance $ct$ from the point $\mathbf{x}$. This means that signals propagate at exactly the speed $c$. (As mentioned in class – though the derivation does not follow in the way indicated there, since the function $k^{m-1}$ becomes odd and one cannot extend the integral to all of $\mathbf{R}^1$ as done here and suggested there – this property of the wave equation does not hold in two dimensions; and the author has seen it suggested that this is the reason why thunder is usually heard to continue even though the lightning flash (and hence the source of the thunder) is essentially instantaneous: a lightning flash – and hence the intial data for the thunder – is essentially a long straight line, meaning that the source will possess cylindrical symmetry, and the wave will be essentially the same as a two-dimensional wave.)

We now give a concrete example.

EXAMPLE. Solve the following problem on $\mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 1, & |\mathbf{x}| \le 1 \\ 0, & |\mathbf{x}| > 1 \end{cases}.$$

Let $g(\mathbf{x}) = \left.\frac{\partial u}{\partial t}\right|_{t=0}$. By our foregoing work, it suffices to evaluate integrals of the type

$$\int_{S_t(\mathbf{x})} g(\mathbf{x}')\, dS';$$

but a little reflection shows that this is just the area of that part of $S_t(\mathbf{x})$ which lies inside the unit *ball* $B_1(0) = \{\mathbf{x}||\mathbf{x}| \le 1\}$. This is thus a problem in geometry rather than calculus. We may distinguish four separate cases: (i) $B_t(\mathbf{x}) \subset B_1(0)$; (ii) $B_1(0) \subset B_t(\mathbf{x})$; (iii) $B_1(0) \cap B_t(\mathbf{x}) = \emptyset$; (iv) everything else. For case (i) to hold we must have $|\mathbf{x}| + t \le 1$, for then $|\mathbf{x} - \mathbf{x}'| < t$ implies $|\mathbf{x}'| < t + |\mathbf{x}| < 1$; also, in this case we have clearly

$$\int_{S_t(\mathbf{x})} g(\mathbf{x}')\, dS' = \text{area}(S_t(\mathbf{x})) = 4\pi t^2.$$

For case (ii) to hold we must have $t - |\mathbf{x}| \ge 1$, for then $|\mathbf{x}'| < 1$ implies $|\mathbf{x}' - \mathbf{x}| \le |\mathbf{x}'| + |\mathbf{x}| < 1 + |\mathbf{x}| < t$; and the integral will vanish unless $\mathbf{x} = 0$ and $t = 1$, in the which case it equals $4\pi$. For case (iii) to hold we must have $|\mathbf{x}| - t \ge 1$, for then $|\mathbf{x}'| < 1$ implies $|\mathbf{x}' - \mathbf{x}| \ge |\mathbf{x}| - |\mathbf{x}'| \ge 1 + t - |\mathbf{x}'| > t$; and in this case the integral is also clearly zero. Finally, in case (iv) we have $|\mathbf{x}| + t > 1$, $|t - |\mathbf{x}|| < 1$, and we see geometrically (try drawing a picture of the situation in two dimensions!) that the intersection of $S_t(\mathbf{x})$ with $B_1(0)$ is a spherical cap with central half-angle $\theta$ satisfying

$$1 = |\mathbf{x}|^2 + t^2 - 2t|\mathbf{x}|\cos\theta,$$

i.e., $\cos\theta = \frac{|\mathbf{x}|^2 + t^2 - 1}{2t|\mathbf{x}|}$. The area of such a spherical cap is given by

$$\begin{aligned} \int_0^{2\pi} \int_0^{\theta} t^2 \sin\theta'\, d\theta'\, d\phi' &= 2\pi t^2 \int_{\cos\theta}^1 dx = 2\pi t^2 (1 - \cos\theta) \\ &= 2\pi t^2 \frac{2t|\mathbf{x}| - |\mathbf{x}|^2 - t^2 + 1}{2t|\mathbf{x}|} = \frac{\pi t}{|\mathbf{x}|}\left(1 - (t - |\mathbf{x}|)^2\right). \end{aligned}$$

We thus see that the second part $u_2$ of the solution $u$ depends only on $\mathbf{x}$ (which makes sense, since the original problem was spherically symmetric), and that we have in particular (remembering the overall factor of $\frac{1}{4\pi t}$)

$$\begin{aligned} u_2(t, \mathbf{x}) &= \frac{1}{4\pi t} \begin{cases} 4\pi t^2, & |\mathbf{x}| + t \le 1 \\ \frac{\pi t}{|\mathbf{x}|}\left(1 - (t - |\mathbf{x}|)^2\right), & |\mathbf{x}| + t > 1,\ |t - |\mathbf{x}|| < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} t, & |\mathbf{x}| + t \le 1 \\ \frac{1 - (t - |\mathbf{x}|)^2}{4|\mathbf{x}|}, & |\mathbf{x}| + t > 1,\ |t - |\mathbf{x}|| < 1 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Since in this case $f = 0$, the first part of the solution will vanish and the above formula for $u_2$ gives in fact the full solution $u$. Let us consider what it means qualitatively. Let us fix some observation point $\mathbf{x}$ and consider $u(t, \mathbf{x})$ as a function of $t$ only. We identify two cases: (i) $|\mathbf{x}| \le 1$; (ii) $|\mathbf{x}| > 1$. In case (i), we see that at time $t = 0$ we have $u = 0$, while for $t \le 1 - |\mathbf{x}|$ we have $u(t, \mathbf{x}) = t$ by the above formula. Now suppose that $t > 1 - |\mathbf{x}|$, but that we still have $|t - |\mathbf{x}|| < 1$: this means that $-1 + |\mathbf{x}| < t < 1 + |\mathbf{x}|$, but the first inequality is trivial since $-1+|\mathbf{x}|<0$, so only the second inequality is meaningful, and we see that overall we have $|t-1|<|\mathbf{x}|$. In this case we have $u(t,\mathbf{x})=\frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}$, which is a segment of a parabola going from

$$u(1-|\mathbf{x}|,\mathbf{x})=\frac{1-(1-2|\mathbf{x}|)^2}{4|\mathbf{x}|}=\frac{4|\mathbf{x}|-4|\mathbf{x}|^2}{4|\mathbf{x}|}=1-|\mathbf{x}|$$

to

$$u(1+|\mathbf{x}|,\mathbf{x})=\frac{1-1}{4|\mathbf{x}|}=0.$$

Finally, if $|x|+t>1$ and $|t-|\mathbf{x}||\geq 1$, which in this case means (as indicated above) that $t>1+|\mathbf{x}|$, then we have $u(t,\mathbf{x})=0$. Thus we have finally

$$u(t,\mathbf{x})=\begin{cases} t, & 0\leq t\leq 1-|\mathbf{x}| \\ \frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}, & 1-|\mathbf{x}|\leq t\leq 1+|\mathbf{x}| \\ 0, & t\geq 1+|\mathbf{x}| \end{cases};$$

note that these three functions agree on the endpoints (except in the special case $\mathbf{x}=0$), so that the resulting function $u$ is continuous in time. This means that $u(t,\mathbf{x})$ first grows linearly, then drops of quadratically to zero, and finally stays at zero for all future time.

Now suppose that $|\mathbf{x}|>1$; in this case, the first case for $u_2$ above never happens, so we are only concerned with the cases $|t-|\mathbf{x}||<1$ and $|t-|\mathbf{x}||\geq 1$. The first case gives $-1+|\mathbf{x}|<t<1+|\mathbf{x}|$, while the second case (naturally) gives everything else; thus we have simply

$$u(t,\mathbf{x})=\begin{cases} \frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}, & -1+|\mathbf{x}|<t<1+|\mathbf{x}| \\ 0, & \text{otherwise} \end{cases}.$$

In this case, $u$ is zero up to time $-1+|\mathbf{x}|$ (this is the minimum time it takes for a signal to pass from the unit ball to the point $\mathbf{x}$); it then exhibits a quadratic increase and decrease, before dropping to zero at time $1+|\mathbf{x}|$ (which is the maximum time it takes for a signal to pass from the unit ball to the point $\mathbf{x}$), after which it remains zero for all time. In other words, then, at points $\mathbf{x}$ outside the unit ball, the solution is a quadratic pulse of width 2 whose height is inversely proportional to the distance $|\mathbf{x}|$ of the point from the origin.

We may use our work in this example to quickly do one more example, as follows.

EXAMPLE. Solve the following problem on $\mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2}=\nabla^2 u,\quad u|_{t=0}=\begin{cases} 1, & |\mathbf{x}|\leq 1 \\ 0, & |\mathbf{x}|>1 \end{cases},\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0.$$

In this case only the first term in the solution for $u$ remains, and we have by equation (1)

$$u(t,\mathbf{x})=\begin{cases} 1, & |\mathbf{x}|+t<1 \\ \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & |\mathbf{x}|+t>1,\ |t-|\mathbf{x}||<1 \\ 0, & |\mathbf{x}|+t>1,\ |t-|\mathbf{x}||>1 \end{cases},$$

where we have dropped the boundary points since the function $u_2(t,\mathbf{x})$ derived above is not in general differentiable there. If we proceed with the same type of analysis that we performed in the previous example, we see that for a fixed $\mathbf{x}$ with $|\mathbf{x}|<1$, we have

$$u(t,\mathbf{x})=\begin{cases} 1, & 0\leq t<1-|\mathbf{x}| \\ \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & 1-|\mathbf{x}|<t<1+|\mathbf{x}| \\ 0, & t>1+|\mathbf{x}| \end{cases};$$

we note that this function is not continuous. Qualitatively, at a point inside the unit ball $u$ is uniformly equal to 1 until the time $1-|\mathbf{x}|$, which is the least amount of time required for a signal to pass from outside the unit ball to the point $\mathbf{x}$; after that it jumps discontinuously to the value $1 - \frac{1}{2|\mathbf{x}|}$, before continually decreasing up to time $t = 1 + |\mathbf{x}|$, at which point it jumps again from the value $-\frac{1}{2|\mathbf{x}|}$ to 0, where it stays for all time.

Similarly, for a fixed $\mathbf{x}$ with $|\mathbf{x}| > 1$, we have

$$u(t, \mathbf{x}) = \begin{cases} \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & -1 + |\mathbf{x}| < t < 1 + |\mathbf{x}| \\ 0, & |t - |\mathbf{x}|| > 1 \end{cases},$$

which is not continuous either. This is a general feature of solutions to the wave equation with discontinuous initial data: whereas the heat equation smooths out initial discontinuities, the wave equation propagates them. Qualitatively, in this case we see that $u$ is initially zero, and stays zero until time $-1+|\mathbf{x}|$, which is the minimum amount of time required for a signal from inside the unit ball to reach the point $\mathbf{x}$; then it jumps discontinuously to the value $\frac{1}{2|\mathbf{x}|}$ before decreasing linearly to the value $-\frac{1}{2|\mathbf{x}|}$ at time $t = 1 + |\mathbf{x}|$ (which, similarly, is the maximum amount of time for a signal from inside the unit ball to reach $\mathbf{x}$), whereupon it jumps discontinuously back to 0. Thus we have again a single pulse, but the front and back edges are now discontinuous jumps, unlike the previous example.

These two examples end the examinable material for this course. (The last result done in class on August 8, about solutions to Laplace's equation, will be added to the notes on Green's functions.) The following appendices are not examinable (though some of the formulas in Appendix I may shed light on why we define convolution the way we do). The author thanks you for your patience, and hopes that you have gained something from your studies through this course. He would be happy to receive feedback on these notes at ncarruth@math.toronto.edu.

APPENDIX I. We would like to know what becomes of convolution under $\mathcal{O}$. To do this, we first consider in more detail exactly how the Fourier transform turns convolution into multiplication. Suppose that $f$ and $g$ are two suitable functions such that all needed Fourier transforms exist and can be inverted. Then we have

$$\begin{aligned}\mathcal{F}[f * g](\mathbf{k}) &= \int_{\mathbf{R}^m} \left[ \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' \right] e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}'\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}g(\mathbf{x}')e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}'\, d\mathbf{x}\end{aligned},$$

from which the result follows after the change of variables $\mathbf{u} = \mathbf{x} - \mathbf{x}'$, $\mathbf{v} = \mathbf{x}'$. We note that the crucial property above was that the expansion functions (the analogoues of the eigenfunctions $e_I$) satisfied the property

$$e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} = e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'};$$

mathematically, if we set for convenience $\mathbf{e_k}(\mathbf{x}) = e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}$, then the $\mathbf{e_k}$ are so-called *homomorphisms* from the Abelian group $\mathbf{R}^m$ (under vector addition) to the group of complex numbers of unit modulus $\{z \in \mathbf{C} | |z| = 1\}$ – in other words, they take addition of vectors to multiplication of complex numbers:

$$\mathbf{e_k}(\mathbf{x} + \mathbf{y}) = \mathbf{e_k}(\mathbf{x})\mathbf{e_k}(\mathbf{y}).$$

Now on a general region $D$, it does not make sense to ask whether the eigenfunctions $\mathbf{e}_I$ satisfy a similar property, since if $\mathbf{x}$, $\mathbf{y} \in D$ there is no reason at all to expect that $\mathbf{x} + \mathbf{y} \in D$.[5] Thus there does not appear to be any way to generalise this property of $\mathcal{F}$ to $\mathcal{O}$.

[5] One could, however, ask whether there were not a more general group structure on $D$. The mathematical field of *harmonic analysis* studies the extension of the transforms here to situations where the domains of the functions are topological groups. These groups are not, however, in general, open subsets of $\mathbf{R}^m$.

With some reflection, though, we note that $\mathbf{e}_\mathbf{k}(\mathbf{x})$ is a homomorphism in $\mathbf{k}$ as well as in $\mathbf{x}$ (this is actually a rather trivial observation, since $\mathbf{k}$ and $\mathbf{x}$ appear in $\mathbf{e}_\mathbf{k}(\mathbf{x})$ interchangeably, i.e., $\mathbf{e}_\mathbf{k}(\mathbf{x}) = \mathbf{e}_\mathbf{x}(\mathbf{k})$):

$$\mathbf{e}_{\mathbf{k}+\mathbf{l}}(\mathbf{x}) = \mathbf{e}_\mathbf{k}(\mathbf{x})\mathbf{e}_\mathbf{l}(\mathbf{x}).$$

From this we can show that the inverse Fourier transform also maps convolutions to products: suppose that we have two Fourier representations

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \quad g(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{g}(\mathbf{k}') e^{-2\pi i \mathbf{k}'\cdot\mathbf{x}}\, d\mathbf{k}';$$

then we may write their product as

$$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) &= \int_{\mathbf{R}^m\times\mathbf{R}^m} \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}') e^{-2\pi i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\, d\mathbf{k}\, d\mathbf{k}' \\ &= \int_{\mathbf{R}^m\times\mathbf{R}^m} \hat{f}(\mathbf{k}''-\mathbf{k}')\hat{g}(\mathbf{k}') e^{-2\pi i \mathbf{k}''\cdot\mathbf{x}}\, d\mathbf{k}''\, d\mathbf{k}' = \int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}''-\mathbf{k}')\hat{g}(\mathbf{k}')\, d\mathbf{k}'\right] e^{-2\pi i \mathbf{k}''\cdot\mathbf{x}}\, d\mathbf{k}'' \\ &= \mathcal{F}^{-1}[f*g](\mathbf{x}), \end{aligned}$$

where as before we have performed the change of variables $\mathbf{k}'' = \mathbf{k}+\mathbf{k}'$. (We note that the same kind of procedure could be used with the forward Fourier transform $\mathcal{F}$.) Now for the $\mathbf{e}_I$ the prospects of generalising this result are brighter since, for the index sets we have studied, if $I,\ J \in \mathcal{I}$, then in fact we also have $I+J \in \mathcal{I}$. This suggests that, while $\mathcal{O}$ might not turn convolutions into products, perhaps $\mathcal{O}^{-1}$ turns (some generalised form of) convolutions into products. We investigate this in more detail. Suppose that we have two expansions

$$u = \sum_{I\in\mathcal{I}} \tilde{u}(I)\mathbf{e}_I, \quad v = \sum_{I\in\mathcal{I}} \tilde{v}(I)\mathbf{e}_I;$$

then we may write, as before,

$$uv = \sum_{I,J\in\mathcal{I}} \tilde{u}(I)\tilde{v}(J)\mathbf{e}_I\mathbf{e}_J.$$

In general, though, there is now no clear way to proceed, since we do not know anything about the $\mathbf{e}_I$. Suppose that we still had the result $\mathbf{e}_I\mathbf{e}_J = \mathbf{e}_{I+J}$ (none of the sets of eigenfunctions we have dealt with actually satisfy this property); then the above sum would become

$$uv = \sum_{I,J\in\mathcal{I}} \tilde{u}(I)\tilde{v}(J)\mathbf{e}_{I+J} = \sum_{K\in\mathcal{I}}\sum_{J\in\mathcal{I}} \tilde{u}(K-J)\tilde{v}(J)\mathbf{e}_K,$$

from which we see that

$$\mathcal{O}[uv](I) = \sum_{J\in\mathcal{I}} \tilde{u}(I-J)\tilde{v}(J).$$

In general, the best we can hope for is some sort of expansion

$$\mathbf{e}_I\mathbf{e}_J = \sum_{K\in\mathcal{I}} \pi_{IJK}\mathbf{e}_K;$$

such an expansion surely exists, assuming anyway that the eigenfunctions $\mathbf{e}_I$ are not too pathological, and allows us to write

$$uv = \sum_{I,J,K\in\mathcal{I}} \tilde{u}(I)\tilde{v}(J)\pi_{IJK}\mathbf{e}_K,$$

where

$$\pi_{IJK} = (\mathbf{e}_I\mathbf{e}_J, \mathbf{e}_K),$$

meaning that

$$\mathcal{O}[uv](K) = \sum_{I,J\in\mathcal{I}} \tilde{u}(I)\pi_{IJK}\tilde{v}(J).$$

This is probably the closest we can come to generalising the property of mapping convolutions into products enjoyed by the Fourier transform. If $\pi_{IJK}$ is zero for most values of the parameters $IJK$, then this result may still be useful; if not, it is probably just a curiousity.

We give an example.

EXAMPLE. Let us consider the simple case of the eigenfunctions of the Laplacian on the unit square with Dirichlet boundary conditions. We have not considered this case directly but a quick review of our derivation of the eigenfunctions of the Laplacian on the unit cube shows that the eigenfunctions are $\mathbf{e}_I = \sin \ell\pi x \sin m\pi y$, where $I = (\ell, m)$, $\ell, m \in \mathbf{Z}$, $\ell, m > 0$. Thus in this case, letting $I = (\ell, m)$, $J = (\ell', m')$, and $K = (\ell'', m'')$, we have

$$\pi_{IJK} = \int_Q \sin \ell\pi x \sin m\pi y \sin \ell'\pi x \sin m'\pi y \sin \ell''\pi x \sin m''\pi y \, dx\, dy.$$

Now

$$\begin{aligned}\int_0^1 \sin \ell\pi x \sin \ell'\pi x \sin \ell''\pi x \, dx &= \frac{1}{2}\int_0^1 [\cos(\ell-\ell')\pi x - \cos(\ell+\ell')\pi x] \sin \ell''\pi x \, dx \\ &= \frac{1}{4}\int_0^1 \sin{(\ell''+\ell-\ell')\pi x} - \sin{(\ell''-\ell+\ell')\pi x} \\ &\qquad - \sin{(\ell''+\ell+\ell')\pi x} + \sin{(\ell''-\ell-\ell')\pi x}\, dx,\end{aligned}$$

which we shall not evaluate explicitly but only determine when it is zero. Clearly, $\int_0^1 \sin n\pi x = \frac{1}{n\pi}(1-(-1)^n)$ is zero exactly when $n$ is even; thus the above integral will be zero unless at least one of the quantities

$$\ell''+\ell-\ell', \quad \ell''-\ell+\ell', \quad \ell''+\ell+\ell', \quad \ell''-\ell-\ell'$$

is odd; but the first two are odd together, as are the last two, and thus the integral will vanish unless at least one of

$$\ell''+\ell-\ell', \quad \ell''-\ell-\ell'$$

is odd. But these are also seen to be odd together, so we find at last that the integral will vanish unless

$$\ell''-\ell-\ell'$$

is odd. Since analogous results hold for the corresponding $y$ integrals, we see that $\pi_{IJK}$ will be zero unless the quantity

$$K-(I+J)$$

is odd (meaning that both of its components are odd). While this is not nearly as nice as requiring it to vanish, it does tell us that $\pi_{IJK}$ vanishes for a sizeable number of indices $IJK$.

Similar triple products can (I believe) be worked out for the Legendre polynomials and the Legendre functions, and probably Bessel functions as well. If anyone is interested in knowing more about this particular topic, please let me know and I can provide more references.

APPENDIX II. SOLUTIONS TO THE NONHOMOGENEOUS WAVE EQUATION. We sketch a solution to the nonhomogeneous wave equation on $\mathbf{R}^3$. Thus consider the problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + F, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g.$$

Fourier transforming as usual, we have

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u} + \hat{F}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

Thus we must now solve an equation of the form

$$y'' + \alpha^2 y = h \tag{2}$$

where $\alpha > 0$ and $h$ is some given function. We may do this by the method of *variation of parameters* (also called *variation of constants*). (See [1], sections 3.4 and 3.6 (especially Theorem 3.6.4 and accompanying discussion) for a treatment of this method in a general setting.) The general solution to the corresponding homogeneous equation

$$y'' + \alpha^2 y = 0$$

is

$$y = a\cos\alpha x + b\sin\alpha x,$$

where $a = y(0)$ and $b = \frac{y'(0)}{\alpha}$. The method of variation of parameters starts by looking for solutions to equation (2) of the form

$$y = a(x)\cos\alpha x + b(x)\sin\alpha x.$$

Differentiating once, we obtain

$$y' = a'\cos\alpha x + b'\sin\alpha x + \alpha\left(-a(x)\sin\alpha x + b(x)\cos\alpha x\right).$$

We require the sum of the first two terms to vanish; then differentiating again, we obtain

$$y'' = \alpha\left(-a'\sin\alpha x + b'\cos\alpha x\right) - \alpha^2\left(a(x)\cos\alpha x + b(x)\sin\alpha x\right),$$

from which we see easily that

$$y'' + \alpha y = h = \alpha\left(-a'\sin\alpha x + b'\cos\alpha x\right).$$

Combining this with the requirement

$$a'\cos\alpha x + b'\sin\alpha x = 0,$$

we see that we now have the system

$$\begin{aligned} \cos\alpha x a' + \sin\alpha x b' &= 0 \\ -\alpha\sin\alpha x a' + \alpha\cos\alpha x b' &= h. \end{aligned}$$

Now the determinant of the coefficient matrix is just the Wronskian of the two solutions:

$$W = \begin{vmatrix} \cos\alpha x & \sin\alpha x \\ -\alpha\sin\alpha x & \alpha\cos\alpha x \end{vmatrix} = \alpha,$$

so that as long as we assume $\alpha \neq 0$ we may solve the above system; in fact, we have (using our formula for the inverse of a two by two matrix)

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\alpha}\begin{pmatrix} \alpha\cos\alpha x & -\sin\alpha x \\ \alpha\sin\alpha x & \cos\alpha x \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} -h\frac{\sin\alpha x}{\alpha} \\ h\frac{\cos\alpha x}{\alpha} \end{pmatrix}.$$

From this we have

$$\begin{aligned} a &= y(0) - \frac{1}{\alpha}\int_0^x h(u)\sin\alpha u\,du \\ b &= \frac{1}{\alpha}y'(0) + \frac{1}{\alpha}\int_0^x h(u)\cos\alpha u\,du, \end{aligned}$$

so that

$$\begin{aligned} y &= y(0)\cos\alpha x + y'(0)\frac{\sin\alpha x}{\alpha} + \frac{1}{\alpha}\int_0^x h(u)\sin\alpha x\cos\alpha u - \sin\alpha u\cos\alpha x\,du \\ &= y(0)\cos\alpha x + y'(0)\frac{\sin\alpha x}{\alpha} + \int_0^x h(u)\frac{\sin\alpha(x-u)}{\alpha}\,du. \end{aligned}$$

We now return to our original problem:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u} + \hat{F}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

The above formula gives

$$\hat{u}(t,\mathbf{k}) = \hat{f}\cos 2\pi|\mathbf{k}|t + \hat{g}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|} + \int_0^t \hat{F}(s,\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|(t-s)}{2\pi|\mathbf{k}|}\,ds.$$

The first two terms are of course the same as those we obtained for the homogeneous equation above. We see that we may invert this formula in much the same way as we did the formula for the solution to the homogeneous problem previously. Specifically, we obtain

$$u(t,\mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,d\mathbf{x}'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\,d\mathbf{x}' + \int_0^t \frac{1}{4\pi(t-s)}\int_{S_{t-s}(\mathbf{x})} F(s,\mathbf{x}')\,dS'\,ds.$$

Let us investigate the final term here, which is the only new thing. We see that the contribution which it gives to $u(\mathbf{x})$ is equal to the integral over all times from 0 to $t$ of a quantity which at time $s$ is (proportional to) the integral over the sphere of radius $t-s$ centred at $\mathbf{x}$ – in other words, the integral over the surface from which a signal will take exactly the time $t-s$ remaining to reach the point $\mathbf{x}$. More succinctly, the contribution $F$ makes to $u$ at the point $\mathbf{x}$ and time $t$ is the integral over the set of all points (through all of space-time, not just space) $(s,\mathbf{x}')$ satisfying $|\mathbf{x}-\mathbf{x}'| = t-s$, i.e., the set of all points just able to send a signal to $\mathbf{x}$ by time $t$.

We may write the above result more simply as follows. First, let us do a change of variables and write $u = t-s$, $\mathbf{x}'' = \mathbf{x}' - \mathbf{x}$; then the last integral above becomes

$$\int_0^t \int_{S_u(\mathbf{x})} \frac{1}{4\pi u}F(t-u,\mathbf{x}')\,dS'\,du = \int_0^t \int_{S_u(0)} \frac{1}{4\pi u}F(t-u,\mathbf{x}''+\mathbf{x})\,dS''\,du;$$

if we now introduce spherical coordinates $(r'',\theta'',\phi'')$ for $\mathbf{x}''$, we may write this integral as (noting that $dS'' = u^2\sin\theta''\,d\theta''\,d\phi''$ since it is the full surface-area element for the sphere of radius $u$)

$$\begin{aligned}\int_0^t \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi u}F(t-u,\mathbf{x}''+\mathbf{x})\sin\theta''\,d\theta''\,d\phi''\,u^2\,du &= \int_{B_t(0)} \frac{F(t-r'',\mathbf{x}''+\mathbf{x})}{4\pi r''}\,dV \\ &= \int_{B_t(\mathbf{x})} \frac{F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\,dV,\end{aligned}$$

where we have changed back to $\mathbf{x}' = \mathbf{x}''+\mathbf{x}$ in the last line, and noted that $r'' = |\mathbf{x}''| = |\mathbf{x}-\mathbf{x}'|$. This expression is related to the so-called *retarded potential* which is used in studying electromagnetic radiation. We recognise the quantity $\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$ as being (up to a sign) the Green's function for the Laplacian on $\mathbf{R}^3$; what is different here is that we are integrating it against a function $F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')$ instead of a function of $\mathbf{x}'$ alone. In other words, roughly speaking, the effect of the source $F$ on the solution $u$ is obtained by integrating against the ordinary Green's function for the Laplacian, but using the *retarded* source function $F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')$ at times which are such that a signal from the point of integration $\mathbf{x}'$ can just reach the observation point $\mathbf{x}$ by the observation time $t$.

We note that the above method of variation of parameters can be used with only slight modifications to solve the nonhomogeneous wave equation on a bounded region, in a manner analogous to our solution to the wave equation on a disk given above.

REFERENCES

Coddington, E. A., and Levinson, N. Theory of Ordinary Differential Equations. New York: McGraw-Hill Book Company, Inc., 1955.