Summary:
- We use the notion of approximate identities introduced last week to prove a version of the Fourier inversion theorem.
- We then use Fourier transforms to study the heat equation, obtaining both integral formulas for solutions to the homogeneous and inhomogeneous equations as well as qualitative information.

A THEOREM ON APPROXIMATE IDENTITIES. We have the following generalisation of Example (c) from last week's lecture notes.

THEOREM 1. Let $\psi : \mathbf{R}^m \to \mathbf{R}^1$ satisfy $\int_{\mathbf{R}^m} |\psi(\mathbf{x})|\, d\mathbf{x} < \infty$, $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$. Then the sequence $\{\psi_n\}_{n=1}^{\infty}$ given by

$$\psi_n(\mathbf{x}) = n^m \psi(n\mathbf{x})$$

is an approximate identity, at least for continuous, bounded functions (i.e., elements of the space $C_b(\mathbf{R}^m)$ to be introduced momentarily).

Proof. The proof is almost identitical to that of the case $m = 1$. Let $f : \mathbf{R}^m \to \mathbf{R}^1$ be bounded and continuous. Then we see that

$$\int_{\mathbf{R}^m} f(\mathbf{x})\psi_n(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x})\psi(n\mathbf{x})n^m\, d\mathbf{x} = \int_{\mathbf{R}^m} f\left(\frac{\mathbf{u}}{n}\right)\psi(\mathbf{u})\, d\mathbf{u},$$

where we have made the change of variables $\mathbf{u} = n\mathbf{x}$, which gives $d\mathbf{u} = n^m\, d\mathbf{x}$ since we are working on $\mathbf{R}^m$. Now this integral can be broken down as follows:

$$\int_{\mathbf{R}^m} f\left(\frac{\mathbf{u}}{n}\right)\psi(\mathbf{u})\, d\mathbf{u} = \int_{\mathbf{R}^m} \left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\psi(\mathbf{u})\, d\mathbf{u} + f(0),$$

since $\int_{\mathbf{R}^m} \psi(\mathbf{u})\, d\mathbf{u} = 1$. It thus suffices to show that the first term on the right-hand side above approaches 0 as $n \to \infty$. Let $M = \sup_{\mathbf{x}\in\mathbf{R}^m} |f(\mathbf{x})| + 1$ (where $\sup_{\mathbf{x}\in\mathbf{R}^m} |f(\mathbf{x})|$ denotes the least upper bound for $|f(\mathbf{x})|$ on $\mathbf{R}^m$), let $\epsilon > 0$, let $\delta > 0$ be such that $|f(\mathbf{x}) - f(0)| < \frac{\epsilon}{2\int_{\mathbf{R}^m} |\psi(\mathbf{u})|\, d\mathbf{u}}$ when $|\mathbf{x}| < \delta$, and let $K \in \mathbf{Z}$, $K > 0$ be such that

$$\int_{|\mathbf{x}|>K} |\psi(\mathbf{x})|\, d\mathbf{x} < \frac{\epsilon}{2M};$$

such a K clearly exists since $\int_{\mathbf{R}^m} |\psi(\mathbf{x})|\, d\mathbf{x} < \infty$. Furthermore, let $N \in \mathbf{Z}$, $N > 0$ be such that $N > \frac{K}{\delta}$, and let $n > N$. Now we have

$$\begin{aligned}\left|\int_{\mathbf{R}^m} \left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\psi(\mathbf{u})\, d\mathbf{u}\right| &\le \int_{\mathbf{R}^m} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} \\ &= \int_{|\mathbf{x}|<K} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} + \int_{|x|>K} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} \\ &\le \frac{\epsilon}{2\int_{\mathbf{R}^m} |\psi(\mathbf{u})|\, d\mathbf{u}} \int_{|\mathbf{x}|<K} |\psi(\mathbf{u})|\, d\mathbf{u} + 2M \int_{|\mathbf{x}|>K} |\psi(\mathbf{u})|\, d\mathbf{u} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

where we have used the fact that $n > N$ implies $\frac{K}{n} < \delta$, and replaced the integral over $|\mathbf{x}| < K$ one over $\mathbf{R}^m$ in the last line. This completes the proof. QED.

The basic idea here is that the function $f\left(\frac{\mathbf{u}}{n}\right)$ looks like a very 'zoomed-in' version of f, so that since ψ needs to be concentrated somewhere finite, if we zoom in f enough it will eventually cover essentially all of the places where ψ is not trivially small; and since f is continuous, zooming in like this makes it look very close to the single number $f(0)$, and since $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$, the resulting integral will be very close to $f(0)$. The foregoing ϵ-δ proof merely makes this rigorous.

A WORD ON FUNCTION SPACES, AND THE NATURE OF THE FOURIER TRANSFORM. We recall that we have defined the space

$$L^1(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R}^1 | \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x} < \infty\}.$$

We now define the space of bounded continuous functions on $\mathbf{R}^m$:

$$C_b(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R}^1 | f \text{ is bounded and continuous on } \mathbf{R}^m\}.$$

(Both spaces could also be defined with real-valued functions replaced by complex-valued ones; in that case, $|f(\mathbf{x})|$ in the definition of $L^1(\mathbf{R}^m)$ means the modulus of the complex number $f(\mathbf{x})$.)

We are now in a position to say more precisely what exactly the Fourier transform *is*. First of all, we recall that a *function* f from a set A to a set B is a rule which assigns to each element $a \in A$ an element $f(a) \in B$. Now the Fourier transform is a *function on functions*, in the sense that for every function in a certain class it gives another function in another class. We have shown how to define $\mathcal{F}[f]$ for any $f \in L^1(\mathbf{R}^m)$; the result is another function on $\mathbf{R}^m$ whose rule is

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

Now we claim that for $f \in L^1(\mathbf{R}^m)$, $\mathcal{F}[f] \in C_b(\mathbf{R}^m)$. That $\mathcal{F}[f]$ is bounded can be seen easily: for any $\mathbf{k} \in \mathbf{R}^m$,

$$|\mathcal{F}[f](\mathbf{k})| = |\int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}| \leq \int_{\mathbf{R}^m} |f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}|\, d\mathbf{x} = \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x},$$

and this last quantity is finite since $f \in L^1(\mathbf{R}^m)$. Since it is independent of $\mathbf{k}$, we see that $\mathcal{F}[f]$ is indeed bounded on $\mathbf{R}^m$, as claimed. To see that it is also continuous on $\mathbf{R}^m$, we may proceed as follows: let $\mathbf{k}_0 \in \mathbf{R}^m$; then

$$\begin{aligned}\lim_{\mathbf{k}\to\mathbf{k}_0} \mathcal{F}[f](\mathbf{k}) &= \lim_{\mathbf{k}\to\mathbf{k}_0} \mathcal{F}[f](\mathbf{k}) = \lim_{\mathbf{k}\to\mathbf{k}_0} \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m} f(\mathbf{x}) \lim_{\mathbf{k}\to\mathbf{k}_0} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}_0\cdot\mathbf{x}}\, d\mathbf{x} = \mathcal{F}[f](\mathbf{k}_0),\end{aligned}$$

where we can interchange the limit with the integral since $f \in L^1(\mathbf{R}^m)$[1]. This shows that $\mathcal{F}[f]$ is continuous on $\mathbf{R}^m$, and hence that $\mathcal{F}[f] \in C_b(\mathbf{R}^m)$, as claimed.

The foregoing shows that we may think of the Fourier transform $\mathcal{F}$ as a *function on functions*, or perhaps better put, a *transformation* or *map* on functions which takes elements of $L^1(\mathbf{R}^m)$ to elements of $C_b(\mathbf{R}^m)$.[2] It can be shown that the Fourier transform actually maps into the subspace of $C_b(\mathbf{R}^m)$ consisting of those functions which *go to zero at $infty$* in a certain sense, but we shall not show that here.

FOURIER INVERSION THEOREM. A version of the Fourier inversion theorem was stated at the end of last week's notes; here we shall prove the following slightly modified version.

THEOREM 2. Suppose that $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$ (i.e., that f is in both L^1 and C^b), and that $\hat{f} \in L^1$. Then we have

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

Proof. This may be shown by using a particular approximate identity. (The one we shall use here is not the only option, incidentally; actually there is a very broad range of possibilities.) For convenience, if $\mathbf{k} \in \mathbf{R}^m$ we shall write $k = |\mathbf{k}|$ for the norm of $\mathbf{k}$. We work from the right-hand side to the left-hand side. Now[3] since $\hat{f} \in L^1(\mathbf{R}^m)$, we may write

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \lim_{n\to\infty} \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k},$$

[1]Again, this technically requires that we use the dominated convergence theorem for the Lebesgue integral.
[2]We note in passing that the inequality $|\mathcal{F}[f](\mathbf{k})| \leq \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x}$ implies that $\mathcal{F}$ is in fact a *continuous* map from L^1 to C_b, at least if we use appropriate norms to give these spaces topologies.
[3]Applying again the dominated convergence theorem of Lebesgue integration theory!

since in the limit the quantity $-\frac{k^2}{n^2} \to 0$, so the exponential approaches 1. Substituting in the definition of $\hat{f}(\mathbf{k})$, we have

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} f(\mathbf{x}') e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \right] e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k};$$

now because of the factor $e^{-\frac{k^2}{n^2}}$, the integrand is in fact integrable over the product $\mathbf{R}^m \times \mathbf{R}^m$, which implies that we can interchange the order of integration, obtaining

$$\int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} e^{-\frac{k^2}{n^2}} e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\, d\mathbf{k} \right] f(\mathbf{x}')\, d\mathbf{x}'.$$

Now the integral in brackets is seen to be the Fourier transform of the Gaussian function $e^{-\frac{k^2}{n^2}}$, evaluated at the point $\mathbf{x}' - \mathbf{x}$. From the results on homework 10, this is seen to be

$$\left(\pi n^2\right)^{\frac{m}{2}} e^{-\pi^2 n^2 |\mathbf{x}'-\mathbf{x}|^2}. \tag{1}$$

Thus the full integral above becomes

$$\int_{\mathbf{R}^m} \left(\pi n^2\right)^{\frac{m}{2}} f(\mathbf{x}') e^{-\pi^2 n^2 |\mathbf{x}-\mathbf{x}'|^2}\, d\mathbf{x}'.$$

Now setting

$$\psi(\mathbf{x}) = \pi^{\frac{m}{2}} e^{-\pi^2 |\mathbf{x}|^2},$$

and noting that $\psi \in L^1(\mathbf{R}^m)$, $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$, and that the function in (1) above is just $\psi_n(\mathbf{x})$ as defined in Theorem 1, we see that, by Theorem 1, we have finally

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \lim_{n\to\infty} \int_{\mathbf{R}^m} \left(\pi n^2\right)^{\frac{m}{2}} e^{-\pi^2 n^2 |\mathbf{x}-\mathbf{x}'|^2} f(\mathbf{x}')\, d\mathbf{x}' = f(\mathbf{x}),$$

as desired. QED.

The transformation on functions which takes a function $f(\mathbf{k})$ in $L^1(\mathbf{R}^m)$ to the function

$$\int_{\mathbf{R}^m} f(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}$$

is called the *inverse Fourier transform* and is denoted $\mathcal{F}^{-1}[f]$. The foregoing shows that, if $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$, then $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$, i.e., that $\mathcal{F}^{-1}$ is indeed a left inverse to $\mathcal{F}$. Identical arguments to those in the proof just given show that also $\mathcal{F}[\mathcal{F}^{-1}[f]] = f$ for such f. These formulas are also correct much more generally: in fact, if $f \in L^1(\mathbf{R}^m)$ is any function satisfying $\int_{\mathbf{R}^m} |f(\mathbf{x})|^2\, d\mathbf{x} < \infty$, then these relations still hold for f. We shall, however, not pursue such questions here but merely regard the above result as being an example of the results which can be obtained. For the most part we shall work with Fourier transforms and their inverses rather more formally.

HEAT EQUATION ON R^m. Consider the following problem on $(0, +\infty) \times \mathbf{R}^m$ (points of which we shall denote as $(t, \mathbf{x})$):

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f.$$

Suppose that $f \in L^1(\mathbf{R}^m)$, and suppose that u and all of its derivatives up to second order are in $L^1(\mathbf{R}^m)$[4]; then, taking the Fourier transform of the above equation, we obtain (assuming that we may interchange the order of integration and differentiation with respect to t)

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2 |\mathbf{k}|^2 \hat{u}, \quad \hat{u}|_{t=0} = \hat{f}.$$

[4]All we really need, of course, is for u to be such that we can take the Fourier transforms needed below. The given conditions are sufficient but probably not necessary.

Now the first equation is almost identical to the one we found when solving the heat equation on the unit cube, and has the solution

$$\hat{u}(t,\mathbf{k}) = \hat{u}(0,\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t} = \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}.$$

Assuming that we may apply Fourier inversion, this gives rise immediately to the integral expression

$$u = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k}.$$

This expression, however, is rather unsatisfactory, since calculating $\hat{f}$ requires us to perform a rather difficult integral, and then we are still faced with evaluating the above integral in order to finally obtain u; in other words, the above expression requires two integrations. We may use properties of the Fourier transform to reduce this to one, as follows. First, we note that

$$\begin{aligned}\mathcal{F}^{-1}[e^{-4\pi^2|\mathbf{k}|^2 t}](\mathbf{x}) &= \int_{\mathbf{R}^m} e^{-4\pi^2|\mathbf{k}|^2 t}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k} = \int_{\mathbf{R}^m} e^{-4\pi^2|\mathbf{k}|^2 t}e^{-2\pi i\mathbf{k}\cdot(-\mathbf{x})}\,d\mathbf{k} \\ &= \left(\frac{\pi}{4\pi^2 t}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|-\mathbf{x}|^2}{4\pi^2 t}} = \frac{1}{(4\pi t)^{\frac{m}{2}}}e^{-\frac{|\mathbf{x}|^2}{4t}}.\end{aligned}$$

This last expression is called the *heat kernel*; let us denote it by $K(t,\mathbf{x})$. Thus we see that $\mathcal{F}[K](t,\mathbf{k}) = e^{-4\pi^2|\mathbf{k}|^2 t}$, so that

$$\hat{u}(t,\mathbf{k}) = \mathcal{F}[f](\mathbf{k})\mathcal{F}[K](t,\mathbf{k}) = \mathcal{F}[f * K],$$

where the convolution is performed only on the spatial variables. Fourier inversion then implies that we have

$$\begin{aligned}u(t,\mathbf{x}) = (f*K)(t,\mathbf{x}) = (K*f)(t,\mathbf{x}) &= \int_{\mathbf{R}^m} K(t,\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' \\ &= \frac{1}{(4\pi t)^{\frac{m}{2}}}\int_{\mathbf{R}^m} e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4t}}f(\mathbf{x}')\,d\mathbf{x}'.\end{aligned}$$

This is the desired formula for u in terms of f.

In order to apply this formula to concrete examples, of course, we would need to find a function f for which the integral above is actually calculable. There are some examples in the textbook for which the above integral can be determined in terms of the error function; for now we shall just comment on some qualitative properties of solutions to the heat equation which emerge from it. The first of these is the result

$$\lim_{t\to\infty} u(t,\mathbf{x}) = 0;$$

this can be seen from the above formula since the quantity $\frac{1}{(4\pi t)^{\frac{m}{2}}} \to 0$ as $t\to\infty$, while the integral simply approaches $\int_{\mathbf{R}^m} f(\mathbf{x}')\,d\mathbf{x}'$, which is finite since $f\in L^1(\mathbf{R}^m)$. It can actually be seen even more clearly from the formula for the Fourier transform for u above, namely

$$\hat{u}(t,\mathbf{k}) = \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}:$$

from this formula it is entirely obvious that $\hat{u}\to 0$ as $t\to\infty$, so assuming that the inverse Fourier transform is continuous in an appropriate sense, the same will be true also of u. Next we note that, at least assuming $f\in C_b$,

$$\lim_{t\to 0+} u(t,\mathbf{x}) = f(\mathbf{x}).$$

To prove this fully rigorously would require an extension of Theorem 1 to the case of nonintegral n; we shall content ourselves by investigating the limit[5]

$$\lim_{n\to\infty} u(\frac{1}{n^2},\mathbf{x}).$$

[5]If the limit above exists, it will certainly be equal to the limit below. However, the limit below can exist without the original limit existing (consider, for example, the function $sin(\frac{2\pi}{t})$, which is zero when $t=\frac{1}{n^2}$ but has no limit as $t\to 0$): this is similar to the fact we learned in multivariable calculus, that a function can have a limit at a point along a certain curve without having a full limit at that point.

This is seen to be

$$\lim_{n\to\infty} \frac{n^m}{(4\pi)^{\frac{m}{2}}} \int_{\mathbf{R}^m} e^{-\frac{n^2|\mathbf{x}-\mathbf{x}'|^2}{4}} f(\mathbf{x}')\, d\mathbf{x}';$$

since

$$\int_{\mathbf{R}^m} e^{-\frac{n^2|\mathbf{x}|^2}{4}}\, d\mathbf{x} = (4\pi)^{\frac{m}{2}},$$

we see that we may apply Theorem 1 to conclude that this limit is in fact $f(\mathbf{x})$, as desired.

The foregoing has the following curious consequence: any function $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$ is the limit of a sequence of functions which have infinitely many derivatives. To see this, we need the following result about convolutions (which is worth knowing in its own right). Suppose that $f, g \in L^1(\mathbf{R}^m)$, and that $\partial_j f \in L^1(\mathbf{R}^m)$ for some j. Then we have

$$\partial_j(f * g)(\mathbf{x}) = \partial_j \int_{\mathbf{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^m} (\partial_j f)(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = ((\partial_j f) * g)(\mathbf{x});$$

in other words, $\partial_j(f * g) = (\partial_j f) * g$. Note that we did not need to assume anything about differentiability (or even continuity) of g here; thus this result shows that the convlution of two functions is at least as *smooth* (i.e., possesses at least as many derivatives) as the smoother of the two factors. Now the heat kernel

$$K(t, \mathbf{x}) = \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4t}}$$

clearly possesses derivatives of all orders in $\mathbf{x}$, for all $t > 0$; since any solution to the heat equation is just the convolution of K with the initial data f, we see that any such solution must have derivatives of all orders in $\mathbf{x}$ for all $t > 0$. In other words, the functions

$$u(\frac{1}{n^2}, \mathbf{x})$$

must have infinitely many derivatives in $\mathbf{x}$ for all n. But these functions converge to f, meaning that f is indeed a limit of functions with infinitely many derivatives, as claimed. We say that the heat equation *smooths out* its initial data. (This is a general property of the class of equations known as parabolic equations of which the heat equation is the simplest example. The wave equation, which we shall study next week, is a member of the class of *hyperbolic equations* and transports singularities rather than smoothing them out.)

Finally, we show how Fourier techniques can be used to solve the inhomogeneous heat equation. To this end, consider the following problem on $\mathbf{R}^m$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + g, \quad u|_{t=0} = f.$$

If we assume as usual that all necessary Fourier transforms exist, then Fourier transforming gives

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u} + \hat{g}, \quad \hat{u}|_{t=0} = \hat{f}.$$

The first equation again becomes a simple linear first-order ordinary differential equation which may be solved using the integrating factor $e^{4\pi^2|\mathbf{k}|^2 t}$. Multiplying both sides by this factor and rearranging, we obtain

$$e^{4\pi^2|\mathbf{k}|^2 t}\hat{g} = e^{4\pi^2|\mathbf{k}|^2 t}\frac{\partial \hat{u}}{\partial t} + 4\pi^2|\mathbf{k}|^2 e^{4\pi^2|\mathbf{k}|^2 t}\hat{u} = \frac{\partial}{\partial t}\left[e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}\right],$$

so replacing t by s and integrating with respect to s from 0 to t,

$$\begin{aligned} e^{4\pi^2|\mathbf{k}|^2 s}\hat{u}(s,\mathbf{k})\Big|_{s=0}^{s=t} &= \int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\hat{g}(s,\mathbf{k})\, ds \\ e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}(t,\mathbf{k}) - \hat{u}(0,\mathbf{k}) &= \int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\hat{g}(s,\mathbf{k})\, ds \\ \hat{u}(t,\mathbf{k}) &= e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \int_0^t e^{-4\pi^2|\mathbf{k}|^2(t-s)}\hat{g}(s,\mathbf{k}), ds. \end{aligned} \tag{2}$$

The first term is just the expression we obtained before, as it should be since that is just the case $g = 0$, and in that case the second term vanishes. Now the second term looks somewhat like a convolution integral, though not quite because of the limits; it turns out that this type of integral is the kind of convolution appropriate for the so-called *Laplace transform* usually encountered in introductory classes on ordinary differential equations.[6] At any rate, assuming that we may interchange the order of the t integral with the $\mathbf{k}$ integral appearing in $\mathcal{F}^{-1}$, we may take the inverse Fourier transform of this expression as before to obtain

$$u(t, \mathbf{x}) = K(t, \mathbf{x}) * f(\mathbf{x}) + \int_0^t K(t-s, \mathbf{x}) * g(s, \mathbf{x})\, ds,$$

where all convolutions are with respect to the variable $\mathbf{x}$.

As with the case of the homogeneous heat equation above, for this formula to be useful in practice we would need functions f and g for which the above integrals are calculable. An (attempt at an) example of this sort is given in Homework 11. For the moment let us do what we did when we discussed the homogeneous heat equation and see what kinds of qualitative information we can determine from this solution. We see that the first term, which is just the solution of the homogeneous equation with the given initial data, goes to zero as $t \to \infty$ and to $f(\mathbf{x})$ as $t \to 0^+$, as before. Now let us consider the second term. Suppose that $g(t, \mathbf{x}) = g_0(\mathbf{x})$ for all $t \geq 0$. Then $\hat{g}(t, \mathbf{k}) = \hat{g}_0(\mathbf{k})$ for all $\mathbf{k}$. Returning now to the expression for the Fourier transform of u in equation (2) above, we see that

$$\begin{aligned}
4\pi^2|\mathbf{k}|^2\hat{u}(t, \mathbf{k}) &= 4\pi^2|\mathbf{k}|^2 \left[e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k}) e^{-4\pi^2|\mathbf{k}|^2 t} \int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\, ds \right] \\
&= 4\pi^2|\mathbf{k}|^2 e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k}) e^{-4\pi^2|\mathbf{k}|^2 t} e^{4\pi^2|\mathbf{k}|^2 s}\Big|_0^t \\
&= 4\pi^2|\mathbf{k}|^2 e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k}) \left[1 - e^{-4\pi^2|\mathbf{k}|^2 t} \right],
\end{aligned}$$

from which it is clear that in the limit $t \to \infty$ we have

$$4\pi^2|\mathbf{k}|^2\hat{u} = \hat{g}_0(\mathbf{k}).$$

But (assuming that the functions involved are such that we can take the inverse Fourier transform of both sides) this is nothing but the equation $-\nabla^2 u = g_0$! From this we see that (at least for suitable functions f and g_0) in the limit as $t \to \infty$, u converges to the solution to the Poisson equation $\nabla^2 u = -g_0$ on $\mathbf{R}^m$. This should be compared with our earlier result, when working on a bounded region, that if the heat equation were solved with nonhomogeneous boundary conditions, in the limit as $t \to \infty$ the solution would converge to the solution to Laplace's equation on that region with the same boundary conditions. In the current case, since we are solving on the whole space $\mathbf{R}^m$, there are no real boundary conditions (the only relevant one are that u should be in L^1), but our work here shows that a similar result holds for the inhomogeneous heat equation.

[6]The Laplace transform takes account of initial conditions while the Fourier transform extends from $-\infty$ to $+\infty$, i.e., over the whole range of the variable. We might have a chance to say a little bit about the Laplace transform towards the end of the course.