Summary:
- We clarify and review a couple points from last week.
- We then show how our work with Poisson's equation gives a series expression for the so-called *Green's function*.
- Using this, we derive other formulas relating to the Green's function, and indicate its conceptual import.
- We then introduce two new integral operations: the *Fourier transform*, which is an integral operator, and *convolution*, which is a generalised product of functions.
- We derive various properties of these operations and provide an indication of their use in solving partial differential equations.

INNER PRODUCTS FOR SPHERICAL BESSEL FUNCTIONS. In the previous week's lectures we derived the orthogonality relation

$$\int_0^1 x^2 j_\ell(\kappa_{\ell i}x)\, j_\ell(\kappa_{\ell j}x)\, dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2} j_{\ell+1}^2(\kappa_{\ell i}), & i = j \end{cases}.$$

This indicates that when expanding functions in series of spherical Bessel functions on the interval $[0,1]$, the inner product we should use is

$$(f,g) = \int_0^1 f(x)\overline{g(x)}x^2\, dx.$$

This should be compared to the inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}x\, dx$$

used when expanding functions in series of ordinary Bessel functions $J_m(\lambda_{mi}x)$ (there we typically used ρ instead of x).

With this inner product in the r coordinate, we noted that the full inner product used when expanding functions in series of the eigenfunctions of the Laplacian on a ball is

$$(f,g) = \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)}\, d\phi \sin\theta\, d\theta\, r^2\, dr = \iiint_B f(\mathbf{x})\overline{g(\mathbf{x})}\, dV,$$

where $\mathbf{x}$ denotes an arbitrary point in three-dimensional space and V is the usual volume element in three-dimensional space. An examination of the inner products used for the various coordinates in the other coordinate systems (rectangular and cylindrical) in which we have constructed eigenfunctions for the Laplacian shows that the same formula holds; specifically, we have respectively

$$(f,g) = \int_Q f(\mathbf{x})\overline{g(\mathbf{x})}\, dV, \quad (f,g) = \int_C f(\mathbf{x})\overline{g(\mathbf{x})}\, dV$$

when expanding in rectangular and cylindrical coordinates, respectively. Thus while the inner products used in the different individual coordinates differ, the inner product on the full set is always given by integrating $f(\mathbf{x})\overline{g(\mathbf{x})}$ over the full set. While this does not add much computationally, it is helpful for remembering the individual inner products we have learned so far.

GREEN'S FUNCTIONS. Recall (see p. 10 of the lecture notes for July 2 – 4) the following manipulations. If $\{\mathbf{e}_I\}$ is a complete set of eigenfunctions for the Laplacian on a set D, say satisfying homogeneous Dirichlet boundary conditions on ∂D, with corresponding eigenvalues λ_I (which we assume to be all nonzero), then the solution to the problem

$$\nabla^2 u = f, \quad u|_{\partial D} = 0$$

has the series solution

$$u = \sum_I \frac{1}{\lambda_I} \frac{(f,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}\mathbf{e}_I.$$

Here I represents an abstract index which may contain multiple separate indices; e.g., in the case of the eigenfunctions on the unit ball I will represent the triple (ℓ, m, i). For simplicity, let us assume that $(\mathbf{e}_I, \mathbf{e}_I) = 1$; this can always be achieved by rescaling the eigenfunctions $\mathbf{e}_I$ if necessary. Then the above formula can be expanded as follows:

$$\begin{aligned} u(\mathbf{x}) &= \sum_I \frac{1}{\lambda_I}(f, \mathbf{e}_I)\mathbf{e}_I(\mathbf{x}) \\ &= \sum_I \frac{1}{\lambda_I} \int_D f(\mathbf{x}')\overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(\mathbf{x}) \\ &= \int_D \left(\sum_I \frac{\mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I} \right) f(\mathbf{x}')\, d\mathbf{x}', \end{aligned} \tag{1}$$

where we assume that the sum is such that we may interchange sum and integral. The (negative[1] of the) function in parentheses above is called the *Green's function* for the problem. We denote it by $G(\mathbf{x}, \mathbf{x}')$, noting that both $\mathbf{x}$ and $\mathbf{x}'$ are actually points in D, hence (at least for $D \subset \mathbf{R}^3$) in $\mathbf{R}^3$. Note that, since G is expressed in terms of the eigenfunctions, it depends in principle upon everything that they depend on, namely (1) the operator (the Laplacian); (2) the region D; (3) the boundary conditions (here, homogeneous Dirichlet). If any of these change, the Green's function will in principle change also.

If the above were all there were to the Green's function, it would not be clear why the above formula is useful: it is not clear that the series expansion above should be summable to anything simpler, and if it isn't then the only real use for the above formula would be to run the above derivation backwards to obtain the series expansion for u from which we started. It turns out, though, that the notion of a Green's function can be discussed profitably independent of any expansion in eigenfunctions, and to this we now turn.

(Before doing this, it is probably helpful to say a few words about the general direction of the course moving forwards. So far we have been focussed almost exclusively on obtaining expansions in complete orthogonal sets which provide formal solutions to our problems. Going forwards, what we shall do gives more directly *integral* representations (rather than series expansions) for the solutions to our problems. Philosophically, though, the two parts are not all that different: in both cases we are seeking representation formulas for solutions.)

Let us return to equation (1) above. First, since for us the eigenfunctions $\mathbf{e}_I$ are all real, we see that we may drop the complex conjugate on $\mathbf{e}_I(\mathbf{x}')$, meaning that we have the series expansion

$$G(\mathbf{x}, \mathbf{x}') = -\sum_I \frac{\mathbf{e}_I(\mathbf{x})\mathbf{e}_I(\mathbf{x}')}{\lambda_I},$$

from which we see easily that $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$, i.e., the Green's function is *symmetric* in its arguments. (This will be important below.) Now if we formally take the Laplacian of G with respect to $\mathbf{x}$, we obtain (denoting this by $\nabla^2_{\mathbf{x}}$, and keeping the complex conjugate since the symmetry of G is not important here)

$$\begin{aligned} \nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') &= -\sum_I \frac{1}{\lambda_I} \nabla^2_{\mathbf{x}} \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} = -\sum_I \frac{1}{\lambda_I} \lambda_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} \\ &= -\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}. \end{aligned}$$

Now this last sum generally does not converge in any usual sense; however, we can make some sense out of it by integrating it against a sufficiently smooth function f and then proceeding formally:

$$\begin{aligned} \int_D \left(\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} \right) f(\mathbf{x}')\, d\mathbf{x}' &= \sum_I \mathbf{e}_I(\mathbf{x}) \int_D f(\mathbf{x}')\overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \\ &= \sum_I (f, \mathbf{e}_I)\mathbf{e}_I(\mathbf{x}) = f(\mathbf{x}), \end{aligned}$$

[1] This negative sign – which was not used in the lecture – seems to be standard, for some reason, but is also extremely annoying from our perspective. We shall, however, include it for ease of reference both to the textbook and to other external sources.

since $\{\mathbf{e}_I\}$ is a complete orthonormal set by assumption. This shows that, at least formally, $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}')$, where $\delta(\mathbf{x})$ is the celebrated *Dirac delta function*, beloved of physicists probably long before it was understood by mathematicians.[2] Heuristically, this is described as a 'function' possessing the following two properties:

1. $\delta(\mathbf{x}) = 0 \,\text{unless}\, \mathbf{x} = 0$;
2. $\displaystyle\int_{\mathbf{R}^3} f(\mathbf{x})\delta(\mathbf{x})\, d\mathbf{x} = f(0)$ for all functions f.

Putting these two conditions together, we see that in effect $\delta(\mathbf{x})$ is zero everywhere except at the origin, where it has an infinitely high peak. We note that the second property implies in particular that $\int_{\mathbf{R}^3} \delta(\mathbf{x})\, d\mathbf{x} = 1$, and also that for any (suitable[3]) f

$$\int_{\mathbf{R}^3} f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}' = f(\mathbf{x});$$

this may be seen by doing a simple substitution with $\mathbf{u} = \mathbf{x} - \mathbf{x}'$. While there is no actual function which satisfies the above two properties, there are many sequences of functions which satisfy them *in the limit*, in the following sense.

DEFINITION. A sequence of functions $\{\delta_n\}$ on some $\mathbf{R}^m$ is said to *have the properties of a delta function in the limit*, or to be an *approximate identity*,[4] if the following two properties hold:

1. $\displaystyle\int_{\mathbf{R}^m} \delta_n(\mathbf{x})\, d\mathbf{x} = 1$ for all n;
2. $\displaystyle\lim_{n\to\infty} \int_{\mathbf{R}^m} f(\mathbf{x})\delta_n(\mathbf{x})\, d\mathbf{x} = f(0)$ for all suitable functions $f(\mathbf{x})$

(see our commentary about suitable functionsn in the footnotes).

The point of this definition is that, while the sequence $\{\delta_n\}$ itself need not have a limit in any normal sense, the functionals (linear maps to $\mathbf{R}$) it induces on spaces of functions have the delta function as a 'limit';[5] more intuitively, while it doesn't make any sense to put the limit above inside the integral, everything works well if we keep it outside the integral. One could in fact probably work out most of the theory using sequences which are approximate identities without mentioning the delta function itself at all; but we shall prefer to take the delta function as something which exists by itself, and only use approximate identities for cases of illustration and in dealing with fine points.

We shall now give several examples of approximate identities, beginning with a simple one to illustrate the idea and then proceeding to more complicated ones which shall be useful in our future work with the

[2]Mathematicians, take note! Just because a physicist fails to give an object a precise mathematical formulation does not mean that one doesn't exist. I heard a quote from a well-known mathematician (I have unfortunately forgotten who) to the effect that, If physicists have been using something consistently for years, mathematicians ought to study how it works (or something like that, I don't remember exactly how the second half went).

[3]We shall be vague about what is meant by 'suitable'. In the mathematically rigorous formulation of delta functions, one actually restricts f to be C^∞, and generally either of compact support – meaning that it vanishes outside of a bounded set – or of 'rapid decrease' (falling to zero faster than any polynomial function) at infinity. It is sometimes appropriate to work with more general functions, though; and the functions which we shall say *have the properties of a delta function in the limit* (see the definition immediately following) satisfy this property for fairly general classes of functions, cf. [1], Theorem 8.15.

[4]This terminology comes from the equation just given, which is seen – cf. the definition of convolution below – to show that the delta function is an identity for the convolution operation. See [1], Theorem 8.15, and ensuing commentary.

[5]Those interested in seeing how to make this notion of limit rigorous may see the appendix [to be added soon], but be warned that it assumes a fairly detailed understanding of point-set toplogy and a high level of mathematical maturity.

Fourier transform. We shall begin by considering delta functions on $\mathbf{R}^1$ and then use these to construct them on $\mathbf{R}^3$.

(Before beginning the examples, we mention the one example we have constructed so far of something which 'looks like a delta function in the limit', namely $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}$ (this is a limit since it is an infinite sum). The above calculation can be written more carefully as follows (using the notation $I \to \infty$ to indicate that all of the indices in I go to infinity; convergence of multiply-indexed sequences is a tricky business and we elide the details here):

$$\lim_{I\to\infty} \int_D \sum_J^I \mathbf{e}_J(\mathbf{x})\overline{\mathbf{e}_J(\mathbf{x}')}f(\mathbf{x}')d\mathbf{x}' = \lim_{I\to\infty} \sum_J^I \mathbf{e}_J(\mathbf{x}) \int_D \overline{\mathbf{e}_J(\mathbf{x}')}f(\mathbf{x}')d\mathbf{x}' = \lim_{I\to\infty} \sum_J^I (\mathbf{e}_J, f)\mathbf{e}_J(\mathbf{x}),$$

and we know that this limit equals $f(\mathbf{x})$ if we take it to be in the L^2 sense. Thus, while our discussion of complete orthogonal sets is insufficient to conclude that $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}$ is an approximate identity in the sense just given (or more precisely, that it would be if we replaced $\mathbf{x}$ by 0), it is somehow one in an L^2 sense (whatever that means[6]). Generally speaking, series such as the foregoing converge pointwisely at points of continuity of f, so that if we restrict to continuous functions then the sum should be an approximate identity. Perhaps the take-home lesson here is that, at least for our purposes, general statements of when a certain sequence is or is not an approximate identity are probably less important than understanding the general idea and specific cases.)

EXAMPLES. (a) For each $n \in \mathbf{Z}$, $n > 0$, define a function $\chi_n : \mathbf{R}^1 \to \mathbf{R}^1$ as follows:

$$\chi_n(x) = \begin{cases} \frac{n}{2}, & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}.$$

Then we see that $\int_{\mathbf{R}^1} \chi_n(x)\,dx = 1$ for all n, and that $\chi_n(x) \to 0$ for all $x \neq 0$ as $n \to \infty$ (to see this, let $x \in \mathbf{R}$, $x \neq 0$, and let $N \in \mathbf{Z}$, $N > \frac{1}{|x|}$; then for all $n > N$ we have $\frac{1}{n} < |x|$, so $x \notin [-\frac{1}{n}, \frac{1}{n}]$ and $\chi_n(x) = 0$). Now let $f : \mathbf{R}^1 \to \mathbf{R}^1$ be continuous at $x = 0$. We claim that

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx \to f(0) \quad \text{as} \quad n \to \infty.$$

To prove this, we first rewrite the left-hand side as follows (recall that $\int_{\mathbf{R}^1} \chi_n(x)\,dx = 1$ for all n):

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx = \int_{\mathbf{R}^1} (f(x) - f(0) + f(0))\chi_n(x)\,dx = \int_{\mathbf{R}^1} (f(x) - f(0))\chi_n(x)\,dx + f(0).$$

Now let $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(0)| < \epsilon$ when $|x| < \delta$, and let $N \in \mathbf{Z}$ be such that $\frac{1}{N} < \delta$. Then for all $n > N$, we have $\frac{1}{n} < \delta$, so for such n

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx = f(0) + \int_{\mathbf{R}^1} (f(x) - f(0))\chi_n(x)\,dx = f(0) + \int_{-\frac{1}{n}}^{\frac{1}{n}} (f(x) - f(0))\frac{n}{2}\,dx,$$

and

$$\left| \int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx - f(0) \right| \leq \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} |f(x) - f(0)|\,dx < \frac{n}{2}\frac{2\epsilon}{n} = \epsilon,$$

which shows that $\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx \to f(0)$, as claimed. While $\lim \chi_n$ does not exist in any normal sense, we see that, in some sense, the sequence χ_n has the properties of the delta function in the limit as $n \to \infty$.

(Intuitively, the idea behind the above ϵ-δ proof is as follows. Since f is continuous at $x = 0$, $f(x) - f(0)$ will be small if x is close to zero. Now if n is large, then $\chi_n(x)$ is zero unless $|x| \leq \frac{1}{n}$; thus $\chi_n(x)$ will be zero

[6]The author suspects that someone more talented than he has already made this precise, but even if that is the case he is not aware of it; for the which ignorance, he apologises.

unless x is small, in the which case $f(x)-f(0)$ will also be small. This suggests that $\int_{\mathbf{R}^1}(f(x)-f(0))\chi_n(x)\,dx$ will be small in this case. Unfortunately, while $f(x)-f(0)$ is small, $\chi_n(x)$ will be large (since $\chi_n(0)\to\infty$ as $n\to\infty$), so this does not follow immediately; but since the interval is also getting small, it turns out – and the calculations above prove rigorously – that in fact this integral is small, as desired. Just for the sake of thouroughness, we give another proof (which in rigour is halfway between the full proof and the intuitive description just given) in the special case where f is continuous on some interval containing 0: in this case, by taking n large enough we may assume that f is continuous on $[\frac{1}{n},\frac{1}{n}]$. Thus $|f(x)-f(0)|$ must also be continuous on this interval, and hence must have a maximum there, call it M_n. Then we may write

$$\int_{\mathbf{R}^1}|f(x)-f(0)|\chi_n(x)\,dx=\int_{-\frac{1}{n}}^{\frac{1}{n}}|f(x)-f(0)|\frac{n}{2}\,dx\le M_n\frac{n}{2}\frac{2}{n}=M_n.$$

But since f is continuous at 0, the quantity M_n must become small as $n\to\infty$, so that the integral $\int_{\mathbf{R}^1}(f(x)-f(0))\chi_n(x)\,dx$ must become small as well, as claimed.)

(b) Again, for each $n\in\mathbf{Z}$, $n>0$, define a function $\phi_n:\mathbf{R}^1\to\mathbf{R}^1$ by

$$\phi_n(x)=\sqrt{n}\pi e^{-nx^2}.$$

We recall the Gaussian integral: for any $a>0$,

$$\int_{\mathbf{R}^1}e^{-ax^2}\,dx=\sqrt{\frac{\pi}{a}};$$

thus we have (as for χ_n) $\int_{-\infty}^{\infty}\phi_n(x)\,dx=1$ for all n. Moreover, $\phi_n(x)\to 0$ for all $x\neq 0$ as $n\to\infty$, as for χ_n. We claim again that for any function f which is continuous at 0 and (in this case) bounded on $\mathbf{R}^1$

$$\int_{\mathbf{R}^1}f(x)\phi_n(x)\,dx\to f(0)\quad\text{as}\quad n\to\infty.$$

The intuition is very similar to that in the previous proof (note that ϕ_n, like χ_n, becomes infinitely sharply peaked at 0 in the limit as $n\to\infty$) and we give only the ϵ-δ proof. Thus let $M=\sup_{x\in\mathbf{R}^1}|f(x)|+1$, $\epsilon>0$, let $\delta>0$ be such that $|f(x)-f(0)|<\epsilon$ for $|x|<\delta$, and let $K\in\mathbf{Z}$, $K>0$ be such that

$$\left|1-\frac{1}{\sqrt{\pi}}\int_{-K}^{K}e^{-x^2}\,dx\right|<\frac{\epsilon}{2M}$$

(such a K certainly exists since $\int_{-\infty}^{+\infty}e^{-x^2}\,dx=\sqrt{\pi}$). Let $N\in\mathbf{Z}$, $N>0$ be such that $\delta\sqrt{N}>K$. Then for $n>N$ we have, doing a change of variables with $u=x\sqrt{n}$,

$$\int_{-\delta}^{\delta}\phi_n(x)\,dx=\sqrt{\frac{n}{\pi}}\int_{-\delta}^{\delta}e^{-nx^2}\,dx=\frac{1}{\sqrt{pi}}\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}}e^{-u^2}\,du,$$

from which we see that

$$\left|\int_{\mathbf{R}^1\setminus[-\delta,\delta]}\phi_n(x)\,dx\right|=\left|1-\int_{-\delta}^{\delta}\phi_n(x)\,dx\right|<\frac{\epsilon}{2M}$$

and moreover that

$$\begin{aligned}\left|f(0)-\int_{-\infty}^{\infty}f(x)\phi_n(x)\,dx\right|&\le\int_{\mathbf{R}^1}|f(x)-f(0)|\phi_n(x)\,dx\\&=\int_{\mathbf{R}^1\setminus[-\delta,\delta]}|f(x)-f(0)|\phi_n(x)\,dx+\int_{-\delta}^{\delta}|f(x)-f(0)|\phi_n(x)\,dx\\&\le 2M\int_{\mathbf{R}^1\setminus[-\delta,\delta]}\phi_n(x)\,dx+\epsilon\int_{-\delta}^{\delta}\phi_n(x)\,dx\\&<\epsilon+\epsilon=2\epsilon,\end{aligned}$$

which shows that $\int_{\mathbf{R}^1} f(x)\phi_n(x)\,dx \to f(0)$ as $n \to \infty$, as desired.[7]

(c) [This example can be skipped at a first reading; in that case, replace ψ_n with ϕ_n in (d) below, and references to (c) with references to (b). For a similar but more careful and general result, see the aforementioned Theorem 8.15 of [1].] Given the foregoing, we are now ready to posit the following general result: suppose that $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ satisfies $\int_{\mathbf{R}^1} \psi(x)\,dx = 1$, $\int_{\mathbf{R}^1} |\psi(x)|\,dx < \infty$ (this latter means that ψ is *in* L^1), and for all $n \in \mathbf{Z}$, $n > 0$ define

$$\psi_n(x) = n\psi(nx).$$

(χ_n in example (a) is certainly of this form; in example (b), we have basically this same form except that we scale by $\sqrt{n}$ instead of n.) Assume now that $\psi_n(x) \to 0$ as $n \to \infty$.[8] Then we claim that for any $f : \mathbf{R}^1 \to \mathbf{R}^1$ which is continuous at $x = 0$ and bounded on $\mathbf{R}^1$,

$$\int_{\mathbf{R}^1} f(x)\psi_n(x)\,dx \to f(0) \quad \text{and} \quad n \to \infty.$$

To see this, we first note that for any n, using a change of variables $u = nx$,

$$\int_{\mathbf{R}^1} \psi_n(x)\,dx = \int_{-\infty}^{\infty} n\psi(nx)\,dx = \int_{-\infty}^{\infty} \psi(u)\,du = 1.$$

Now let $M = \sup_{x\in\mathbf{R}^1} |f(x)| + 1$, let $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(0)| < \epsilon$ when $|x| < \delta$, and let $K \in \mathbf{Z}$, $K > 0$ be such that

$$\int_{\mathbf{R}^1\setminus[-K,K]} |\psi(x)|\,dx < \frac{\epsilon}{2M};$$

such a K clearly exists since $\int_{-\infty}^{\infty} |\psi(x)|\,dx < \infty$. Now choose $N \in \mathbf{Z}$, $N > 0$, such that $N\delta > K$, and let $n > N$. Then we have

$$\begin{aligned}\left|\int_{\mathbf{R}^1\setminus[-\delta,\delta]} \psi_n(x)\,dx\right| &\le \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |\psi_n(x)|\,dx \\ &= \int_{\mathbf{R}^1\setminus[-n\delta,n\delta]} |\psi(u)|\,du < \frac{\epsilon}{2M},\end{aligned}$$

so

$$\begin{aligned}\left|f(0) - \int_{-\infty}^{\infty} f(x)\psi_n(x)\,dx\right| &\le \int_{\mathbf{R}^1} |f(x) - f(0)||\psi_n(x)|\,dx \\ &= \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |f(x) - f(0)||\psi_n(x)|\,dx + \int_{-\delta}^{\delta} |f(x) - f(0)||\psi_n(x)|\,dx \\ &\le 2M \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |\psi_n(x)|\,dx + \epsilon \int_{-\infty}^{\infty} |\psi_n(x)|\,dx < 2\epsilon,\end{aligned}$$

and $\int_{\mathbf{R}^1} f(x)\psi_n(x)\,dx \to f(0)$ as $n \to \infty$, as claimed.

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[7] Note that we did not really need f to be bounded; we just needed f to be such that the tails

$$\int_{\mathbf{R}^1\setminus[-\delta,\delta]} f(x)\phi_n(x)\,dx$$

would go to zero as $n \to \infty$. Since $\phi_n(x)$ goes to zero like e^{-nx^2}, it is sufficient, for example, that f go to infinity no more than exponentially fast as $x \to \pm\infty$. This material is interesting, but we pass over it for now.

[8] It is possible to give sufficient conditions for this to hold: for example, suppose that there were $M > 0$, $\alpha > 1$ such that $|\psi(x)| < \frac{M}{x^\alpha}$ for all x (or even for all x sufficiently large); then clearly $\psi_n(x) < n\frac{M}{(nx)^\alpha} = \frac{M}{x}n^{1-\alpha} \to 0$ as $n \to \infty$, since $\alpha > 1$ – this is the condition used in [1], Theorem 8.15. Other conditions could presumably also be found.

(d) We would now like to construct a delta function on $\mathbf{R}^3$. We may construct it as the improper limit of a sequence like those just given in the following way. Let ψ be any function satisfying the requirements given in (c), and define

$$\Psi_{\ell mn}(x,y,z) = \psi_\ell(x)\psi_m(y)\psi_n(z).$$

(Using three different indices is just for convenience in making the calculation below simpler; it could probably be done without it, at least for f uniformly continuous on some neighborhood of the origin.) Then if $f : \mathbf{R}^3 \to \mathbf{R}^1$ is any function which is bounded on $\mathbf{R}^3$ and continuous at $x = 0$, we have

$$\int_{\mathbf{R}^3} f(\mathbf{x})\Psi_{\ell mn}(\mathbf{x})\,d\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\psi_\ell(x)\psi_m(y)\psi_n(z)\,dx\,dy\,dz.$$

Now the hypotheses allow us to interchange the limit on n with the integral over z;[9] taking this limit, the above integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,0)\psi_\ell(x)\psi_m(y)\,dx\,dy.$$

We may then similarly take the limit on m, and finally on ℓ, to see that

$$\int_{\mathbf{R}^3} f(\mathbf{x})\Psi_{\ell mn}(\mathbf{x})\,d\mathbf{x} \to f(0,0,0) \quad \text{as} \quad \ell, m, n \to \infty.$$

From this it can be shown that the sequence

$$\hat{\Psi}_n(\mathbf{x}) = \Psi_{nnn}(\mathbf{x})$$

behaves also like a delta function in the limit as $n \to \infty$. In general, then, if we have a sequence in $\mathbf{R}^1$ of the form given in (c) which behaves like a delta function in the limit, then we may obtain a sequence in $\mathbf{R}^n$ (for any n) which behaves like a delta function in the limit by taking a product of n copies of the sequence in $\mathbf{R}^1$, one in each variable separately. We write this symbolically in $\mathbf{R}^3$ as

$$\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$$

(one tends to write δ for the delta function in any dimension, and even for multiple different dimensions in one single equation as above, without regards to niceties of notation!).

(e) Finally, for use in a moment we would like to find an expression for the delta function in $\mathbf{R}^3$ which is adapted to spherical coordinates in the way the expressions in (d) are adapted to rectangular coordinates. We shall apply the method of (d) to the function from (b). We have

$$\phi_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2},$$

so

$$\Phi_n(\mathbf{x}) = \left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-n(x^2+y^2+z^2)}$$

will behave like a delta function in the limit as $n \to \infty$. In spherical coordinates, this can be written as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} f(r,\theta,\phi)\left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-nr^2} r^2 \sin\theta\,dr\,d\theta\,d\phi;$$

if we extend f to be even in r on $\mathbf{R}^1$, then this may be rewritten as

$$\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi)\left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-nr^2} r^2 \sin\theta\,dr\,d\theta\,d\phi$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi) 2\sqrt{\frac{n}{\pi}} e^{-nr^2}\frac{nr^2}{2\pi r^2} r^2\sin\theta\,dr\,d\theta\,d\phi.$$

[9] One might need the dominated convergence theorem and Lebesgue integration. The deponent verb may have all the forms of the gerund.

Now we note that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = -\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2}\,dx\bigg|_{a=1} = -\frac{d}{da}\sqrt{\frac{\pi}{a}}\bigg|_{a=1} = \frac{1}{2}\sqrt{\pi}$$

thus the function $\psi = \frac{2}{\sqrt{\pi}}x^2e^{-x^2}$ is of the form covered by (c), so that (scaling by $\sqrt{n}$ instead of n, as in (b)) the sequence $\psi_n(x) = 2\sqrt{\frac{n}{\pi}}nx^2e^{-nx^2}$ will behave like a delta function in the limit $n \to \infty$. But the above integral is exactly

$$\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi)\frac{1}{4\pi r^2}\psi_n(r)r^2\,dr\sin\theta\,d\theta\,d\phi.$$

Thus we identify the three-dimensional delta function $\delta(\mathbf{x})$ with the function

$$\frac{1}{4\pi r^2}\delta(r)$$

in spherical coordinates.[10]

With our understanding of delta functions (hopefully!) increased by these examples, we return to our study of Green's functions. We recall that we have derived the relation (which holds in some sort of L^2 sense, perhaps not in the precise sense of our definition of approximate identity above)

$$\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}').$$

For the rest of our work with Green's functions, we shall take this equation (and not the expansion in eigenfunctions) as the starting point; in other words, for us a *Green's function* will be any function on $\mathbf{R}^m$ (generally we have $m = 3$) which satisfies the above equation, in the sense that for any suitable function f

$$\int_{\mathbf{R}^m} -\left(\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')\right)f(\mathbf{x}')\,d\mathbf{x}' = f(\mathbf{x}).$$

We shall show later how to take boundary conditions into account. We show how it may be used to prove the above representation formula for solutions to Poisson's equation. Suppose that $f : \mathbf{R}^3 \to \mathbf{R}$ is a function such that $\int_{\mathbf{R}^3} G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}'$ exists for all $\mathbf{x} \in \mathbf{R}^3$; then, assuming that we may interchange integration and differentiation, we have

$$\nabla^2_{\mathbf{x}}\int_{\mathbf{R}^3} G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = \int_{\mathbf{R}^3}\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = \int_{\mathbf{R}^3}\delta(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = -f(\mathbf{x})$$

[10]There are several objections which could legitimately be brought up to this derivation and formula. One of them involves the fact that $\delta(r) = 0$ unless $r = 0$, which means that we should be able to replace r by 0 in the above expression: but this would involve dividing by 0, which is meaningless. This can be answered by noting that delta functions *proper* (as opposed to the sequences we have been constructing which lead to them) only really exist when appearing under integral signs, and the one here only exists when appearing under a three-dimensional integral; in such a case, there will always be the factor r^2 from the volume element in spherical coordinates to cancel the r^2 in the denominator here. Another, more subtle, objection is that the final integral we just derived involved integrating over r from $-\infty$ to ∞, whereas the integral we started with only involved an integral from 0 to $+\infty$; thus we seem to have counted things twice. This can be answered (though not, I admit, entirely resolved; to entirely resolve either of these objections we would probably have to work in a much more rigorous setting) by the following observation: note that the normalisation we used for ψ was also obtained by integrating from $-\infty$ to ∞; thus we have effectively divided by an extra factor of 2, which should cancel the problematic one. It should perhaps also be pointed out that the delta function requires integrating over an interval containing zero, not just on half of such an interval, so that the integral from 0 to ∞ might be said, in some sense, to collect only half of the delta function (though I do not think this can be made precise in any real sense).

This shows that the function

$$u = -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' \tag{1}$$

satisfies Poisson's equation on $\mathbf{R}^3$. (See Theorem 8.3 of the textbook for a more careful treatment by another method of this result; but note that the textbook's definition of a Green's function interchanges $\mathbf{x}$ and $\mathbf{x}'$ compared with ours – symmetry of the Green's function (see below) can be used to turn the result back into something closer to what we have here.)

It might be helpful to give some intuitive content to this relation. Consider Poisson's equation. The Green's function $G(\mathbf{x}, \mathbf{x}')$ gives the solution at the point $\mathbf{x}$ due to a (negative) unit (i.e., negative delta function) source at the point $\mathbf{x}'$. (In physical terms, if we are solving for the electrostatic potential, so that the right-hand side of Poisson's equation is essentially the charge density, then the Green's function gives the electrostatic potential at $\mathbf{x}$ due to a point charge of unit size at $\mathbf{x}'$; if we are solving for the steady-state temperature distribution in a body with internal sources, then the Green's function gives the temperature at a point $\mathbf{x}$ due to a single point source of unit strength at $\mathbf{x}'$; and so on.) Since Poisson's equation is *linear*, we expect that the solution for a sum of such sources, say at the points $\mathbf{x}_i'$ should be the sum $-\sum_i f_i G(\mathbf{x}, \mathbf{x}_i')$, where f_i represents the size of the source at $\mathbf{x}_i'$. Now if we are given a continuous source, then it makes sense[11] that the sum should become an integral, and that the whole solution should be

$$-\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}',$$

exactly as we showed just now. In other words, the representation formula for the solution to Poisson's equation using the Green's function is just what is obtained by taking a superposition of solutions due to individual point sources.

To go back to our more formal investigations for a moment, if we look at the relation

$$\nabla_{\mathbf{x}}^2 \int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' = -f(\mathbf{x})$$

more carefully, we see that it means that the operator

$$f(\mathbf{x}) \mapsto -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'$$

is a left inverse to the Laplacian, in the sense that, denoting the map by $\mathcal{G}$, we have $\nabla^2 \mathcal{G}[f] = f$ for all suitable functions f. If we assume that G is symmetric (for the Green's function on all of $\mathbf{R}^3$, this will follow from the calculation we give in a moment; for the Green's function on a bounded region, this was already noted as following from the expansion in eigenfunctions, or see Theorem 8.4 in the textbook), then we can show that $\mathcal{G}$ is also a right inverse:

$$\begin{aligned}
-\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') \nabla_{\mathbf{x}'}^2 f(\mathbf{x}')\, d\mathbf{x}' &= -\int_{\mathbf{R}^3} \nabla_{\mathbf{x}'} \cdot (G(\mathbf{x}, \mathbf{x}') \nabla_{\mathbf{x}'} f(\mathbf{x}')) - \nabla_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') \cdot \nabla_{\mathbf{x}'} f(\mathbf{x}')\, d\mathbf{x}' \\
&= \int_{\mathbf{R}^3} \nabla_{\mathbf{x}'} \cdot (\nabla_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')) - \nabla_{\mathbf{x}'}^2 G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' \\
&= -\int_{\mathbf{R}^3} \nabla_{\mathbf{x}'}^2 G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' \\
&= -\int_{\mathbf{R}^3} \nabla_{\mathbf{x}'}^2 G(\mathbf{x}', \mathbf{x}) f(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^3} \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}')\, d\mathbf{x}' = f(\mathbf{x}),
\end{aligned} \tag{1'}$$

where we have used the fact that the delta function is even, and also that $G(\mathbf{x}, \mathbf{x}') \to 0$ as $\mathbf{x}$ or $\mathbf{x}' \to \infty$.

GREEN'S FUNCTION ON $\mathbf{R}^3$. The Green's function on $\mathbf{R}^3$ can most conveniently be computed by using the expression for the delta function derived in Example (d) above. We have the equation

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}').$$

[11] One could derive this in more detail, but we shall pass over it here because of space and time considerations.

Now in this equation $\mathbf{x}'$ is considered to be a fixed parameter. We now introduce spherical coordinates on $\mathbf{R}^3$ centred at $\mathbf{x}'$; i.e., let (r, θ, ϕ) of a point $\mathbf{x}$ satisfy (writing $\mathbf{x} = (x, y, z)$, $\mathbf{x}' = (x', y', z')$)

$$r\sin\theta\cos\phi = x - x', \quad r\sin\theta\sin\phi = y - y', \quad r\cos\theta = z - z'.$$

Then the delta function $\delta(\mathbf{x} - \mathbf{x}')$ can be written, by Example (d) above, as

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(r)}{4\pi r^2}.$$

This suggests that the Green's function will only depend on r (at least as far as its x dependence is concerned); thus the equation for G reduces to

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} = -\frac{\delta(r)}{4\pi r^2}.$$

Now

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right),$$

so multiplying by r^2 we obtain

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = -\frac{\delta(r)}{4}\pi.$$

Integrating from 0 to some value r gives

$$r^2\frac{\partial G}{\partial r} = -\frac{1}{4\pi},$$

whence

$$\frac{\partial G}{\partial r} = -\frac{1}{4\pi r^2}, \quad G = \frac{1}{4\pi r} + C$$

for some constant C. Requiring G to vanish as $\mathbf{x} \to \infty$ gives $C = 0$. Thus we have the Green's function

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi r} = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|},$$

which is seen to be symmetric in $\mathbf{x}$ and $\mathbf{x}'$, as claimed.

GREEN'S FUNCTION ON A FINITE REGION. Suppose that we are now interested in solving Poisson's equation on a bounded region D; in other words, consider the equation

$$\nabla^2 u = f, \quad u|_{\partial D} = 0.$$

We claim that an appropriate Green's function for this problem is given by the solution to the problem

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x}, \mathbf{x}'), \quad G|_{\mathbf{x}\in\partial D} = 0;$$

in other words, we claim that the solution to this problem is given by

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}'.$$

(Note the relation of this to our discussion of the intuitive content of the representation formula (1) above: in the case at hand, the solution u must also satisfy a homogeneous Dirichlet boundary condition – in the context of the two physical examples we gave above, this amounts to saying that the region D is surrounded by a grounded conductor or by a substance of constant temperature 0 – and so we require the solutions to the point source problems – the Green's function – to satisfy the same condition.) While we could give a proof of this using definition of G in terms of an orthogonal expansion given initially, we would like to prove it using just the definition of G given here. In order to do this, we shall first derive a calculus formula known

as *Green's second identity* (see pp. 490 – 491 in the textbook). Suppose that f and g are suitably smooth functions on a domain D. Then

$$\int_D f\nabla^2 g - g\nabla^2 f\, d\mathbf{x} = \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} dS,$$

where $\frac{\partial f}{\partial n} = \mathbf{n}\cdot\nabla f$ denotes the outwards normal derivative of f at ∂D.

This is basically an integration-by-parts formula and may be derived as follows. We have

$$\begin{aligned}\int_D f\nabla^2 g\, d\mathbf{x} &= \int_D \nabla\cdot(f\nabla g) - \nabla f\cdot\nabla g\, d\mathbf{x} = \int_{\partial D}\mathbf{n}\cdot(f\nabla g)\, dS - \int_D \nabla f\cdot\nabla g\, d\mathbf{x}\\ &= \int_{\partial D} f\frac{\partial g}{\partial n}\, dS - \int_D \nabla f\cdot\nabla g\, d\mathbf{x};\end{aligned}$$

interchanging f and g and subtracting then gives

$$\begin{aligned}\int_D f\nabla^2 g - g\nabla^2 f\, d\mathbf{x} &= \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial g}{\partial n}\, dS - \int_D \nabla f\cdot\nabla g - \nabla g\cdot\nabla f\, d\mathbf{x}\\ &= \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\, dS,\end{aligned}$$

as claimed.

Now, formally, if we pretend that Green's identity applies also to the Green's function in $\mathbf{x}'$ (which it won't, because of the singularity; see Theorem 8.3 in the textbook for a more careful treatment of the result here), then we may write, applying Green's identity (integrating in $\mathbf{x}'$) with $f = u(\mathbf{x}')$ and $g = G(\mathbf{x},\mathbf{x}')$ (please note that the results we derived above, giving $\mathcal{G}\nabla^2 = \nabla^2\mathcal{G} =$ identity, were on $\mathbf{R}^3$ and hence do not apply in the present case; in essence, we are trying to re-derive at least one of them in the present case),

$$\begin{aligned}\int_D u(\mathbf{x}')\nabla^2_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\nabla^2_{\mathbf{x}'}u(\mathbf{x}')\, d\mathbf{x}' &= -u(\mathbf{x}) - \int_D G(\mathbf{x},\mathbf{x}')\nabla^2_{\mathbf{x}'}u(\mathbf{x}')\, d\mathbf{x}'\\ &= \int_{\partial D} u(\mathbf{x}')\frac{\partial G}{\partial n'} - G(\mathbf{x},\mathbf{x}')\frac{\partial u}{\partial n'}\, dS'\end{aligned}$$

where primes denote derivatives and integrals with respect to $\mathbf{x}'$. Thus we obtain

$$u(\mathbf{x}) = -\int_D G(\mathbf{x},\mathbf{x}')\nabla^2_{\mathbf{x}'}u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x},\mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS'. \tag{2}$$

Now in the case of the Poisson equation above,

$$\nabla^2 u = f,\quad u|_{\partial D} = 0,$$

we see that the volume integral becomes simply

$$-\int_D G(\mathbf{x},\mathbf{x}')\nabla^2_{\mathbf{x}'}u(\mathbf{x}')\, d\mathbf{x}',$$

while the surface integral vanishes, since $u|_{\partial D} = 0$ and $G|_{\mathbf{x}'\in\partial D} = 0$. Thus we have

$$u(\mathbf{x}) = -\int_D G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}',$$

as claimed.

The above formula (2) above is of wider applicability. We pause for a moment to derive from it an important result about solutions to Laplace's equation. Suppose that D is some open set in $\mathbf{R}^3$, and that u satisfies $\nabla^2 u = 0$ on D. Let G denote the Green's function on $\mathbf{R}^3$ derived in the previous section. Let

$\mathbf{x} \in D$, and let $r > 0$ be such that $B_r(\mathbf{x}) \subset D$; we can do this since D is open (more intuitively, $\mathbf{x}$ cannot be on the boundary of D so it has to be inside D). Then applying formula (2) using G as our Green's function, $B_r(\mathbf{x})$ as our region (in place of D), and $\mathbf{x}$ as our evaluation point, we obtain

$$u(\mathbf{x}) = -\int_{B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') \nabla^2_{x'} u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') \frac{\partial u}{\partial n'} - u(\mathbf{x}') \frac{\partial G}{\partial n'}\, dS'.$$

Now on $B_r(\mathbf{x})$ we have $\nabla^2 u = 0$, so the first integral vanishes; also, for $\mathbf{x}' \in \partial B_r(\mathbf{x})$ we have

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} = \frac{1}{4\pi r},$$

which is a constant, so that the middle term above becomes

$$\begin{aligned} \int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') \frac{\partial u}{\partial n'}\, dS' &= \frac{1}{4\pi r} \int_{\partial B_r(\mathbf{x})} \mathbf{n} \cdot \nabla u\, dS' \\ &= \frac{1}{4\pi r} \int_{B_r(\mathbf{x})} \nabla^2 u\, dS' = 0, \end{aligned}$$

where we have used the divergence theorem in the penultimate inequality. Thus we are left with only the last term, i.e.,

$$u(\mathbf{x}) = -\int_{\partial B_r(\mathbf{x})} u(\mathbf{x}') \frac{\partial G}{\partial n'}\, dS'.$$

Now to calculate $\frac{\partial G}{\partial n'}$ we may proceed geometrically as follows. This derivative is the derivative in the direction normal to the sphere $\partial B_r(\mathbf{x})$; alternatively, if we set up a spherical coordinate system (r'', θ'', ϕ'') for $\mathbf{x}'$ centred at the point $\mathbf{x}$, then $\frac{\partial}{\partial n'}$ will simply be the radial derivative $\frac{\partial}{\partial r''}$. But we have also

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} = \frac{1}{4\pi r''},$$

so that

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r''} = -\frac{1}{4\pi r''^2} = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|^2},$$

and we obtain finally

$$u(\mathbf{x}) = \int_{\partial B_r(\mathbf{x})} \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|^2} u(\mathbf{x}')\, dS' = \frac{1}{4\pi r^2} \int_{\partial B_r(\mathbf{x})} u(\mathbf{x}')\, dS',$$

i.e., the value of u at any point $\mathbf{x}$ is equal to the average of u over a sphere centred at $\mathbf{x}$, as long as the corresponding ball bounded by the sphere is entirely contained in the region D on which u satisfies Laplace's equation. This implies that u cannot have a local maximum or local minimum in D, unless it is constant. The main idea is as follows: if u had a local maximum at some point $\mathbf{x}$, and r were any number for which the above formula applied and small enough that $\mathbf{x}$ was a maximum point for u on $B_r(\mathbf{x})$, then u on $\partial B_r(\mathbf{x})$ would have to be equal everywhere to $u(\mathbf{x})$, as it cannot anywhere be greater so were it somewhere less its average would also be less, which would contradict the above equation. The same logic holds for a local minimum.

The foregoing shows that if u is a continuous solution to Laplace's equation on a bounded region, then it must take its extreme values on the boundary of the region (since it must take them somewhere, and if it took either of them at an interior point then it would have a local extremum; and in that case it would be constant, so its maximum value on D would equal its minimum value there, and both would be taken on the boundary – and everything would be quite trivial, of course!).

Returning to our main topic, now, we note the similarity of (2) to formula (1′) above: the only additional terms are those for the boundary conditions. In other words, formula (2) allows us to pass backwards from the Laplacian of u to obtain u, as long as we are given suitable information about u on the boundary. Note

though (see [2], p. 37, on which the following discussion is based) that as it stands the information which seems to be required is too much: for example, suppose that we are trying to solve Laplace's equation; we know from our previous studies that giving just the value of u on the boundary suffices to obtain a unique solution to Laplace's equation, and hence to obtain also the normal derivatives $\frac{\partial u}{\partial n}$ on the boundary. But the formula above seems to require us to give both; since they cannot be specified independently, the formula does not seem to be much use as it stands. The resolution lies in the manipulations following equation (2): we are free to impose boundary conditions on G also, and may impose them to make either of the terms in the surface integral vanish. Thus, if we are given u on the boundary, we shall use a Green's function which vanishes on the boundary, so that the term involving $\frac{\partial u}{\partial n}$ does not appear; if we are instead given $\frac{\partial u}{\partial n}$ on the boundary, then we shall use a Green's function which satisfies $\frac{\partial G}{\partial n}\big|_{x\in\partial D} = 0$, so that (using symmetry of G) the term involving u on the boundary vanishes.

As an example, consider now the problem

$$\nabla^2 u = f, \quad u|_{\partial D} = g;$$

if G is the Green's function defined above, then by (2) we may write

$$u(\mathbf{x}) = -\int_D G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' - \int_{\partial D} g(\mathbf{x}')\frac{\partial G}{\partial n'}\,dS'.$$

This gives an alternative representation of the solution to the above problem (which we already know how to solve in terms of orthogonal expansions, at least for $D = Q, C, B$). In the case where $f = 0$, we obtain the formula

$$u(\mathbf{x}) = -\int_{\partial D} g(\mathbf{x}')\frac{\partial G}{\partial n'}\,dS'$$

for the solution to the boundary-value problem for Laplace's equation

$$\nabla^2 u = 0, \quad u|_{\partial D} = g.$$

Returning to general theory, we may now ask how we are to find a Green's function satisfying an appropriate boundary condition on ∂D. This may be done by a method very similar to that used to solve Poisson's equation with nonhomogeneous boundary conditions. We first set $G_0(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'}$, which is just the Green's function for $\mathbf{R}^3$ and hence satisfies $\nabla^2_{\mathbf{x}} G_0 = -\delta(\mathbf{x}-\mathbf{x}')$. Then we let $u(\mathbf{x},\mathbf{x}')$ solve the boundary-value problem for Laplace's equation given by

$$\nabla^2_{\mathbf{x}} u(\mathbf{x},\mathbf{x}') = 0, \quad u|_{\mathbf{x}\in\partial D} = -G_0(\mathbf{x},\mathbf{x}');$$

if we let $G(\mathbf{x},\mathbf{x}') = G_0(\mathbf{x},\mathbf{x}') + u(\mathbf{x},\mathbf{x}')$, then we see that G satisfies

$$\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \quad G|_{\mathbf{x}\in\partial D} = 0,$$

and hence that G is the desired Green's function.

If we wished instead to solve Poisson's equation with Neumann boundary conditions, we would use the function u satisfying instead the problem

$$\nabla^2_{\mathbf{x}} u(\mathbf{x},\mathbf{x}') = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\mathbf{x}\in\partial D} = -\frac{\partial G_0}{\partial n};$$

then $G = G_0 + u$ would be the desired Green's function.

All of this, of course, is just empty air unless we have a method to actually calculate a Green's function; in other words, unless we can actually solve the boundary-value problems for Laplace's equation given above. In general, of course, the only methods we know for solving Laplace's equation involve expansions in orthogonal sets, so it seems that the only formulas we can obtain for Green's functions at the moment are still as expansions in orthogonal sets, and it isn't clear that we have gained much. It turns out, though,

that for specific geometries there are other techniques for finding a Green's function; for example, for a sphere one can calculate the Green's function using the so-called *method of images* – see Example 8.2.1 in the textbook. (Regardless of this, the foregoing is important to know from a theoretical point of view. And actually we have gained something practical by the manipulations above, since the solutions to Laplace's equation are written in terms of orthogonal sets on sets of dimension one less than the space in which we work: for example, if we use an orthogonal expansion to find the Green's function on the sphere, we would be expanding in the basis $\{P_{\ell m}(\cos\theta)\cos m\phi, P_{\ell m}(\cos\theta)\sin m\phi\}$ rather than the full basis of eigenfunctions of the Laplacian on the ball. This is a gain in simplicity, at least.)

While it would be beneficial and interesting to take some time to give concrete examples involving Green's functions, considerations of time and space impel us to pass over this and continue to our next topic, Fourier transforms. We shall try to come back and give an example or two of the above theory at some point in the future.

FOURIER TRANSFORMS. We first say a few words about orthogonal expansions in general. Consider an expansion in the basis of eigenfunctions for the Laplacian on Q obeying homogeneous Dirichlet boundary conditions: this is a series of the form $\sum_{\ell mn} a_{\ell mn}\sin\ell\pi x\sin m\pi y\sin n\pi z$, where ℓ, m and n range over all positive integers. Now if we use instead periodic boundary conditions then we would obtain expressions of the above form but with both sin and cos terms; if we were to express everything in terms of complex exponentials, we would get a sum of the form

$$\sum_{\ell,m,n=-\infty}^{\infty} a_{\ell mn}e^{2i\ell\pi x}e^{2im\pi y}e^{2in\pi z} = \sum_{\ell,m,n=-\infty}^{\infty} a_{\mathbf{l}}e^{2\pi i\mathbf{l}\cdot\mathbf{x}},$$

where $\mathbf{l} = (\ell, m, n)$. The point here is that we can express any function on the bounded region Q as a *series* in a *discrete* set of functions $\{e^{2\pi i\mathbf{l}\cdot\mathbf{x}}\}$. This is related to the fact that the eigenvalues of the Laplacian on Q (with suitable boundary conditions) form a discrete set (for periodic boundary conditions, $4\pi^2\left(\ell^2+m^2+n^2\right)$). Now it turns out that, in general, the set of eigenvalues of the Laplacian on a bounded set is discrete. (For those who have or will study quantum mechanics, this is closely related to the statement that a bound partical has only a discrete set of energy levels.) On an unbounded set, though, the set of eigenvalues (more properly, in this case, the *spectrum*[12]) of the Laplacian becomes continuous, and the sum over eigenvalues in the above expression must be replaced by an integral. (See section 5.1.1 in the textbook for a more detailed explanation of this crossover from series to integral.) In the case of the complex exponential basis, this gives rise to the *Fourier transform*.

We begin by recall the complex exponential basis on $[0,1]$: if f is any suitably well-behaved function on $[0,1]$, then we have

$$f = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx},$$

where

$$\hat{f}_k = \int_0^1 f(x)e^{-2\pi ikx}\,dx$$

(the signs in the exponents differ since when we take an inner product we always take the conjugate of the second function). This may clearly be extended to the unit cube, giving

$$f(\mathbf{x}) = \sum_{\ell,m,n=-\infty}^{\infty} \hat{f}_{\mathbf{k}} e^{2\pi i\mathbf{k}\cdot\mathbf{x}},$$

where

[12]If A is a linear operator on a vector space V, the *spectrum* of A is defined as the set of numbers λ such that $A-\lambda I$ is not invertible. If V is finite-dimensional, this is the same as the set of eigenvalues of A; but in infinite-dimensional spaces, such as those we work with here, this is not necessarily the case.

$$\hat{f}_{\mathbf{k}} = \int_Q f(x) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$$

where we now write $\mathbf{k} = (\ell, m, n)$. This motivates the following definition of the Fourier transform. We first make one comment about terminology. We shall be dealing mainly with functions f which have the property that $\int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x} < \infty$; such functions are said to be in L^1 (on $\mathbf{R}^m$); in symbols,

$$L^1(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{C} | \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x} < \infty\}.$$

DEFINITION. Suppose that $f \in L^1(\mathbf{R}^m)$. Then we define the *Fourier transform* of f, which we denote $\hat{f}$ or $\mathcal{F}[f]$, to be the function from $\mathbf{R}^m$ to $\mathbf{C}$ given by

$$\hat{f}(\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

(Note that this definition differs from that in the textbook by the factor of 2π in the exponent. As we shall see, factors of 2π appear in various places in the study of Fourier transforms; there is no way to get rid of all of them, and the different conventions just push them around into different locations. From a purely mathematical point of view, there are many reasons why the above definition recommends itself. See [1], p. 278, for discussion.)

The Fourier transform enjoys the following properties. We write $\mathbf{x} = (x^1, \ldots, x^m)$, $\mathbf{k} = (k^1, \ldots, k^m)$; the superscripts are just that – superscripts – not powers; think of them like subscripts, but written up, not down.[13]

PROPERTIES OF THE FOURIER TRANSFORM. (a) The Fourier transform is linear:

$$\mathcal{F}[af + bg](\mathbf{k}) = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}).$$

(b) If f is differentiable and $\frac{\partial f}{\partial x^j} \in L^1$ for some j (note that this latter does not follow from $f \in L^1$!), then

$$\mathcal{F}[\partial_j f](\mathbf{k}) = 2\pi i k^j \mathcal{F}[f](\mathbf{k}).$$

(c) If $x^j f \in L^1$ for some j, then

$$\mathcal{F}[2\pi i x^j f](\mathbf{k}) = -\frac{\partial \hat{f}}{\partial k^j}.$$

(d) For any $\boldsymbol{\alpha} \in \mathbf{R}^m$,

$$\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) = e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\alpha}} \hat{f}(\mathbf{k}).$$

(e) Similarly,

$$\mathcal{F}[e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x})](\mathbf{k}) = \hat{f}(\mathbf{k} - \boldsymbol{\alpha}).$$

Proof. (a) This is entirely straightforward, almost trivial:

$$\begin{aligned}\mathcal{F}[af + bg](\mathbf{k}) &= \int_{\mathbf{R}^m} [af(\mathbf{x}) + bg(\mathbf{x})]\, e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= a\int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} + b\int_{\mathbf{R}^m} g(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}).\end{aligned}$$

(b) We first note a technical point. Contrary to what I said in class, $f \in L^1$ does *not* imply that $f \to 0$ as $\mathbf{x} \to \infty$ (think of the function on $\mathbf{R}^1$ defined by

$$f(x) = \begin{cases} 1, & x \in [n, n + \frac{1}{n^2}] \text{ for some } n \in \mathbf{Z} \\ 0, & \text{otherwise} \end{cases};$$

[13]The reason for this is that from a differential-geometric perspective, quantities with indices *up* are distinct from quantities with indices *down*. This is related to the first notion of 'tensor' which I mentioned in class on Thursday, as a collection of numbers transforming in a certain way under a coordinate transformation.

it is simple to see that $f \in L^1$, but clearly f has no limit as $x \to \infty$). However, $\frac{\partial f}{\partial x^j} \in L^1$ implies that $\lim_{x^j \to \infty} f(\mathbf{x})$ exists: for

$$\lim_{x^j \to \infty} f(\mathbf{x}) = f(0) + \lim_{x^j \to \infty} \int_0^{x^j} \frac{\partial f}{\partial x^j}(x^1, \cdots, y^j, \cdots, x^n)\, dy^j,$$

and since the latter integral is absolutely convergent (as $\partial_j f \in L^1$), it must be convergent, meaning that $\lim_{x^j \to \infty} f(\mathbf{x})$ exists. Since $f \in L^1$, this limit must be zero. The result may now be proven using integration by parts:

$$\begin{aligned}\mathcal{F}[\partial_j f](\mathbf{k}) &= \int_{\mathbf{R}^m} \frac{\partial f}{\partial x^j} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} \frac{\partial}{\partial x^j}\left(f e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\right) + 2\pi i k^j f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= 2\pi i k^j \mathcal{F}[f](\mathbf{k}),\end{aligned}$$

where the boundary term vanishes since, as just noted, $\lim_{x^j \to \infty} f(\mathbf{x}) = 0$.

(c) We have

$$\begin{aligned}\mathcal{F}[2\pi i x^j f](\mathbf{k}) &= \int_{\mathbf{R}^m} 2\pi i x^j f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = -\int_{\mathbf{R}^m} f(\mathbf{x}) \frac{\partial}{\partial k^j} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= -\frac{\partial}{\partial k^j} \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = -\frac{\partial \hat{f}}{\partial k^j},\end{aligned}$$

where we may interchange differentiation and integration since $x^j f \in L^1$[14]

(d) This and (e) are very straightforward calculations. Here we do a change of variables $\mathbf{y} = \mathbf{x} - \boldsymbol{\alpha}$:

$$\begin{aligned}\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) &= \int_{\mathbf{R}^m} f(\mathbf{x} - \boldsymbol{\alpha}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m} f(\mathbf{y}) e^{-2\pi i \mathbf{k}\cdot(y+\boldsymbol{\alpha})}\, d\mathbf{x} = e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\alpha}} \hat{f}(\mathbf{k}).\end{aligned}$$

(e)
$$\begin{aligned}\mathcal{F}[e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x})](\mathbf{k}) &= \int_{\mathbf{R}^m} e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i (\mathbf{k}-\boldsymbol{\alpha})\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \hat{f}(\mathbf{k} - \boldsymbol{\alpha}).\end{aligned}$$

Parts (a) and (b) are probably the most important for us at the moment. To show how (b) is used more generally, suppose that $P(\mathbf{k})$ is some polynomial on $\mathbf{R}^m$, i.e., $P(\mathbf{k})$ is a linear combination of monomial terms of the form

$$\left(k^1\right)^{\alpha_1} \left(k^2\right)^{\alpha_2} \cdots \left(k^m\right)^{\alpha_m},$$

where $\alpha_1, \ldots, \alpha_m$ are all nonnegative integers. Suppose that $Q(\mathbf{k})$ is the above monomial. Then we define the differential operator $Q(\nabla)$ by

$$Q(\nabla)(f)(\mathbf{x}) = \frac{\partial^{\alpha_1} f}{\partial x^{1\alpha_1}} \cdots \frac{\partial^{\alpha_m} f}{\partial x^{m\alpha_m}}.$$

Part (b) can then be used to show that for any polynomial $P(\mathbf{k})$,

$$\mathcal{F}\left[P(\nabla) f\right](\mathbf{k}) = P(2\pi i \mathbf{k}) \hat{f}(\mathbf{k});$$

i.e., the Fourier transform turns differential operators into multiplication operators. Let us consider a few examples.

[14] Again, to get things in this generality one probably needs the Lebesgue integral and dominated convergence theorem. We apologise.

EXAMPLES. (a) Let us consider an example in $\mathbf{R}^1$ for simplicity. If f is such that all of the relevant Fourier transforms are defined, then

$$\mathcal{F}[f'(x)](k) = -2\pi i k \hat{f}(k),$$
$$\mathcal{F}[f''(x)](k) = -4\pi^2 k^2 \hat{f}(k).$$

(b) Let us consider how the Fourier transform acts on the Laplacian of a function in $\mathbf{R}^3$. Thus suppose that f is a function on $\mathbf{R}^3$ and is such that all of the relevant Fourier transforms are defined. Then

$$\begin{aligned}\mathcal{F}[\nabla^2 f](\mathbf{k}) &= \mathcal{F}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right] = (-2\pi i k^1)^2 \hat{f}(\mathbf{k}) + (-2\pi i k^2)^2 \hat{f}(\mathbf{k}) + (-2\pi i k^3)^2 \hat{f}(\mathbf{k}) \\ &= -4\pi^2 (k^1)^2 \hat{f}(\mathbf{k}) - 4\pi^2 (k^2)^2 \hat{f}(\mathbf{k}) - 4\pi^2 (k^3)^2 \hat{f}(\mathbf{k}) = -4\pi^2 |\mathbf{k}|^2 \hat{f}(\mathbf{k}),\end{aligned}$$

where $|\mathbf{k}| = (k^1)^2 + (k^2)^2 + (k^3)^2$ is the Euclidean norm of $\mathbf{k}$.

The above three examples will be the most important examples of this kind of thing for us going forwards.

CONVOLUTION. To derive the next property satisfied by the Fourier transform, we need to define another operation on functions known as *convolution*. Unlike the Fourier transform, which takes a single function to a single function, convolution is a product, which takes two functions to one function. It is defined as follows: suppose that $f, g \in L^1(\mathbf{R}^m)$; then we define their *convolution* $f * g$ (note that this is a star, an asterisk, with *six points*, and may not be properly written with less!) to be the function on $\mathbf{R}^m$

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}'.$$

Convolution satisfies the following properties:

(a) It is *bilinear*:

$$([af + bh] * g)(\mathbf{x}) = a(f * g)(\mathbf{x}) + b(h * g)(\mathbf{x}), \quad (f * [ag + bh])(\mathbf{x}) = a(f * g)(\mathbf{x}) + b(f * h)(\mathbf{x}).$$

(b) It is commutative:

$$(f * g)(\mathbf{x}) = (g * f)(\mathbf{x})$$

and associative:

$$[(f * g) * h](\mathbf{x}) = [f * (g * h)](\mathbf{x}).$$

Proof. (a) This is again almost trivial; we show only the first one as the second is identical:

$$\begin{aligned}([af + bh] * g)(\mathbf{x}) &= \int_{\mathbf{R}^m} (af + bh)(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}' = a \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}' + b \int_{\mathbf{R}^m} h(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}' \\ &= a(f * g)(\mathbf{x}) + b(h * g)(\mathbf{x}).\end{aligned}$$

(b) The first of these is straightforward, doing a change of variables to $\mathbf{y} = \mathbf{x} - \mathbf{x}'$:

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')\, d\mathbf{x}' = \int_{R^m} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = (g * f)(\mathbf{x}).$$

The second may be shewn as follows, using the change of variables $\mathbf{y} = \mathbf{x}' + \mathbf{x}''$, $\mathbf{y}' = \mathbf{x}'$; we note that this has unit determinant. We ignore the difficulties in rewriting the iterated integrals as a single integral.

$$\begin{aligned}[(f * g) * h](\mathbf{x}) &= \int_{\mathbf{R}^m} (f * g)(\mathbf{x} - \mathbf{x}') h(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^m} \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}' - \mathbf{x}'') g(\mathbf{x}'') h(\mathbf{x}')\, d\mathbf{x}''\, d\mathbf{x}' \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y} - \mathbf{y}') h(\mathbf{y}')\, d\mathbf{y}'\, d\mathbf{y} = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{y})(g * h)(\mathbf{y})\, d\mathbf{y} = [f * (g * h)](\mathbf{x}),\end{aligned}$$

as desired.

For us, the main interest in convolution is not the above algebraic properties (though it is important to know about these), but rather the way in which convolution interacts with the Fourier transform. This is given in the following theorem.

THEOREM. Suppose that $f, g \in L^1(\mathbf{R}^m)$, $f * g \in L^1(\mathbf{R}^m)$. Then

$$\mathcal{F}[f * g](\mathbf{k}) = \hat{f}(\mathbf{k})\hat{g}(\mathbf{k});$$

i.e., the Fourier transform turns convolutions into products.

Proof. Assuming (as in the proof of associativity above) that we may combine the iterated integrals appearing here into a single integral over the product space, we have, using the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}'$, $\mathbf{y}' = \mathbf{x}'$,

$$\begin{aligned}\mathcal{F}[f * g](\mathbf{k}) &= \int_{\mathbf{R}^m} (f * g)(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}'\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m\times\mathbf{R}^m} f(\mathbf{y})g(\mathbf{y}')e^{-2\pi i\mathbf{k}\cdot(\mathbf{y}+\mathbf{y}')}\, d\mathbf{y}'\, d\mathbf{y} = \int_{\mathbf{R}^m} f(\mathbf{y})e^{-2\pi i\mathbf{k}\cdot\mathbf{y}}\, d\mathbf{y} \int_{\mathbf{R}^m} g(\mathbf{y}')e^{-2\pi i\mathbf{k}\cdot\mathbf{y}'}\, d\mathbf{y}' \\ &= \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}),\end{aligned}$$

as claimed.

We note in passing that an analogous result would hold if we replaced $e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}$ in $\mathcal{F}$ by $e^{2\pi i\mathbf{k}\cdot\mathbf{x}}$; this will be used shortly.

One use of the Fourier transform for solving partial differential equations can now be indicated briefly. Suppose that we are interested in solving Poisson's equation on $\mathbf{R}^m$; i.e., that we have the problem $\nabla^2 u = f$. If $f \in L^1$, then assuming $u \in L^1$ we may take the Fourier transform of both sides to obtain

$$-4\pi^2|\mathbf{k}|^2\hat{u}(\mathbf{k}) = \hat{f}(\mathbf{k}),$$

whence

$$\hat{u} = -\frac{1}{4\pi^2|\mathbf{k}|^2}\hat{f}(\mathbf{k}).$$

By the foregoing theorem, then, if we can find a function g whose Fourier transform is $-\frac{1}{4\pi^2|\mathbf{k}|^2}$, then we will have

$$u(\mathbf{x}) = (g * f)(\mathbf{x}) = \int_{\mathbf{R}^m} g(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}'.$$

Our work with Green's functions suggests that in $\mathbf{R}^3$ we have $g(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}$. Note, though, that the Fourier transform result above is independent of m.

Even more generally, suppose that we were interested in the equation

$$P(\nabla)u = f$$

for some polynomial P; if $f \in L^1$, then taking the Fourier transform gives $P(\mathbf{k})u = f$, or $u(\mathbf{k}) = \frac{1}{P(\mathbf{k})}f(\mathbf{k})$. Hence, if $\frac{1}{P(\mathbf{k})}$ is the Fourier transform of a function g, then $u(\mathbf{x}) = (g * f)(\mathbf{x})$ will be the solution to the original problem. Actually calculating g, however, is quite another matter. We shall try to say more about this later.

FOURIER INVERSION THEOREM. Our study of the basic theory of the Fourier transform will be essentially completed once we have established the *Fourier inversion theorem*, which tells us how to invert the Fourier transform. Its statement is as follows; a proof will be given next week.

THEOREM. Suppose that $f \in L^1$ is such that f is bounded on $\mathbf{R}^m$ (this condition can be removed by doing more careful calculations in Example (b) on p. 5) and $\hat{f} \in L^1$. Then we have[15]

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

[15]This formula, like so many of the other formulas we have given in this course, may not hold at every point. It will, however, hold at points at which f is continuous; it will, in fact, hold *almost everywhere*, meaning everywhere except on a set of *measure zero*. For a related result, see Theorem 5.1 in the textbook.

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