Summary:
- We use the eigenfunctions and eigenvalues of the Laplacian on a cylinder, derived last week, to solve a problem involving Poisson's equation on a cylinder.
- We then derive the eigenfunctions and eigenvalues of the Laplacian on the unit ball.
- We give a method for solving Poisson's equation and the heat equation with inhomogeneous boundary conditions, and give an example in spherical coordinates.

EXAMPLE. Solve the following problem on the cylinder $C = \{(\rho, \theta, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = z\rho^2 \cos 2\phi, \quad u|_{\partial C} = 0.$$

From last time, we know that the eigenfunctions of the Laplacian on C are

$$\mathbf{e}_{nmi} = \begin{cases} J_m(\lambda_{mi}\rho) \cos m\phi \sin n\pi z \\ J_m(\lambda_{mi}\rho) \sin m\phi \sin n\pi z \end{cases}$$

with corresponding eigenvalues

$$\lambda_{nmi} = -\lambda_{mi}^2 - n^2\pi^2.$$

Thus we must expand the function $z\rho^2 \cos 2\phi$ in this basis. To do this, we compute as follows:

$$\begin{aligned}\left(z\rho^2 \cos 2\phi, J_m(\lambda_{mi}\rho) \cos m\phi \sin n\pi z\right) &= \int_0^1 \int_0^1 \int_0^{2\pi} z\rho^2 \cos 2\phi J_m(\lambda_{mi}\rho) \cos m\phi \sin n\pi z \, d\phi \, dz \, \rho d\rho \\ &= \int_0^1 z \sin n\pi z \, dz \int_0^1 \rho^3 J_m(\lambda_{mi}\rho) \, d\rho \int_0^{2\pi} \cos 2\phi \cos m\phi \, d\phi,\end{aligned}$$

which is seen to be zero when $m \neq 2$, while when $m = 2$ it becomes

$$\frac{(-1)^{n+1}}{n\pi} \frac{J_3(\lambda_{mi})}{\lambda_{mi}} \pi,$$

whence the coefficient of $J_m(\lambda_{mi}\rho) \cos m\phi \sin n\pi z$ in the expansion of $z\rho^2 \cos 2\phi$ when $m \neq 2$ is zero, while when $m = 2$ it is

$$\frac{\frac{1}{n\pi}(-1)^{n+1} \frac{J_3(\lambda_{2i})}{\lambda_{2i}} \pi}{\frac{1}{2} \cdot \frac{1}{2} J_3^2(\lambda_{2i}) \cdot \pi} = \frac{4(-1)^{n+1}}{n\pi\lambda_{2i} J_3(\lambda_{2i})}.$$

A similar calculation shows immediately that the coefficient of $J_m(\lambda_{mi}\rho) \sin m\phi \sin n\pi z$ is zero, since $\cos 2\phi$ is orthogonal to $\sin m\phi$ for all m. Thus we have finally

$$z\rho^2 \cos 2\phi = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi\lambda_{2i} J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho) \sin n\pi z \cos 2\phi.$$

Given this, the solution to our original problem is almost immediate: we assume as usual that we have an expansion of the form

$$u(\rho, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho) \sin n\pi z \left(c_{nmi} \cos m\phi + d_{nmi} \sin m\phi\right);$$

then, assuming that we may differentiate term-by-term, we have, since both $J_m(\lambda_{mi}\rho) \cos m\phi \sin n\pi z$ and $J_m(\lambda_{mi}\rho) \sin m\phi \sin n\pi z$ are eigenfunctions of the Laplacian with the same eigenvalue $\lambda_{nmi} = -\lambda_{mi}^2 - n^2\pi^2$,

$$\begin{aligned}\nabla^2 u &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} \left(-\lambda_{mi}^2 - n^2\pi^2\right) J_m(\lambda_{mi}\rho) \sin n\pi z \left(c_{nmi} \cos m\phi + d_{nmi} \sin m\phi\right) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi\lambda_{2i} J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho) \sin n\pi z \cos 2\phi,\end{aligned}$$

whence we see that $d_{nmi} = 0$ for all n, m, i, while $c_{nmi} = 0$ for all n and i unless $m = 2$ and

$$c_{n2i} = -\frac{4(-1)^{n+1}}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)\left(\lambda_{2i}^2 + n^2\pi^2\right)},$$

so that finally we have the solution

$$u(\rho, \theta, z) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \frac{4(-1)^n}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)\left(\lambda_{2i}^2 + n^2\pi^2\right)} J_2\left(\lambda_{2i}\rho\right) \sin n\pi z \cos 2\phi.$$

EXAMPLE. Solve the following problem on $(0, +\infty) \times C$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = z\rho^2 \cos 2\phi, \quad u|_{(0,+\infty)\times\partial Q} = 0.$$

From the previous example, we have the expansion

$$z\rho^2 \cos 2\phi = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} \frac{8}{(2k+1)\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)} J_2\left(\lambda_{2i}\rho\right) \sin{(2k+1)\pi z} \cos 2\phi.$$

Expanding u as

$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{mi}\rho\right) \sin n\pi z \left(c_{nmi}(t) \cos m\phi + d_{nmi}(t) \sin m\phi\right)$$

and substituting this into the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$ as before, we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{mi}\rho\right) \sin n\pi z \left(c'_{nmi}(t) \cos m\phi + d'_{nmi}(t) \sin m\phi\right) \\
&\qquad = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} \left(-\lambda_{mi}^2 - n^2\pi^2\right) J_m\left(\lambda_{mi}\rho\right) \sin n\pi z \left(c_{nmi}(t) \cos m\phi + d_{nmi}(t) \sin m\phi\right),
\end{aligned}$$

whence equating coefficients of like terms gives the equations

$$\begin{aligned}
c'_{nmi} &= -\left(\lambda_{mi}^2 + n^2\pi^2\right) c_{nmi} \\
d'_{nmi} &= -\left(\lambda_{mi}^2 + n^2\pi^2\right) d_{nmi}.
\end{aligned} \tag{1}$$

Now the initial condition gives

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{mi}\rho\right) \sin n\pi z \left(c_{nmi}(0) \cos m\phi + d_{nmi}(0) \sin m\phi\right) \\
&\qquad = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \frac{4\left(1 - (-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)} J_2\left(\lambda_{2i}\rho\right) \sin n\pi z \cos 2\phi,
\end{aligned}$$

so

$$c_{nmi}(0) = \begin{cases} 0, & m \neq 2, \\ \frac{4(1-(-1)^n)}{n\pi\lambda_{2i}J_3(\lambda_{2i})}, & m = 2 \end{cases}$$
$$d_{nmi}(0) = 0,$$

whence the system (1) gives $d_{nmi}(t) = 0$ for all n, m, i and all t, while $c_{nmi}(t) = 0$ for all t unless $m = 2$ and finally

$$c_{n2i}(t) = \frac{4\left(1 - (-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)} e^{-\left(\lambda_{2i}^2 + n^2\pi^2\right)t},$$

so that the solution to our original problem is finally

$$u(\rho, \theta, z) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{n\pi\lambda_{2i} J_3\left(\lambda_{2i}\right)} e^{-\left(\lambda_{2i}^2 + n^2\pi^2\right)t} J_2\left(\lambda_{2i}\rho\right) \sin n\pi z \cos 2\phi$$
$$= \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \frac{8}{(2k+1)\pi\lambda_{2i} J_3\left(\lambda_{2i}\right)} e^{-\left(\lambda_{2i}^2 + (2k+1)^2\pi^2\right)t} J_2\left(\lambda_{2i}\rho\right) \sin\left(2k+1\right)\pi z \cos 2\phi.$$

EIGENVALUES AND EIGENFUNCTIONS FOR THE LAPLACIAN ON THE UNIT BALL. We now turn our attention to the task of finding the eigenfunctions and eigenvalues of the Laplacian on the unit ball with homogeneous[1] Dirichlet boundary conditions. In other words, let $B = \{(r, \theta, \phi) | r < 1\}$ denote the unit ball in spherical coordinates, and consider the problem

$$\nabla^2 u = \lambda u, \quad u|_{\partial B} = 0.$$

We approach this problem as before, by separating variables; thus we set

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi),$$

and recalling that in spherical coordinates the Laplacian is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

we obtain, substituting into $\nabla^2 u = \lambda u$ and dividing by u

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2 \sin^2\theta}\frac{\Phi''}{\Phi} = \lambda. \tag{2}$$

Since only the quantity $\frac{\Phi''}{\Phi}$ depends on ϕ, this quantity must be constant. Considerations identical to those used when solving Laplace's equation in spherical and cylindrical coordinates and when finding the eigenvalues of the Laplacian in cylindrical coordinates show that we must in fact have $\frac{\Phi''}{\Phi} = -m^2$, where $m \in \mathbf{Z}$, $m \geq 0$, which has solutions $\{hi = \cos m\phi$, $\Phi = \sin m\phi$ (the latter only for $m > 0$). Substituting this back into equation (2) above, we obtain

$$\lambda = \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} - \frac{m^2}{r^2 \sin^2\theta}$$
$$= \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right);$$

as when we solved Laplace's equation in spherical coordinates (see notes for May 23 – 30), this implies that the quantity in parentheses above is constant. By analogy with what we did there, we set it equal to $-\ell(\ell+1)$, where $\ell \in \mathbf{Z}$, $\ell \geq 0$. Then Θ must satisfy the equation

$$\Theta'' + \cot\theta\Theta' + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0,$$

whence we see that $\Theta(\theta) = P_{\ell m}(\cos\theta)$, as when solving Laplace's equation. We are thus left only with the following equation for R:

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} - \frac{1}{r^2}\ell(\ell+1) = \lambda,$$
$$R'' + \frac{2}{r}R' + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)R = 0.$$

[1]Since eigenvalue problems must of necessity be linear, it makes no sense to ask for an eigenfunction of the Laplacian satisfying inhomogeneous boundary conditions; or at any rate, while one could certainly write out the equations, it is hard to see how the resulting solutions could be of use.

As it stands, this is close to Bessel's equation (see notes for June 11 – 13, p. 2, Equation (2))

$$P'' + \frac{1}{\rho}P' + \left(\lambda^2 - \frac{m^2}{\rho^2}\right)P = 0,$$

but it is not identical. We may transform it into Bessel's equation by the following method. Let $S = r^{\frac{1}{2}}R$, so that $R = r^{-\frac{1}{2}}S$; then we have

$$\begin{aligned} R' &= -\frac{1}{2}r^{-\frac{3}{2}}S + r^{-\frac{1}{2}}S', \\ R'' &= \frac{3}{4}r^{-\frac{5}{2}}S - r^{-\frac{3}{2}}S' + r^{-\frac{1}{2}}S'', \end{aligned}$$

whence we see that

$$\begin{aligned} 0 &= R'' + \frac{2}{r}R' + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)R \\ &= \left(\frac{3}{4}r^{-\frac{5}{2}}S - r^{-\frac{3}{2}}S' + r^{-\frac{1}{2}}S''\right) + \frac{2}{r}\left(-\frac{1}{2}r^{-\frac{3}{2}}S + r^{-\frac{1}{2}}S'\right) + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)r^{-\frac{1}{2}}S \\ &= r^{-\frac{1}{2}}\left(S'' + \left(-r^{-1} + \frac{2}{r}\right)S' + \left(\frac{3}{4}r^{-2} - \frac{1}{r^2} - \lambda - \frac{\ell(\ell+1)}{r^2}\right)S\right) \\ &= r^{-\frac{1}{2}}\left(S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\ell(\ell+1)+\frac{1}{4}}{r^2}\right)S\right) = r^{-\frac{1}{2}}\left(S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\left(\ell+\frac{1}{2}\right)^2}{r^2}\right)S\right), \end{aligned}$$

so that S must satisfy the equation

$$S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\left(\ell+\frac{1}{2}\right)^2}{r^2}\right)S = 0.$$

Now the boundary condition $u|_{\partial B} = 0$ means that R must satisfy $R(1) = 0$; since $S = r^{\frac{1}{2}}R$, this implies that $S(1) = 0$ also. Thus S cannot be a modified Bessel function, which implies that we must have $\lambda < 0$ and (up to a multiplicative constant) $S = J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r)$. Again, $S(1) = 0$ implies that $\sqrt{\lambda} = \kappa_{\ell i}$ for some i, where $\kappa_{\ell i}$ denotes the ith positive zero of $J_{\ell+\frac{1}{2}}(x)$ (thus, if we were to extend our earlier notation and let $\lambda_{\nu i}$ denote the ith positive zero of $J_\nu(x)$ for any real $\nu \geq 0$, we have $\kappa_{\ell i} = \lambda_{\ell+\frac{1}{2},i}$; this latter expression is the notation which we used in class). We thus obtain that up to a multiplicative constant

$$R = r^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}\left(\kappa_{\ell i}r\right).$$

It turns out to be convenient to take the multiplicative constant to be $\sqrt{\frac{\pi}{2}}$. The resulting functions are called *spherical Bessel functions* and are denoted by j_ℓ, $\ell \in \mathbf{Z}$, $\ell \geq 0$; explicitly,

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}J_{\ell+\frac{1}{2}}(x).$$

We thus obtain finally that the eigenfunctions for the Laplacian on the unit ball are

$$\mathbf{e}_{m\ell i} = \begin{cases} j_\ell\left(\kappa_{\ell i}r\right)P_{\ell m}(\cos\theta)\cos m\phi \\ j_\ell\left(\kappa_{\ell i}r\right)P_{\ell m}(\cos\theta)\sin m\phi \end{cases},$$

with corresponding eigenvalue

$$\lambda_{m\ell i} = -\kappa_{\ell i}^2.$$

We note that the eigenvalue does not depend on m (though it does depend on both ℓ and i).

We now derive the orthogonality properties of the j_ℓ. First, we note without proof that the Bessel functions J_ν satisfy the same orthogonality relations as the J_m for all real (not just integer) $\nu \geq 0$, namely

$$\int_0^1 x J_\nu\left(\lambda_{\nu i} x\right) J_\nu\left(\lambda_{\nu j} x\right)\, dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2} J_{\nu+1}^2\left(\lambda_{\nu i}\right), & i = j \end{cases}.$$

From this we may derive the orthogonality property of the spherical Bessel functions, as follows:

$$\begin{aligned} \int_0^1 x^2 j_\ell\left(\kappa_{\ell i} x\right) j_\ell\left(\kappa_{\ell j} x\right)\, dx &= \frac{\pi}{2\sqrt{\kappa_{\ell i}\kappa_{\ell j}}} \int_0^1 x J_{\ell+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},i} x\right) J_{\ell+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},j} x\right)\, dx \\ &= \begin{cases} 0, & i \neq j \\ \frac{\pi}{4\lambda_{\ell+\frac{1}{2},i}} J_{\ell+\frac{1}{2}+1}^2\left(\lambda_{\ell+\frac{1}{2},i}\right), & i = j \end{cases}, \end{aligned}$$

whence we see that $\{j_\ell\left(\kappa_{\ell i} x\right)\}$ is an orthogonal set on the interval $[0,1]$ with the normalisation integral

$$\begin{aligned} \int_0^1 x^2 j_\ell^2\left(\kappa_{\ell i} x\right)\, dx &= \frac{\pi}{4\lambda_{\ell+\frac{1}{2},i}} J_{\ell+\frac{1}{2}+1}^2\left(\lambda_{\ell+\frac{1}{2},i}\right) = \frac{1}{2}\left(\sqrt{\frac{\pi}{2\lambda_{\ell+\frac{1}{2},i}}} J_{\ell+1+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},i}\right)\right)^2 \\ &= \frac{1}{2} j_{\ell+1}^2\left(\kappa_{\ell i}\right). \end{aligned}$$

From this it follows, as before, that $\{\mathbf{e}_{m\ell i}\}$ is a complete orthogonal set on the unit ball B with respect to the inner product

$$\begin{aligned} (f(r,\theta,\phi), g(r,\theta,\phi)) &= \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)}\, d\phi \sin\theta d\theta\, r^2 dr \\ &= \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)} r^2 \sin\theta\, d\phi\, d\theta\, dr. \end{aligned}$$

(Note that the quantity $r^2 \sin\theta\, d\phi\, d\theta\, dr$ is just the volume element dV in spherical coordinates; in other words, the integral above is simply $\iiint_B f\overline{g} dV$.) This allows us to solve Poisson's equation and the heat equation on B, as we did with the unit cube Q and the cylinder C before.

EXAMPLE. Solve the following problem on B:

$$\nabla^2 u = r\sin\theta\sin\phi, \quad u|_{\partial B} = 0.$$

We begin, as usual, be expanding the function on the right-hand side in the basis of eigenfunctions $\{\mathbf{e}_{m\ell i}\}$ appropriate to the problem; thus we write

$$r\sin\theta\sin\phi = \sum_{m=0}^\infty \sum_{\ell=m}^\infty \sum_{i=1}^\infty j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\left(a_{m\ell i}\cos m\phi + b_{m\ell i}\sin m\phi\right),$$

where

$$\begin{aligned} b_{m\ell i} &= \frac{\left(r\sin\theta\sin\phi, j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\sin m\phi\right)}{\left(j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\sin m\phi, j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\sin m\phi\right)} \\ a_{m\ell i} &= \frac{\left(r\sin\theta\sin\phi, j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\cos m\phi\right)}{\left(j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\cos m\phi, j_\ell\left(\kappa_{\ell i} r\right) P_{\ell m}(\cos\theta)\cos m\phi\right)}. \end{aligned}$$

Since $(\sin\phi, \cos m\phi) = 0$ for all m, we see that we have $a_{m\ell i} = 0$ for all m, ℓ, i; similarly, $b_{m\ell i} = 0$ for all ℓ and i unless $m = 1$, in the which case we may compute (recalling that $\{P_{\ell m}(x)\}_{\ell=m}^\infty$ is a complete orthogonal set on $[-1,1]$ for all $m \geq 0$, and that $P_{11} = \sin\theta$)

$$\begin{aligned} \left(r\sin\theta\sin\phi, j_\ell\left(\kappa_{\ell i} r\right) P_{\ell 1}(\cos\theta)\sin\phi\right) &= \int_0^1 \int_0^\pi \int_0^{2\pi} r\sin\theta\sin\phi j_\ell\left(\kappa_{\ell i} r\right) P_{\ell 1}(\cos\theta)\sin\phi\, d\phi \sin\theta d\theta\, r^2 dr \\ &= \int_0^1 r^3 j_\ell\left(\kappa_{\ell i} r\right)\, dr \int_0^\pi \sin\theta P_{\ell 1}(\cos\theta)\sin\theta\, d\theta \int_0^{2\pi} \sin^2\phi\, d\phi \end{aligned}$$

which is zero unless $\ell = 1$, while if $\ell = 1$ it is (using the normalisation $\int_{-1}^{1} P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$, which in our case becomes $\int_{-1}^{1} P_{11}^2(x)\,dx = \frac{4}{3}$, and remembering that $\kappa_{1i} = \lambda_{\frac{3}{2},i}$)

$$\frac{4\pi}{3}\int_0^1 r^3 j_1(\kappa_{1i}r)\,dr = \frac{4\pi}{3}\sqrt{\frac{\pi}{2\kappa_{1i}}}\int_0^1 r^{\frac{5}{2}} J_{\frac{3}{2}}\left(\lambda_{\frac{3}{2},i}r\right)dr$$
$$= \frac{4\pi}{3}\sqrt{\frac{\pi}{2\kappa_{1i}}}\frac{J_{\frac{5}{2}}\left(\lambda_{\frac{3}{2},i}\right)}{\lambda_{\frac{3}{2},i}} = \frac{4\pi}{3\kappa_{1i}}j_2(\kappa_{1i})$$

whence using the normalisation integrals for j_1, P_{11}, and $\sin\phi$ we obtain

$$b_{11i} = \frac{2}{\kappa_{1i}j_2(\kappa_{1i})},$$

while $b_{m\ell i} = 0$ unless $m = \ell = 1$. (Note the similarity of the above form to that derived for ordinary (nonspherical) Bessel functions when expanding expressions like ρ^m on a cylinder.) Thus we have finally the expansion

$$r\sin\theta\sin\phi = \sum_{i=1}^{\infty}\frac{2}{\kappa_{1i}j_2(\kappa_{1i})}j_1(\kappa_{1i}r)\sin\theta\sin\phi.$$

Writing now

$$u = \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}\sum_{i=1}^{\infty} j_\ell(\kappa_{\ell i}r)\,P_{\ell m}(\cos\theta)\left(c_{m\ell i}\cos m\phi + d_{m\ell i}\sin m\phi\right),$$

we see that

$$\nabla^2 u = \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}\sum_{i=1}^{\infty} -\kappa_{\ell i}^2 j_\ell(\kappa_{\ell i}r)\,P_{\ell m}(\cos\theta)\left(c_{m\ell i}\cos m\phi + d_{m\ell i}\sin m\phi\right);$$

equating this to the expansion for the function $r\sin\theta\sin\phi$ obtained above gives, as usual, $c_{m\ell i} = -\frac{1}{\kappa_{\ell i}^2}a_{m\ell i} = 0$ for all m, ℓ, i, while $d_{m\ell i} = -\frac{1}{\kappa_{\ell i}^2}b_{m\ell i}$ is zero unless $m = \ell = 1$, in the which case

$$d_{11i} = -\frac{2}{\kappa_{1i}^3 j_2(\kappa_{1i})},$$

and we have finally

$$u = \sum_{i=1}^{\infty} -\frac{2}{\kappa_{1i}^3 j_2(\kappa_{1i})}j_1(\kappa_{1i}r)\sin\theta\sin\phi.$$

A similar example could clearly be worked for the heat equation, along the lines of the pair of examples given in cylindrical coordinates above; we leave the formulation and solution of such a problem to the reader.

INHOMOGENEOUS BOUNDARY CONDITIONS. Consider now the problem (say on B)

$$\nabla^2 u = f, \quad u|_{\partial B} = g,$$

where neither f nor g is identically zero. This problem may be solved by first solving the two ancillary problems

$$\nabla^2 u_1 = f, \quad u_1|_{\partial B} = 0,$$
$$\nabla^2 u_2 = 0, \quad u_2|_{\partial B} = g,$$

the second of which may be solved using the methods developed for solving Laplace's equation on a ball, and the first of which may be solved using the eigenfunctions just derived. If we then set $u = u_1 + u_2$, we see that

$$\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2 = f + 0 = f,$$
$$u|_{\partial B} = u_1|_{\partial B} + u_2|_{\partial B} = 0 + g = g;$$

in other words, $u = u_1 + u_2$ is a solution to our original problem.

This method clearly applies to any of the regions Q, C, B we have studied.

In the hope that the foregoing is sufficiently clear as it stands, we skip giving any examples to talk about a similar method for the heat equation. In this case, we are given the problem

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial B} = g.$$

As in the case of Poisson's equation just considered, this may be solved by decomposing u as a sum of solutions to two ancillary problems. The decomposition is a bit more subtle in this case. We first give some motivation. Recall from our previous work that solutions to the heat equation with homogeneous boundary data converge to 0 as $t \to +\infty$. A more careful consideration of the series solutions given above shows that in fact also $\frac{\partial u}{\partial t} \to 0$ as $t \to +\infty$.[2] Now if $\frac{\partial u}{\partial t} = 0$, then the heat equation becomes simply $\nabla^2 u = 0$, i.e., it becomes Laplace's equation. Noting that $u = 0$ is the unique solution to Laplace's equation on B satisfying $u|_{\partial B} = 0$, we see that in this (admittedly very special!) case the solution to the heat equation with boundary data $u|_{\partial B} = 0$ converges to the solution to *Laplace's* equation on B with the same boundary data.

It turns out that this is true for inhomogeneous boundary data also, as we shall now show. Thus let U_1 be the solution to the problem on B

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = g$$

(which is just a boundary-value problem for Laplace's equation on the unit ball, and hence is a problem we know how to solve). Now let us define $u_1 : (0, +\infty) \times B \to \mathbf{R}^1$ by $u_1(t, x, y, z) = U_1(x, y, z)$; then we see that u_1 is a solution to the problem

$$\frac{\partial u_1}{\partial t} = \nabla^2 u_1, \quad u_1|_{t=0} = U_1, \quad u_1|_{(0,+\infty)\times\partial B} = g,$$

since in this case $\frac{\partial u_1}{\partial t} = 0$. (Note that the initial condition is a bit silly since in fact $u_1 = U_1$ for all t; but it is certainly true nonetheless.) Since the problem we wish to solve is

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial B} = g,$$

this suggests taking the other part of the solution to be the function u_2 satisfying

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = f - U_1, \quad u|_{(0,+\infty)\times\partial B} = 0,$$

which we can solve using the eigenfunction methods developed earlier. Letting u_1 and u_2 be these two solutions, and taking $u = u_1 + u_2$, we see that

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} = 0 + \nabla^2 u_2 = \nabla^2 u_1 + \nabla^2 u_2 = \nabla^2 u,\\ u|_{t=0} = u_1|_{t=0} + u_2|_{t=0} = U_1 + f - U_1 = f,\\ u|_{(0,+\infty)\times\partial B} = u_1|_{(0,+\infty)\times\partial B} + u_2|_{(0,+\infty)\times\partial B} = g + 0 = g,\end{gathered}$$

so that $u = u_1 + u_2$ is indeed a solution to our original problem, as desired.

EXAMPLE. We give a simple example of the foregoing to illustrate the procedure. Consider the following problem on B:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = r\sin\theta\sin\phi, \quad u|_{(0,+\infty)\times\partial B} = \cos\theta.$$

[2] Note that this does *not* follow from the preceding statement: consider, for example, the function $f(t) = \frac{1}{t}\sin t^3$; we have clearly $f(t) \to 0$ as $t \to +\infty$, while $f'(t) = -\frac{1}{t^2}\sin t^3 + 3t\sin t^3$, which does not converge to any limit as $t \to +\infty$.

We first solve the problem

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = \cos\theta;$$

now on ∂B we have $\cos\theta = z$, since $\partial B = \{(r,\theta,\phi)|r=1\}$; since z satisfies $\nabla^2 z = 0$, we see that the solution to this equation is just $U_1 = z = r\cos\theta$. Thus we are left with the problem

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u|_{t=0} = r\sin\theta\sin\phi - r\cos\theta, \quad u|_{(0,+\infty)\times\partial B} = 0.$$

Now some reflection[3] indicates that the initial data here may be expanded as

$$\sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta)$$

(the point is that the sum above is just to expand the function r in the basis $\{j_1(\kappa_{1i} r)\}$, and hence is insensitive to which combination of $\{P_{1m}\cos m\phi, P_{1m}\sin m\phi\}$ the function r is multiplied by). Thus by standard methods (whose details we invite the reader to fill in as an exercise!) we have

$$u_2(t,x,y,z) = \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} e^{-\kappa_{1i}^2 t} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta),$$

and thus we have finally the solution

$$\begin{aligned} u = u_1 + u_2 &= r\cos\theta + \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} e^{-\kappa_{1i}^2 t} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta) \\ &= \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} j_1(\kappa_{1i} r)\left(e^{-\kappa_{1i}^2 t}\sin\theta\sin\phi + \left(1 - e^{-\kappa_{1i}^2 t}\right)\cos\theta\right). \end{aligned}$$

We note that this solution does indeed converge to the solution $u_1 = r\cos\theta$ to Laplace's equation with the given inhomogeneous boundary conditions, as claimed. We also note the nice interpolation that occurs term-by-term in the above sum between the initial data (for which the angular dependence is $\sin\theta\sin\phi$) and the final value (for which the angular dependence is $\cos\theta$).

[3]The author is reminded of a comment in the aforementioned textbook *Classical Electrodynamics* by J. D. Jackson to the effect that 'adroit use of the recurrence relation leads to ...', and of the exasperated reaction of his electrodynamics instructor upon finding this sentence: 'Oh, J. D.!' The author apologises for making a slightly similar remark here. He hopes that working out the details is somewhat more straightforward than for the corresponding result in Jackson!