Summary:

- We introduce the modified Bessel functions I_m in greater detail, and show how they can be used to solve certain boundary-value problems for Laplace's equation on a cylinder.
- We then show how to use J_m and I_m together to solve the most general kind of boundary-value problem for Laplace's equation on a cylinder.
- We show how to solve Laplace's equation on a rectangular prism using rectangular coordinates in three dimensions, and point out that the most general problem requires using three separate series.
- We then give a brief introduction to the eigenvalue problem for the Laplacian, including why it is useful.

MODIFIED BESSEL FUNCTIONS. Recall that Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

while substituting in the separated $u = P(\rho)\Phi(\phi)Z(z)$ and dividing by u gives the equation

$$\frac{P''}{P} + \frac{P'}{\rho P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0,$$

from which we see that we must have both $\frac{\Phi''}{\Phi}$ and $\frac{Z''}{Z}$ constant. If we are considering problems on the whole range $[0, 2\pi]$ of ϕ, then Φ must be periodic with period 2π, and this means that $\frac{\Phi''}{\Phi} = -m^2$ for some $m \in \mathbf{Z}$, $m \geq 0$. This leaves the question as to what $\frac{Z''}{Z}$ is. Previously we considered the case where $\frac{Z''}{Z} > 0$ and then showed that this together with the boundary condition $u|_{\rho=1} = 0$ gave rise to solutions for P of the form $J_m(\lambda_{mi}\rho)$, where λ_{mi} is the ith zero of J_m. At the end of the last set of lecture notes (June 11 – 13), we gave a brief discussion of the case where $\frac{Z''}{Z} < 0$. We would now like to treat this in greater detail.

Thus suppose that $\frac{Z''}{Z} = -\mu^2$, where we may assume $\mu \geq 0$. This means that $Z(z) = c\cos\mu z + d\sin\mu z$ for some constants c and d, and that P satisfies the equation

$$P'' + \frac{1}{\rho}P' - \left(\mu^2 + \frac{m^2}{\rho^2}\right)P = 0.$$

We see that this is formally the same as the equation satisfied by $J_m(\lambda\rho)$, but with $\lambda = i\mu$. This suggests that a solution to this equation which is well-behaved at 0 is

$$P(\rho) = J_m(i\mu\rho).$$

However, we have so far only defined J_m for real values of the independent variable, so it is not clear a priori what this expression should mean. Recall though that we defined J_m via the power series

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m},$$

which converges for all real x. It can be shown that this power series also converges for all complex x also, and thus we define $J_m(x)$ for any complex number x to be equal to the sum of the above power series. (This is analogous to how we used the power series expansion $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ to define e^x when x is a complex number; in the case $x = i\theta$, that gives rise to the formula $e^{i\theta} = \cos\theta + i\sin\theta$, cf. the review sheet on complex numbers.) Thus the solution above is

$$\begin{aligned} P(\rho) = J_m(i\mu\rho) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{i\mu\rho}{2}\right)^{2k+m} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} i^{2k+m}\left(\frac{\mu\rho}{2}\right)^{2k+m} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(-1)^k i^m\left(\frac{\mu\rho}{2}\right)^{2k+m} \\ &= i^m \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}\left(\frac{\mu\rho}{2}\right)^{2k+m}. \end{aligned}$$

Since it is convenient to have functions of a real variable take real values, we drop the factor of i^m and define the *modified Bessel function of degree* m to be

$$I_m(x) = i^{-m} J_m(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}.$$

It is useful to note the similarity between the pair $J_m(x)$, $I_m(x)$ and the pair $\sin x$, $\sinh x$; see the notes for June 11 – 13, p. 10 for discussion.

Given the foregoing, then, we see that the general separated solution to Laplace's equation on a cylinder (well-behaved at $\rho = 0$) in the case where $\frac{Z''}{Z} = -\mu^2$ is given by

$$I_m(\mu\rho)(a\cos m\phi + b\sin m\phi)(c\cos \mu z + d\sin \mu z). \tag{1}$$

We now face the problem of determining which values for μ are appropriate. Recall that when dealing with the case $\frac{Z''}{Z} = \lambda^2 > 0$, the values for λ were determined by the boundary condition $u|_{\rho=a} = 0$, which forced $J_m(\lambda a) = 0$, which meant that $\lambda a = \lambda_{mi}$ for some i (where λ_{mi}, again, is the ith zero of J_m), or $\lambda = \frac{\lambda_{mi}}{a}$. This suggests that in the present case μ should be determined by a boundary condition in z.[1] We now give an example to show which kinds of boundary-value problems make use of separated solutions of the foregoing type.

EXAMPLE. Solve on $\{(\rho, \phi, z) | \rho < 1, \ \ 0 < z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u_{\rho=1} = 1.$$

Since we have the conditions $u|_{z=0} = u|_{z=1} = 0$, we see that the solution must be oscillatory in the z-direction, so that we must use the above form of separated solution, i.e., our general solution will be a series in solutions of the type in equation (1). Applying the z boundary conditions $u|_{z=0} = u|_{z=1} = 0$ gives $c = 0$, $\sin \mu = 0$, so $\mu = n\pi$, where $n \in \mathbf{Z}$ and we may take $n > 0$ (this is exactly the same as what we did to implement the boundary conditions $u|_{x=0} = u|_{x=1} = 0$ when we solved Laplace's equation in rectangular coordinates earlier on in the course). Thus the general solution to Laplace's equation on the above region which satisfies the first two boundary conditions above will be (absorbing d into a and b)

$$u = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} I_m(n\pi\rho)(a_{nm} \cos m\phi + b_{nm} \sin m\phi) \sin n\pi z.$$

We note that $\{\cos m\phi \sin n\pi z, \sin m\phi \sin n\pi z | n, m \in \mathbf{Z}, m \geq 0, n > 0\}$ is a complete orthogonal set on $\{(\phi, z) | \phi \in [0, 2\pi], z \in [0, 1]\}$ with respect to the inner product

$$(f(\phi, z), g(\phi, z)) = \int_0^{2\pi} \int_0^1 f(\phi, z)\overline{g(\phi, z)}\, dz\, d\phi;$$

this can be shewn exactly as was done for the set $\{P_{\ell m} \cos m\phi, P_{\ell m} \sin m\phi | m, \ell \in \mathbf{Z}, m \geq 0, \ell \geq m\}$ previously (by first expanding in ϕ, obtaining z-dependent coefficients, and then expanding each of these coefficients in a series in $\sin n\pi z$, for example). The relevant normalisation integrals are

$$(\cos m\phi \sin n\pi z, \cos m\phi \sin n\pi z) = \int_0^{2\pi} \cos^2 m\phi\, d\phi \int_0^1 \sin^2 n\pi z\, dz = \frac{\pi}{2},$$
$$(\sin m\phi \sin n\pi z, \sin m\phi \sin n\pi z) = \int_0^{2\pi} \sin^2 m\phi\, d\phi \int_0^1 \sin^2 n\pi z\, dz = \frac{\pi}{2}.$$

[1]Note that this is in accordance with how we have determined separation constants so far: they are determined by boundary conditions in the *oscillatory* directions, not in the nonoscillatory ones.

We now need only to determine a_{nm} and b_{nm} by implementing the final boundary condition $u|_{\rho=1} = 1$. This gives

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m(n\pi)(a_{nm}\cos m\phi + b_{nm}\sin m\phi)\sin n\pi z = 1;$$

by our general results on expansions in complete orthogonal sets, we may write

$$\begin{aligned} a_{nm}I_m(n\pi) &= \frac{(1, \cos m\phi\sin n\pi z)}{(\cos m\phi\sin n\pi z, \cos m\phi\sin n\pi z)} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \cos m\phi\sin n\pi z\, dz\, d\phi \\ &= \frac{2}{\pi}\int_0^{2\pi}\cos m\phi\, d\phi\int_0^1 \sin n\pi z\, dz = \begin{cases} \frac{2}{n\pi}(1-(-1)^n), & m = 0 \\ 0, & m \neq 0 \end{cases}, \\ b_{nm}I_m(n\pi) &= \frac{(1, \sin m\phi\sin n\pi z)}{(\sin m\phi\sin n\pi z, \sin m\phi\sin n\pi z)} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \sin m\phi\sin n\pi z\, dz\, d\phi = 0, \end{aligned}$$

where we have used orthogonality of the set $\{\cos m\phi, \sin m\phi | m \in \mathbf{Z}, m \geq 0\}$ together with the fact that $\cos 0 \cdot \phi = \cos 0 = 1$ and the integral $\int_0^1 \sin n\pi z\, dz = \frac{1}{n\pi}(1-(-1)^n)$. Thus our final solution is given by (noting that $1-(-1)^n = 0$, n even, 2, n odd)

$$u = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{I_0((2k+1)\pi\rho)}{I_0((2k+1)\pi)}\sin(2k+1)\pi z.$$

The above method can clearly be used with any problem of the form

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u|_{\rho=1} = f(\phi, z),$$

for suitably well-behaved functions $f(\phi, z)$. Should we be working on a cylinder like $\{(\rho, \phi, z)|\rho < a, 0 < z < b\}$, the only difference would be that we would take $\mu = \frac{n\pi}{b}$ instead of $\mu = n\pi$. The a factor would only show up in the coefficients, not in the separation constants (just as, when we solved problems with $u|_{\rho=1} = 0$, the length of the cylinder did not show up in the separation constants, only the radius). We now consider how to treat still more general problems.

GENERAL BOUNDARY VALUE PROBLEMS ON A CYLINDER. We shall proceed by means of an example.

EXAMPLE. Solve on $\{(\rho, \phi, z)|\rho < 2, 0 < z < 3\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = 0, \quad u|_{z=3} = \rho^2\cos 2\phi, \quad u|_{\rho=2} = z\phi.$$

This problem does not look quite exactly like anything we have encountered before. By this point we have had a great deal of experience solving problems of the form

$$\nabla^2 u = 0, \quad u|_{z=0} = 0, \quad u|_{z=3} = \rho^2\cos 2\phi, \quad u|_{\rho=2} = 0, \tag{2}$$

and in the previous example we saw how to solve problems like

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=3} = 0, \quad u|_{\rho=2} = z\phi, \tag{3}$$

but the current problem is not of either of these forms: actually it looks rather like a mix of the two! It turns out that this is exactly the key to solving it, too: since the equation $\nabla^2 u = 0$ is linear and homogeneous, the sum of any two solutions is still a solution; thus if we let u_1 denote the solution to problem (2) and u_2 the solution to problem (3), then $u = u_1 + u_2$ will still solve $\nabla^2 u = 0$, and a moment's thought shows that it satisfies all of the boundary conditions of the original problem.

[We pause to note that this is a very general technique. As we have had occasion to note multiple times, when solving Laplace's equation we must have at least one direction which is not oscillatory. But nonoscillatory functions do not form complete orthogonal sets, so this means that there will be at least one part of the boundary on which we cannot specify arbitrary boundary data (and must in general have homogeneous boundary data). We can solve general problems with nonhomogeneous boundary data on all boundaries using the above method: split the problem up into multiple (in three dimensions we never need more than 3) subproblems, each of which has nonhomogeneous data on at most one set of boundaries; if this is done correctly, so that the nonhomogeneous data do not add on top of each other when the solutions are added, the sum of the solution to each subproblem will be the solution to the original problem, just as here.]

Let us consider first problem (2):

$$\nabla^2 u_1 = 0, \quad u_1|_{z=0} = 0, \quad u_1|_{z=3} = \rho^2 \cos 2\phi, \quad u|_{\rho=2} = 0.$$

We see that the general solution satisfying the third boundary condition will be of the form

$$u_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{\lambda_{mn}}{2} \rho \right) (a_{mn} \cos m\phi + b_{mn} \sin m\phi) \left(c_{mn} \cosh \frac{\lambda_{mn}}{2} z + d_{mn} \sinh \frac{\lambda_{mn}}{2} z \right).$$

Before proceeding we pause to indicate another way of writing out this sum which is more convenient in cases where we have inhomogeneous data on both ends of the cylinder (here, where we have $u_1|_{z=0} = 0$, it does not make that much difference). This comes from noting that sometimes it can be hard or even impossible to determine the individual quantities a_{mn}, b_{mn}, etc.: what we obtain naturally are various *products* of these quantities, e.g., $a_{mn}c_{mn}$, etc.. (This impossibility of determining the individual factors in these products is the reason why we constantly speak of 'absorbing' (e.g.) d_{mn} into a_{mn} and b_{mn}, etc..) However, a moment's thought shows that we actually *don't care* about the individual quantities either: the only things that matter for the solution are exactly the products $a_{mn}c_{mn}$, etc., which we are able to calculate. Thus it makes sense to get rid of unknowable and irrelevant quantities and write out the sum only in terms of knowable and relevant ones. Further, since we typically think of expanding in ϕ first, it makes sense to write the series in such a way that the $\cos \phi$ terms and $\sin \phi$ terms are clearly separated. Thus instead of the above form, we consider the alternate form

$$\begin{aligned} u_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{\lambda_{mn}}{2} \rho \right) \Bigg[&\left(\alpha_{mn} \cosh \frac{\lambda_{mn}}{2} z + \beta_{mn} \sinh \frac{\lambda_{mn}}{2} z \right) \cos m\phi \\ &+ \left(\gamma_{mn} \cosh \frac{\lambda_{mn}}{2} z + \delta_{mn} \sinh \frac{\lambda_{mn}}{2} z \right) \sin m\phi \Bigg]. \end{aligned}$$

This is exactly equivalent to the above form, with the definitions

$$\alpha_{mn} = a_{mn}c_{mn}, \quad \beta_{mn} = a_{mn}d_{mn}, \quad \gamma_{mn} = b_{mn}c_{mn}, \quad \delta_{mn} = b_{mn}d_{mn},$$

and moreover it is exactly these four quantities which can be determined uniquely in terms of the boundary data.

With this expression in hand, we may now determine the coefficients from the boundary data, as follows (recall that the normalisation for $J_m \left(\frac{\lambda_{mn}}{2} \rho \right)$ is $\left(J_m \left(\frac{\lambda_{mn}}{2} \rho \right), J_m \left(\frac{\lambda_{mn}}{2} \rho \right) \right) = \frac{1}{2} 2^2 J_{m+1}^2 (\lambda_{mn}) = 2 J_{m+1}^2 (\lambda_{mn})$):

$$\begin{aligned} 0 = u_1|_{z=0} &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\frac{\lambda_{mn}}{2} \rho \right) [\alpha_{mn} \cos m\phi + \gamma_{mn} \sin m\phi], \\ \alpha_{mn} &= \frac{1}{2\pi J_{m+1}^2 (\lambda_{mn})} \left(0, J_m \left(\frac{\lambda_{mn}}{2} \rho \right) \cos m\phi \right) = 0, \\ \gamma_{mn} &= \frac{1}{2\pi J_{m+1}^2 (\lambda_{mn})} \left(0, J_m \left(\frac{\lambda_{mn}}{2} \rho \right) \sin m\phi \right) = 0, \end{aligned}$$

a result we could also have obtained by inspection (though it is important to remember the logic that goes behind it). The other boundary condition then gives

$$0 = u_1|_{z=1} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\left[\beta_{mn}\sinh\frac{\lambda_{mn}}{2}\cos m\phi + \delta_{mn}\sinh\frac{\lambda_{mn}}{2}\sin m\phi\right],$$

$$\begin{aligned}\beta_{mn}\sinh\frac{\lambda_{mn}}{2} &= \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}\left(\rho^2\cos 2\phi, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\cos m\phi\right) = \begin{cases}\frac{\left(\rho^2, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\right)}{2J_{m+1}^2(\lambda_{mn})}, & m = 2,\\ 0, & m \neq 2\end{cases},\\ \delta_{mn}\sinh\frac{\lambda_{mn}}{2} &= \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}\left(\rho^2\cos 2\phi, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\sin m\phi\right) = 0,\end{aligned}$$

where we have used orthogonality of the set $\{\cos m\phi, \sin m\phi\}$. Now we may calculate further (making the change of variables $x = \frac{\lambda_{2n}}{2}\rho$)

$$\begin{aligned}\left(\rho^2, J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\right) &= \int_0^2 \rho^2 J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\rho\,d\rho = \frac{16}{\lambda_{2n}^4}\int_0^{\lambda_{2n}} x^3 J_2(x)\,dx = \frac{16}{\lambda_{2n}^4}x^3 J_3(x)\Big|_0^{\lambda_{2n}}\\ &= \frac{16 J_3(\lambda_{2n})}{\lambda_{2n}},\end{aligned}$$

whence we have

$$\begin{aligned}\beta_{2n}\sinh\frac{\lambda_{2n}}{2} &= \frac{8}{\lambda_{2n}J_3(\lambda_{2n})},\\ \beta_{2n} &= \frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})},\end{aligned}$$

and finally

$$u_1 = \sum_{n=1}^{\infty}\frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})}J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\sinh\frac{\lambda_{2n}}{2}z\cos 2\phi.$$

We now turn to problem (3). In this case, as shewn in the previous example, the general solution satisfying the first two boundary conditions will be of the form

$$u_2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi\rho)\left(a_{mn}\cos m\phi + b_{mn}\sin m\phi\right)\sin\frac{n\pi}{3}z.$$

The final boundary condition gives

$$z\phi = u_2|_{\rho=2} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(2n\pi)\left(a_{mn}\cos m\phi + b_{mn}\sin m\phi\right)\sin\frac{n\pi}{3}z.$$

As before, we may calculate the coefficients a_{mn} and b_{mn} using our general formula for coefficients in orthogonal expansions, viz. (assuming for the moment that $m > 0$) –

$$\begin{aligned}a_{mn}I_m(2n\pi) &= \frac{2}{3\pi}\left(z\phi, \cos m\phi\sin\frac{n\pi}{3}z\right) = \frac{2}{3\pi}\int_0^{2\pi}\int_0^3 z\phi\cos m\phi\sin\frac{n\pi}{3}z\,dz\,d\phi\\ &= \frac{2}{3\pi}\int_0^{2\pi}\phi\cos m\phi\,d\phi\int_0^3 z\sin\frac{n\pi}{3}z\,dz\\ &= \frac{2}{3\pi}\left(\frac{1}{m}\phi\sin m\phi + \frac{1}{m^2}\cos m\phi\right)\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3 = 0,\\ b_{mn}I_m(2n\pi) &= \frac{2}{3\pi}\left(z\phi, \sin m\phi\sin\frac{n\pi}{3}z\right) = \frac{2}{3\pi}\int_0^{2\pi}\int_0^3 z\phi\sin m\phi\sin\frac{n\pi}{3}z\,dz\,d\phi\\ &= \frac{2}{3\pi}\int_0^{2\pi}\phi\sin m\phi\,d\phi\int_0^3 z\sin\frac{n\pi}{3}z\,dz\\ &= \frac{2}{3\pi}\left(-\frac{1}{m}\phi\cos m\phi + \frac{1}{m^2}\sin m\phi\right)\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3 = (-1)^n\frac{36}{\pi mn},\end{aligned}$$

while for $m = 0$ we have $b_{0n} = 0$ by definition and

$$\begin{aligned} a_{0n} I_0(2n\pi) &= \frac{1}{3\pi}\left(z\phi, \sin\frac{n\pi}{3}z\right) = \frac{1}{3\pi}\,\frac{1}{2}\phi^2\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3 \\ &= \frac{4\pi}{3}(-1)^{n+1}\frac{9}{n\pi} = (-1)^{n+1}\frac{12}{n}. \end{aligned}$$

This gives finally

$$\begin{aligned} a_{0n} &= (-1)^{n+1}\frac{12}{nI_0(2n\pi)}, \quad a_{mn} = 0, \;\; m \neq 0, \\ b_{0n} &= 0, \quad b_{mn} = (-1)^n\frac{36}{\pi mn I_m(2n\pi)}, \;\; m \neq 0, \end{aligned}$$

and hence the solution

$$u_2 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{12}{nI_0(2n\pi)}I_0(n\pi\rho)\sin\frac{n\pi}{3}z + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\frac{36}{\pi mn I_m(2n\pi)}I_m(n\pi\rho)\sin m\phi \sin\frac{n\pi}{3}z.$$

Thus we obtain as the final solution to our original problem

$$\begin{aligned} u = \sum_{n=1}^{\infty}\frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})}J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\sinh\frac{\lambda_{2n}}{2}z\cos 2\phi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{12}{nI_0(2n\pi)}I_0(n\pi\rho)\sin\frac{n\pi}{3}z \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\frac{36}{\pi mn I_m(2n\pi)}I_m(n\pi\rho)\sin m\phi\sin\frac{n\pi}{3}z. \end{aligned}$$

LAPLACE'S EQUATION IN THREE-DIMENSIONAL RECTANGULAR COORDINATES. In three-dimensional rectangular coordinates, Laplace's equation has the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We attempt to solve this by the method of separation of variables. Thus we look for solutions of the form $u = X(x)Y(y)Z(z)$; substituting in and dividing by u, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0. \tag{4}$$

By standard arguments ($\frac{X''}{X}$ depends only on x, and nothing else on the left-hand side depends on x, and analogously for the remaining terms) we have that there must be constants μ_1, μ_2, μ_3 such that

$$X'' = \mu_1 X, \quad Y'' = \mu_2 Y, \quad Z'' = \mu_3 Z.$$

Note that we have not yet attempted to determine the signs of these constants. Substituting in to equation (4), we have

$$\mu_1 + \mu_2 + \mu_3 = 0.$$

Thus we see that at least one of μ_1, μ_2, μ_3 must be positive and at least one must be negative. (We ignore for the moment the case where all of them are zero.) Which are positive and which are negative depends on the type of problem we wish to solve. We shall indicate the general method for determining this by means of a specific example.

EXAMPLE. Solve on $\{(x, y, z)|x, y, z \in [0, 1]\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{z=0} = u|_{z=1} = 0, \quad u|_{y=0} = 0, \quad u|_{y=1} = 1.$$

We begin by looking for separated solutions of $\nabla^2 u = 0$ which satisfy the homogeneous boundary conditions; thus we look for solutions $X(x)Y(y)Z(z)$ which satisfy $X(0) = X(1) = Z(0) = Z(1) = 0$. Now it can be shewn that any linear combination of sinh and cosh can vanish at at most one point (I should have given the proof a long time ago; it is very simple: if $a\cosh x + b\sinh x = 0$, then letting $\alpha = \frac{1}{2}(a+b)$ and $\beta = \frac{1}{2}(a-b)$, we have

$$\begin{aligned} \alpha e^x + \beta e^{-x} &= 0 \\ \alpha e^{2x} + \beta &= 0 \\ e^{2x} &= -\frac{\beta}{\alpha}, \end{aligned}$$

which has at most one real solution x (and none if $\frac{\beta}{\alpha} > 0$)). Similarly, any linear function can vanish at at most one point. This implies that neither X nor Z can be a linear combination of sinh and cosh, nor can they be linear; since X and Z are either linear combinations of sinh and cosh (when $\mu_i > 0$), or are linear (when $\mu_i = 0$), or are linear combinations of sin and cos (when $\mu_i < 0$), the latter case must obtain. This implies that $\frac{X''}{X}$ and $\frac{Z''}{Z}$ must both be negative, i.e., that $\mu_1 = -\lambda_1^2$, $\mu_3 = -\lambda_3^2$ for some λ_1, $\lambda_3 > 0$. Hence we must have $\mu_2 > 0$, say $\mu_2 = \lambda_2^2$, $\lambda_2 > 0$. The equation $\mu_1 + \mu_2 + \mu_3 = 0$ then gives

$$\lambda_2^2 = \lambda_1^2 + \lambda_3^2.$$

(This illustrates the general procedure for determining when we take $\mu_i > 0$ and when we take $\mu_i < 0$: the μ_i corresponding to coordinates which have homogeneous boundary data at both ends will be negative, while the remaining one will be positive. If we have inhomogeneous data along more than one coordinate direction, we should split the problem up into multiple subproblems as we did in the previous example.)

The general separated solution is thus

$$(a\cos\lambda_1 x + b\sin\lambda_1 x)(c\cos\lambda_3 z + d\sin\lambda_3 z)(e\cosh\lambda_2 y + f\sinh\lambda_2 y).$$

Now $X(0) = X(1) = 0$ implies that $a = 0$, $\lambda_1 = n\pi$, exactly as we found when we solved Laplace's equation on a rectangle; similarly, now, $Z(0) = Z(1) = 0$ implies that $c = 0$, $\lambda_3 = m\pi$. Thus the general separated solution satisfying the first four boundary conditions is

$$\sin n\pi x \sin m\pi z \left(e\cosh y\pi\sqrt{n^2+m^2} + f\sinh y\pi\sqrt{n^2+m^2} \right),$$

and the general solution will be a series in these solutions, i.e.,

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin n\pi x \sin m\pi z \left(a_{nm}\cosh y\pi\sqrt{n^2+m^2} + b_{nm}\sinh y\pi\sqrt{n^2+m^2} \right).$$

The boundary conditions in y now give

$$0 = u|_{y=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin n\pi x \sin m\pi z (a_{nm}),$$

whence we see that (since, similarly to what we mentioned in the first example above, $\{\sin n\pi x \sin m\pi z | n, m \in \mathbf{Z}, n, m > 0\}$ is a complete orthogonal set on $[0,1]\times[0,1]$ with respect to the standard inner product, with normalisation constant $(\sin n\pi x \sin m\pi z, \sin n\pi x \sin m\pi z) = \frac{1}{4}$)

$$a_{nm} = 4(0, \sin n\pi x \sin m\pi z) = 0.$$

(We could have implemented this condition at the level of the separated solutions, and written our original series for u without the cosh term; we have proceeded this way in order to emphasise that when the boundary data on one side of the cube are inhomogeneous, the direction perpendicular to that side (here, y) should be treated differently than the other sides. In particular, the full procedure as illustrated here would allow us to also treat the case where the boundary data at $y = 0$ were not homogeneous, and this could not in general be implemented at the level of the separated solution.) Similarly, the other boundary condition gives

$$1 = u|_{y=1} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin n\pi x \sin m\pi z \sinh \pi\sqrt{n^2 + m^2},$$

whence we obtain

$$\begin{aligned} b_{nm} \sinh \pi\sqrt{n^2+m^2} &= 4(1, \sin n\pi x \sin m\pi z) = 4\int_0^1\int_0^1 \sin n\pi x \sin m\pi z\, dx\, dz \\ &= 4\int_0^1 \sin n\pi x\, dx \int_0^1 \sin m\pi z\, dz = \frac{4}{nm}(1-(-1)^n)(1-(-1)^m), \end{aligned}$$

which is 0 if either of n or m is even and $\frac{16}{nm}$ when both are odd. Thus we have

$$b_{2k+1,2\ell+1} = \frac{16}{(2k+1)(2\ell+1)\sinh \pi\sqrt{(2k+1)^2+(2\ell+1)^2}}$$

and finally the solution

$$u = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{16}{(2k+1)(2\ell+1)\sinh \pi\sqrt{(2k+1)^2+(2\ell+1)^2}} \sin{(2k+1)\pi x} \sin{(2\ell+1)\pi z} \sinh y\pi\sqrt{(2k+1)^2+(2\ell+1)^2}.$$

(The example I did in class was actually much simpler than this, involving just a single separated solution as the final answer. I didn't realise I was doing a different problem until I was almost finished typing it up though – and anyway it doesn't hurt to see another (and more complicated!) example.)

EIGENFUNCTIONS OF THE LAPLACIAN. The next topics which we wish to treat are Green's functions, the heat equation, and the wave equation (though we may take some time off to talk about Fourier transforms at some point). The study of all of these, especially of the first two, benefit from a knowledge of the eigenfunctions of the Laplacian, so we now turn to that topic. First we give an example from linear algebra as motivation. (See also the examples we gave related to diagonalisation in the first week or two of the course.)

EXAMPLE. Let A be an $n \times n$ matrix, and x and y be column vectors of length n. Consider the equation $Ax = y$. If we know the inverse matrix A^{-1}, then we can solve this by writing $x = A^{-1}y$. In general, though, finding the inverse of a matrix is hard. If A were diagonal, though, it would be easy, since the inverse of a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

is

$$D = \begin{bmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}.$$

More abstractly, suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ were a basis of eigenvectors for the matrix A; suppose also that A is symmetric, so that this set can be taken to be orthogonal.

[This can be shewn in an analogous fashion to how we showed that the Legendre polynomials and Bessel functions formed orthogonal sets. For simplicity we work with the standard real inner product. Symmetry of A means that for any vectors v and w, we have

$$(v, Aw) = \sum_{i=1}^{n} v_i (Aw)_i = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i A_{ij} w_j = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i A_{ji} w_j = (Av, w),$$

so if $\mathbf{e}_i$ and $\mathbf{e}_j$ are eigenvectors corresponding to distinct eigenvalues, say λ_i and λ_j, then we may write

$$(\mathbf{e}_i, A\mathbf{e}_j) = \lambda_j(\mathbf{e}_i, \mathbf{e}_j) = (A\mathbf{e}_i, \mathbf{e}_j) = \lambda_i(\mathbf{e}_i, \mathbf{e}_j),$$

so $(\lambda_j - \lambda_i)(\mathbf{e}_i, \mathbf{e}_j) = 0$ and $(\mathbf{e}_i, \mathbf{e}_j) = 0$ since $\lambda_i \neq \lambda_j$. (In the event that $\mathbf{e}_i$ and $\mathbf{e}_j$ belong to the same eigenvalue, they can be taken orthogonal by applying the Graham-Schmidt process if needed.)]

Then we can write

$$y = \sum_{i=1}^{n} \frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}\mathbf{e}_i,$$
$$x = \sum_{i=1}^{n} x_i\mathbf{e}_i,$$

whence the system $Ax = y$ becomes

$$\sum_{i=1}^{n} \lambda_i x_i \mathbf{e}_i = \sum_{i=1}^{n} \frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}\mathbf{e}_i.$$

Since $\{\mathbf{e}_i\}$ is a basis, this implies that $\lambda_i x_i = \frac{(y,\mathbf{e}_i)}{(\mathbf{e}_i,\mathbf{e}_i)}$, so

$$x_i = \frac{1}{\lambda_i}\frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)},$$
$$x = \sum_{i=1}^{n} \frac{1}{\lambda_i}\frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}.$$

Note that this procedure did not require us to invert any matrix; in fact, the computations involved were nothing more than the taking of inner products and multiplication and division. (Finding the eigenvectors of A, of course, is highly nontrivial, so this method is not necessarily any faster overall at solving a single system.)

The idea behind this example may be applied to, among other things, the study of a generalisation of Laplace's equation called *Poisson's equation*. So far we have only studied the homogeneous equation $\nabla^2 u = 0$; however, there are many cases (such as, for example, when one has a source of heat inside a region and wishes to find the equilibrium temperature distribution, or when one has a nonzero charge density inside a region and wishes to find the electrostatic potential) when one wishes to solve an equation of the form $\nabla^2 u = f$ for some function f. Generally one still has boundary conditions which u is also required to satisfy. Suppose now that there were a complete orthogonal set of (nonzero) functions $\{e_i\}$, where i is an abstract index, such that $\nabla^2 e_i = \Lambda_i e_i$, and such that each e_i satisfied the relevant boundary conditions. Then we would be able to expand the function f as

$$f = \sum_i \frac{(f, e_i)}{(e_i, e_i)}e_i,$$

and also any potential solution u as

$$u = \sum_i u_i e_i.$$

Substituting both of these into the equation $\nabla^2 u = f$, we obtain

$$\sum_i \Lambda_i u_i e_i = \sum_i \frac{(f, e_i)}{(e_i, e_i)}e_i;$$

since the set $\{e_i\}$ is orthogonal and does not contain 0, this implies that for each i

$$\Lambda_i u_i = \frac{(f, e_i)}{(e_i, e_i)}. \tag{5}$$

If $\Lambda_i \neq 0$ for all i, then we may solve this for u_i and then substitute in to the expansion $u = \sum_i u_i e_i$ to obtain a series expansion for the solution u to Poisson's equation in the functions e_i, much as we have been doing for solutions to Laplace's equation (though the eigenfunctions e_i may well be different from the orthogonal bases we have used so far). If $\Lambda_i = 0$ for some i then things are more complicated. From equation (5) it is evident that in this case there can be no solution (at least, not one expressible as a series in the $\{e_i\}$) if $(f, e_i) \neq 0$. If, however, we happen to have $(f, e_i) = 0$ whenever $\Lambda_i = 0$, then clearly there will still be a solution; though it is not necessarily unique, since the u_i will not be determined by equation (5). We may obtain a unique solution by requiring $u_i = 0$ for such i. Thus we see that the equation $\nabla^2 u = f$ will have a unique solution if we restrict both f and u to lie in the space of functions which are orthogonal to all eigenfunctions of ∇^2 with zero eigenvalues. We shall probably have more to say on this point later.

Let us assume for the moment, for simplicity, that none of the eigenvalues are zero (or that we have restricted f and u as just indicated, and then restricted i to run over the eigenfunctions corresponding to nonzero eigenvalues). Then we may write the solution u as

$$u = \sum_i \frac{1}{\Lambda_i} \frac{(f, e_i)}{(e_i, e_i)} e_i;$$

now if our inner product (f, e_i) were given by an integral, say (writing things schematically for generality) $(f, e_i) = \int_D f \overline{e_i}\, dx'$, then we may express this equation as follows (formally interchanging summation and integration):

$$u(x) = \sum_i e_i(x) \frac{1}{\Lambda_i (e_i, e_i)} \int_D f(x') \overline{e_i(x')}\, dx' = \int_D \left(\sum_i \frac{e_i(x) \overline{e_i(x')}}{(e_i, e_i)} \frac{1}{\Lambda_i} \right) f(x')\, dx'.$$

A function $G(x, x')$ such as that in the parentheses above is called a *Green's function* for the given boundary-value problem. We shall study such functions systematically starting next week. The above expression gives (at least formally) the Green's function in terms of the eigenfunctions and eigenvalues of the Laplacian for the given boundary conditions.

[The formula above has a formal analogue in linear algebra as well. We may write the formula as

$$u(x) = \int_D G(x, x') f(x')\, dx'.$$

Now the solution to an equation $Ax = y$ can be written as

$$x_i = \sum_j A^{-1}_{ij} y_j;$$

if we think of i as corresponding to x, j as corresponding to x', and $\sum$ as corresponding to $\int$, then we see that in some sense G corresponds to $\left(\nabla^2\right)^{-1}$; i.e., the integral operator given above involving G is an 'inverse' to the Laplacian.]

Another place where the eigenfunctions of ∇^2 are useful is in studying the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$. Suppose that we are interested in studying this equation subject to certain boundary conditions on u (which are constant in time), and suppose that we have a complete orthogonal set of eigenfunctions $\{e_i\}$ for the Laplacian ∇^2 subject to these boundary conditions. Then we could write for each time t, as before,

$$u(t, \mathbf{x}) = \sum_i u_i(t) e_i(\mathbf{x}),$$

and substituting this into the heat equation gives

$$\sum_i u_i'(t) e_i = \sum_i \Lambda_i u_i(t) e_i,$$

whence we have $u_i'(t) = \Lambda_i u_i(t)$, i.e., the system completely decouples, exactly as we discussed in the first couple weeks of class. This last equation has solution $u_i(t) = u_{i,0} e^{\Lambda_i t}$, and thus our solution u is

$$u = \sum_i u_{i,0} e^{\Lambda_i t} e_i,$$

where the constants $u_{i,0}$ are to be determined from the initial condition $u|_{t=0}$, exactly as we determine coefficients in orthogonal expansions for Laplace's equation using boundary conditions. We shall go over all of this in more detail later on in the course.