Summary:

- By separating variables in Laplace's equation in cylindrical coordinates, we derive Bessel's equation, and use it to derive the Taylor series expansion for Bessel functions on nonnegative integer order.
- We then discuss the orthogonality and completeness properties of these functions.
- Finally, we then use these Taylor series expansions to deduce differentiation and recursion relations for the Bessel functions of nonnegative integer order, and say a few words about modified Bessel functions.

SEPARATION OF VARIABLES IN CYLINDRICAL COORDINATES. Recall (see the lecture notes for the week of May 23) that the Laplacian in cylindrical coordinates (ρ, ϕ, z) (which is related to Cartesian coordinates (x, y, z) by $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$) is given by

$$\nabla^2 f(\rho, \phi, z) = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

We are interested (as usual) in solving the equation $\nabla^2 u = 0$, on a region possessing cylindrical symmetry, subject to certain conditions imposed on the boundary of that region. As before, we shall proceed by looking first for separated solutions $u = P(\rho)\Phi(\phi)Z(z)$,[1] and then investigating whether the full solution can be expressed as a series of solutions of this type.

Substituting this *ansatz* into Laplace's equation and dividing by u as usual, we obtain the equation

$$\frac{1}{P}\frac{d^2 P}{d\rho^2} + \frac{1}{\rho}\frac{1}{P}\frac{dP}{d\rho} + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0. \tag{1}$$

We note that the last term depends only on Z, and is moreover the only term on the left-hand side dependant on Z, and must therefore be constant. Similarly, the term $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ is the only term dependant on ϕ and must therefore also be constant. To proceed further, we must (as for the case of spherical coordinates) know something more about the region over which we wish to solve Laplace's equation. Let us suppose that we are interested in solving over a region which involves a full range of the angular variable ϕ (for example, a cylinder $\{(\rho, \phi, z) | \rho < a, b \le z \le c, 0 \le \phi \le 2\pi\}$). Then, just as for spherical coordinates, u and therefore Φ must be periodic in ϕ with period 2π. Now Φ satisfies the equation $\frac{d^2\Phi}{d\phi^2} = \mu\Phi$ for some constant μ; requiring Φ to be periodic forces $\mu \le 0$, say $\mu = -m^2$, giving $\Phi = a\cos m\phi + b\sin m\phi$ for some a and b; further requiring the period to be 2π gives $m \in \mathbf{Z}$. We may take $m \ge 0$ without loss of generality.[2]

The treatment of the constant corresponding to Z is more involved. To provide some context, we first recall our treatment of Laplace's equation on a square. Recall that in that case separated solutions of the form $u = X(x)Y(y)$ satisfied the equation

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0,$$

from which it is easy to see that both terms must be constant, meaning that we must have $X'' = \mu X$, $Y'' = -\mu Y$. The question then arises as to whether μ should be positive or negative (or 0, but we shall not consider that case here). Clearly, $\mu > 0$, say $\mu = m^2$, implies that we have the general solutions $Y = a\sin my + b\cos my$, $X = c\sinh mx + b\cosh mx$, i.e., Y will be oscillatory while X will be exponential, while $\mu < 0$, say $\mu = -m^2$, implies the exact opposite: $X = c\sin mx + b\cos mx$, $Y = a\sinh my + b\cosh my$, i.e., X will be oscillatory while Y will be exponential. For the boundary-value problems which we have considered so far, we had conditions like $X(0) = X(1) = 0$, which forced us to choose X to be oscillatory

[1] Here P is the capital form of the Greek letter ρ, *not* the capital form of the English letter p.

[2] Note that if $m = 0$, the general solution for Φ is not $a\cos\phi + b\sin m\phi = a$ but rather $a + b\phi$; since ϕ is not periodic as a function of ϕ, we must have $b = 0$, meaning that the solution is in fact just $\Phi = a$ for some constant a. For notational simplicity we shall write $\Phi = a\cos m\phi + b\sin m\phi$ as the general solution for all m, even $m = 0$, with the implicit understanding that when $m = 0$ we shall always (for definiteness) take $b = 0$ (otherwise b would be undefined in this case). This device can be avoided by considering the complex basis $e^{im\phi}$ instead, but we shall not do that here.

and hence Y to be exponential. Had we had instead conditions like $Y(0) = Y(1) = 0$, we would have been forced to take instead Y to be oscillatory and hence X to be exponential.

It turns out that the same duality holds in the present case.[3] Thus, depending on the given boundary conditions, we may be forced to take Z to be oscillatory, in the which case P will be non-oscillatory; or we may be forced to take P to be oscillatory, in the which case Z will be non-oscillatory. (In general, we will have a sum of solutions, one of each type.) Without prejudicing the final result, then, let us write for the moment

$$\frac{d^2Z}{dz^2} = \epsilon\lambda^2 Z,$$

where $\lambda \in \mathbf{R}$, $\lambda \geq 0$, and $\epsilon = \pm 1$.[4] Substituting this and the equation for Φ into equation (1) above, we obtain for P the equation

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(\epsilon\lambda^2 - \frac{m^2}{\rho^2}\right) P = 0. \tag{2}$$

If $\lambda = 0$ this equation has the general solution (much as for the r-dependent factor in separated solutions in spherical coordinates) $P = ar^{\alpha_+} + br^{\alpha_-}$, where $\alpha_\pm$ are the solutions to $\alpha(\alpha+2) = m^2$. In this case, we have also $Z = c + dz$, whence we obtain the general separated solution

$$u = \left(ar^{\alpha_+} + br^{\alpha_-}\right)(c + dz)\left(e\cos m\phi + f\sin m\phi\right).$$

Suppose now that $\lambda > 0$, and define a new function $Q : [0,\infty) \to \mathbf{R}$ by $Q(x) = P\left(\frac{x}{\lambda}\right)$; equivalently, by $P(\rho) = Q(\lambda\rho)$. Substituting this into equation (2) above for P gives

$$\begin{aligned} 0 &= \lambda^2 Q''(\lambda\rho) + \frac{\lambda}{\rho}Q'(\lambda\rho) + \left(\epsilon\lambda^2 - \frac{m^2}{\rho^2}\right) Q(\lambda\rho) \\ &= \lambda^2\left[Q''(\lambda\rho) + \frac{1}{\lambda\rho}Q'(\lambda\rho) + \left(\epsilon - \frac{m^2}{\lambda^2\rho^2}\right) Q(\lambda\rho)\right], \end{aligned}$$

whence we obtain, writing $x = \lambda\rho$,

$$\frac{d^2Q}{dx^2} + \frac{1}{x}\frac{dQ}{dx} + \left(\epsilon - \frac{m^2}{x^2}\right) Q = 0.$$

When $\epsilon = 1$ this is called (see [1], p. 38) *Bessel's equation for functions of order m*. We now restrict to this case for the moment; thus we consider the equation

$$\frac{d^2Q}{dx^2} + \frac{1}{x}\frac{dQ}{dx} + \left(1 - \frac{m^2}{x^2}\right) Q = 0. \tag{3}$$

[3] You may wonder why this did not play so central a part in our treatment of Laplace's equation in spherical coordinates. In spherical coordinates, assuming we solve on a region which covers a full range of ϕ and θ, we have natural boundary conditions on the corresponding factors of the separated solution Φ (identical to that here) and Θ (that it be finite at both $\theta = 0$ and $\theta = \pi$, i.e., at both poles, or equivalently, on the z-axis) which turned out to force both of them to be oscillatory. Thus the only remaining factor, $R(r)$, was forced to be the non-oscillatory (though not, we might note, in this case, exponential). This would not have happened had we solved Laplace's equation on a wedge, say for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ – in that case we would have to consider (in general) both oscillatory and non-oscillatory solutions in the θ direction, which could lead to oscillatory solutions in the r direction. [Note. You may recall that when we discussed the equation for R we had a restriction on the separation constant (namely – see the lecture notes for the week of May 23, p. 8 – $\alpha > \frac{1}{4}$); were this condition not satisfied, the solutions in the r direction could become oscillatory. We shall not pursue this further here.]

[4] The observant reader may note that we could drop ϵ by letting λ be a complex number, with say $\Re\lambda \geq 0$ for definiteness. It turns out that the so-called *modified Bessel functions*, which are the non-oscillatory counterparts of the oscillatory Bessel functions to be derived presently, are obtained from the latter by just this kind of transformation. We shall have more to say about all this below.

We wish to derive a power-series representation for the solutions to this equation. To do this, it is convenient to make another change of variables by setting $Q(x) = x^m q(x)$ for some function q; this gives

$$Q' = mx^{m-1}q + x^m q', \qquad Q'' = m(m-1)x^{m-2}q + 2mx^{m-1}q' + x^m q'',$$

whence, upon substituting into equation (3), we obtain

$$\begin{aligned} 0 &= x^m q'' + 2mx^{m-1}q' + m(m-1)x^{m-2}q + x^{m-1}q' + mx^{m-2}q + x^m q - m^2 x^{m-2}q \\ &= x^m q'' + (2m+1)x^{m-1}q' + \left(m(m-1) + m - m^2\right) x^{m-2}q + x^m q \\ &= x^m \left(q'' + \frac{2m+1}{x}q' + q\right), \end{aligned}$$

whence finally

$$q'' + \frac{2m+1}{x}q' + q = 0. \tag{4}$$

Now suppose that q can be expanded in a Taylor series about $x = 0$ as

$$q = \sum_{n=0}^{\infty} a_n x^n;$$

substituting into equation (4) then gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + (2m+1)na_n x^{n-2} + a_n x^n \\ &= (2m+1)a_1 x^{-1} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (2m+1)na_n x^{n-2} + a_{n-2}x^{n-2} \\ &= (2m+1)a_1 x^{-1} + \sum_{n=0}^{\infty} x^n \left((n+1)(n+2)a_{n+2} + (2m+1)(n+2)a_{n+2} + a_n\right), \end{aligned}$$

since the first two terms in the series on the first line vanish for $n = 0$ and $n = 1$ except for the $(2m+1)a_1 x^{-1}$ term. Since the final series above contains no terms with negative powers of x, the term $(2m+1)a_1 x^{-1}$ must vanish, meaning that (since here m is a nonnegative integer) we must have $a_1 = 0$. The series itself must then vanish, which gives the recurrence relation

$$\begin{aligned} (n+2)(2m+n+2)a_{n+2} + a_n &= 0, \\ a_{n+2} &= -\frac{1}{(n+2)(2m+n+2)}a_n, \\ a_n &= -\frac{1}{n(2m+n)}a_{n-2}, \end{aligned}$$

where in the last line we have simply replace $n+2$ by n everywhere. Since we have $a_1 = 0$ by the foregoing, this recurrence relation implies that $a_n = 0$ for all odd n, so that the power series for q only has even-order terms. Moreover, inspection of the recurrence relation above shows that we have the general formula

$$\begin{aligned} a_{2k} &= \frac{(-1)^k (2m)!!}{(2k)!!(2m+2k)!!}a_0 \\ &= \frac{(-1)^k 2^m m!}{2^k k! 2^{m+k}(m+k)!}a_0 = \frac{(-1)^k m!}{4^k k!(m+k)!}a_0, \end{aligned}$$

where as for odd numbers we define $(2k)!! = (2k)(2k-2)\cdots 4\cdot 2 = 2^k k!$. (This formula may be proved by mathematical induction as follows: when $k=0$ the coefficient above is simply $\frac{(-1)^0 m!}{4^0 0!(m+0)!} = 1$, proving the base case; supposing it holds for $2k-2$, we have

$$\begin{aligned} a_{2k} &= -\frac{1}{(2k)(2m+2k)}\frac{(-1)^{k-1}m!}{4^{k-1}(k-1)!(m+k-1)!}a_0 \\ &= \frac{(-1)^k m!}{4^k k(k-1)!(m+k)(m+k-1)!}a_0 \\ &= \frac{(-1)^k m!}{4^k k!(m+k)!}c_0, \end{aligned}$$

proving that it holds for $2k$ as well, and hence for all indices.) As with our definition of the Legendre polynomials, we are free to define a_0; we set $a_0 = \frac{1}{2^m m!}$, so that

$$\begin{aligned} a_{2k} &= \frac{(-1)^k}{2^{2k+m}k!(m+k)!}, \\ q &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+m}k!(m+k)!}x^{2k}, \\ Q = x^m q &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}. \end{aligned}$$

This function represented by this series is called the *Bessel function of order* m and denoted $J_m(x)$.

Pulling everything back together, then, we see that the solution of equation (2) with $\epsilon = 1$ which has a power series expansion about $x=0$ is given by $J_m(\lambda\rho)$. Now when $\epsilon = 1$ we have for Z the equation

$$\frac{d^2Z}{dz^2} = \lambda^2 Z,$$

which has the general solution $Z = c\cosh\lambda z + d\sinh\lambda z$. Thus the general separated solution to Laplace's equation in this case is

$$u = J_m(\lambda\rho)\left(a\cos m\phi + b\sin m\phi\right)\left(c\cosh\lambda z + d\sinh\lambda z\right).$$

We have already restricted m to be a nonnegative integer, but note that there is as yet no restriction on λ. This is analogous to the situation we were in when solving Laplace's equation in a square in rectangular coordinates: the general solution was in terms of functions $\sin mx$, $\cos mx$, etc., where m was fixed only by the boundary conditions in the x direction. Thus we expect λ to be fixed by the boundary conditions obtaining in ρ. By requiring our solution to be regular at $x=0$, we have already given one boundary condition. Now consider the boundary condition $u|_{\rho=a} = 0$; this gives for λ the equation

$$J_m(a\lambda) = 0.$$

It can be shewn that this equation has an infinite number of solutions. In the case $a=1$, we label them $\lambda_{m,i}$, $i=1,2,\ldots$; in the case of general a, then, the correct values of λ are $\frac{1}{a}\lambda_{m,i}$, $i=1,2,\ldots$. Unfortunately, unlike for sine and cosine, there is no explicit formula for the $\lambda_{m,i}$, so we shall have to be content with just this notation. (It can be shewn – though we shall not do so here – that the zeroes are discrete (meaning that they do not 'pile up', i.e., have no accumulation point), and that as $i\to\infty$ for fixed m, the spacing becomes constant (see [1], p. 506).)

ORTHOGONALITY AND COMPLETENESS. We would now like to know something about the orthogonality and completeness properties of these Bessel functions. We first note one possible point of confusion which has not arisen in any of our previous studies. Legendre polynomials are complete in the sense that any suitably well-behaved function on $[-1, 1]$ can be expanded in a series

$$\sum_{\ell=0}^{\infty} a_\ell P_\ell(x);$$

similarly, associated Legendre functions $P_{\ell m}$ for fixed m are complete in the sense that any suitably well-behaved function on $[-1, 1]$ can be expanded in the analogous series

$$\sum_{\ell=m}^{\infty} a_\ell P_{\ell m}(x).$$

This might lead us to expect that the completeness result for Bessel functions would state that any suitably well-behaved function on some appropriate interval (perhaps their domain of definition, $[0, \infty)$) can be expanded in a series of the form

$$\sum_{m=0}^{\infty} a_m J_m(x)$$

(this is termed a *Neumann series*). It turns out that various results of this form are true (see [1], Chapter XVI). However, some reflection shows that they are not actually relevant for our current setting.[5] Roughly, this is because the index m is already 'used' in some sense by the orthogonal basis $\{\cos m\phi, \sin m\phi\}$. More precisely, we expect that a general solution to the boundary-value problem we are looking at can be expressed in the form

$$u = \sum_i \sum_{m=0}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)(a_{m,i}\cos m\phi + b_{m,i}\sin m\phi)\left(c_{m,i}\cosh\frac{1}{a}\lambda_{m,i}z + d_{m,i}\sinh\frac{1}{a}\lambda_{m,i}z\right).$$

Now since this series by construction satisfies the boundary condition $u|_{\rho=a} = 0$, the only boundary conditions we might have to fit are on surfaces of constant z, say $z = L$. Suppose for the sake of definiteness that we were given the condition $u|_{z=L} = 1$. Then we would need to find an expansion (on the interval $[0, a]$, we should note)

$$1 = \sum_i \sum_{m=0}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)(a'_{m,i}\cos m\phi + b'_{m,i}\sin m\phi)$$

(the constants $a'_{m,i}$ and $b'_{m,i}$ will be related but not identical to the constants $a_{m,i}$ and $b_{m,i}$ in the full expansion). As before, we may think of fixing ρ and using orthogonality of the basis $\{\cos m\phi, \sin m\phi\}$ to determine which m-valeus are present; clearly, we have only $m = 0$. Thus we are left with the expansion problem

$$1 = \sum_i a''_{0,i} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right);$$

in other words, working the expansion out in the ϕ direction gets rid of the index m. (This is analogous to what we did when considering expansions of functions on the sphere in terms of the basis $\{P_{\ell m}(\cos\theta)\cos m\phi$, $P_{\ell m}(\cos\theta)\sin m\phi\}$, whereby we fixed θ and expanded in $\{\cos m\phi, \sin m\phi\}$ to obtain functions $c_m(\theta)$, $d_m(\theta)$, which were then expanded in a series of $P_{\ell m}(\cos\theta)$ with m fixed.) This result also points us in the direction of the correct completeness result for Bessel functions in our current situation; namely, we expect that for any nonnegative integer m, a suitably well-behaved function on the interval $[0, a]$ will have an expansion of the form

$$\sum_{i=1}^{\infty} a_i J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right), \tag{5}$$

[5] This is not to say that they are not useful for solving boundary-value problems – just that they are not needed for the type of boundary-value problem we are investigating at the moment.

where $\{\frac{1}{a}\lambda_{m,i}\}_{i=1}^{\infty}$ is the set of zeroes of $J_m(ax) = 0$. It turns out that this result is correct (though we shall not prove it here); the expansion above is called a *Fourier-Bessel series.* (See [1], Chapter XVIII, especially 18.24.)[6]

Given the correctness of the above series expansion, we can determine the coefficients a_i if we have an appropriate orthogonality result for the functions $J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)$ on the interval $[0, a]$. We shall prove an appropriate orthogonality result in the same way we proved orthogonality for the associated Legendre functions; see the lecture notes for the week of June 4, pp. 7–8. We first rewrite Bessel's equation as (here $P = J_m(\lambda\rho)$)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - \frac{m^2}{\rho^2}P = -\lambda^2 P.$$

Let us denote the operator on the left-hand side by L, meaning that we denote the entire left-hand side by LP; thus Bessel's equation becomes simply the eigenvalue equation for L, $LP = -\lambda^2 P$. We will now show that L is self-adjoint with respect to an appropriate inner product on $[0, a]$. For integrable functions f and g on $[0, a]$, let

$$(f, g) = \int_0^a \rho f(\rho)\overline{g(\rho)}\, d\rho.$$

Now suppose that f and g satisfy the boundary condition $f(a) = 0$, $g(a) = 0$. Then[7]

$$\begin{aligned}(Lf, g) &= \int_0^a \rho\left(\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \frac{m^2}{\rho^2}f\right)\overline{g}\, d\rho \\ &= \int_0^a \frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \frac{m^2}{\rho}f(\rho)\overline{g(\rho)}\, d\rho \\ &= -\int_0^a \rho\frac{df}{d\rho}\overline{\frac{dg}{d\rho}} + \frac{m^2}{\rho}f(\rho)\overline{g(\rho)}\, d\rho \\ &,\end{aligned}$$

where we have performed an integration by parts and used the fact that $g(a) = 0$. Since this expression (up to conjugation, which doesn't matter when f and g are real as they are for us at this point) is clearly symmetric in f and g, we conclude that

$$(Lf, g) = (f, Lg)$$

(alternatively, this can be shewn by performing another integration by parts, as was done when dealing with associated Legendre functions). Now suppose that $f(\rho) = J_m(\frac{1}{a}\lambda_{m,i}\rho)$, $g(\rho) = J_m(\frac{1}{a}\lambda_{m,i'}\rho)$, $i \neq i'$; then the above equation gives (since f and g here clearly satisfy the boundary condition $f(a) = g(a) = 0$)

$$\begin{aligned}(Lf, g) &= -\frac{1}{a^2}\lambda_{m,i}^2(f, g) = (f, Lg) \\ &= -\frac{1}{a^2}\lambda_{m,i'}^2(f, g),\end{aligned}$$

[6]Note that Neumann series and Fourier-Bessel series do not exhaust the possibilities for series expansions in terms of Bessel functions; there are also, for example, *Kapteyn series* and *Schlömilch series* (see [1], Chapters XVII and XIX), but we shall not discuss them here.

[7]We are eliding one subtle point, namely whether the function $\frac{1}{\rho}f(\rho)\overline{g(\rho)}$ is integrable on $[0, a]$. Since we are interested in cases where $f(\rho) = J_m(\lambda\rho)$, $g(\rho) = J_m(\lambda'\rho)$ for some λ, λ', and since J_m has a zero of order m at $\rho = 0$ (i.e., $J_m(\rho) = \rho^m q(\rho)$, where $q(\rho)$ is finite at $\rho = 0$), for us these functions will be integrable when f is as long as $m \neq 0$. But when $m = 0$ this term is not present in L. Thus the calculations below are valid for the cases in which we are interested. [It would be good to see a fuller treatment of this point, but that would be (a) most importantly, outside the expertise of the current author, and (b) probably beyond the scope of the course.]

whence (since $\lambda_{m,i} \neq \lambda_{m,i'}$ as $i \neq i'$) we must have $(f, g) = 0$, showing orthogonality with respect to the given inner product. To calculate the expansion coefficients we need only the normalisation. This is found to be (see [1], 18.24)

$$\int_0^a \rho J_m^2 \left(\frac{1}{a} \lambda_{m,i} \rho \right) \, d\rho = \frac{1}{2} a^2 J_{m+1}^2 \left(\lambda_{m,i} \right).$$

Thus we may finally write, in expansion (5) above,

$$a_i = \frac{\left(f(\rho), J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \right)}{\left(J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right), J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \right)} = \frac{2}{a^2 J_{m+1}^2 \left(\lambda_{m,i} \right)} \int_0^a \rho f(\rho) J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \, d\rho.$$

We now indicate in general how all of this may be used to solve boundary-value problems. Suppose that we are to solve Laplace's equation on the cylinder $\{(\rho, \phi, z) | \rho < a, \quad 0 \leq z \leq b\}$, with the boundary conditions

$$u|_{\rho=a} = 0, \qquad u|_{z=0} = 0, \qquad u|_{z=b} = f(\rho, \phi).$$

The first condition allows us to conclude that the series will be of the form

$$u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \left(a_{m,i} \cos m\phi + b_{m,i} \sin m\phi \right) \left(c_{m,i} \cosh \frac{1}{a} \lambda_{m,i} z + d_{m,i} \sinh \frac{1}{a} \lambda_{m,i} z \right),$$

while the second condition then allows us to conclude (since $\cosh 0 = 1$) that $c_{m,i} = 0$ for all m, i; absorbing $d_{m,i}$ into $a_{m,i}$ and $b_{m,i}$, we are left with the expansion

$$u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \sinh \frac{1}{a} \lambda_{m,i} z \left(a_{m,i} \cos m\phi + b_{m,i} \sin m\phi \right).$$

We may now handle this expansion and the final boundary condition in an analogous way to how we handled the expansion and condition

$$u = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) r^{\ell} \left(a_{\ell,m} \cos m\phi + b_{\ell,m} \sin m\phi \right), \qquad u|_{r=a} = f(\theta, \phi).$$

More specifically, we need to expand $f(\rho, \phi)$ in the basis $\{J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \cos m\phi, J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right) \sin m\phi\}$; we may do this by first fixing some ρ, and then expanding along ϕ to obtain ρ-dependent coefficients

$$a_m(\rho) = \frac{1}{\pi} \int_0^{2\pi} f(\rho, \phi) \cos m\phi \, d\phi, \qquad b_m(\rho) = \frac{1}{\pi} \int_0^{2\pi} f(\rho, \phi) \sin m\phi \, d\phi$$

for $m > 0$, while for $m = 0$ we have $b_0 = 0$ by convention and

$$a_0(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho, \phi) \, d\phi.$$

(This separate formula for a_0 was what the factor $\frac{1}{2}$ on the constant term in the Fourier expansions we saw earlier on in class was meant to solve, but we have not adopted that convention here.) This allows us to write

$$f = \sum_{m=0}^{\infty} a_m(\rho) \cos m\phi + b_m(\rho) \sin m\phi.$$

In order to write this as a series along the lines of that for u above, we must now expand $a_m(\rho)$ and $b_m(\rho)$ in series of $\{J_m \left(\frac{1}{a} \lambda_{m,i} \rho \right)\}$, where now m is fixed and only i varies; this is exactly analogous to how we had to expand the coefficient functions $a_m(\theta)$ and $b_m(\theta)$ in the basis $\{P_{\ell,m}(\cos\theta)\}_{\ell=m}^{\infty}$, with fixed m. This will give expansions

$$a_m(\rho) = \sum_{i=1}^{\infty} a_{mi} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right),$$

and similarly for $b_m(\rho)$. Equating coefficients then allows us to determine u, as usual.
DIFFERENTIATION FORMULAS AND RECURRENCE RELATIONS. In order to calculate the integrals needed to find the coefficients in expansions such as those above, we need results on Bessel functions similar to those we derived for the Legendre polynomials previously. We now take up this question.

PROPOSITION. The Bessel functions satisfy the following four identities:

1. $J_{m-1}(x) - J_{m+1}(x) = 2J_m'(x)$, $m > 0$; $J_0'(x) = -J_1(x)$,

while for $m > 0$ we have

2. $J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$;
3. $J_{m-1}(x) = J_m'(x) + \frac{m}{x} J_m(x)$;
4. $J_{m+1}(x) = -J_m'(x) + \frac{m}{x} J_m(x)$;
5. $\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$;
6. $\frac{d}{dx}\left(x^{-m} J_m(x)\right) = -x^{-m} J_{m+1}(x)$.

Proof. Recall the series expansion

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}.$$

Differentiating this expression term-by-term, we obtain

$$J_m'(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(k+m+k)\left(\frac{x}{2}\right)^{2k+m-1},$$

where we have written $2k + m = k + m + k$. We expand out these two series separately since they will be useful in proving the second identity also. We see that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}k\left(\frac{x}{2}\right)^{2k+m-1} &= -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!(m+1+(k-1))!}\left(\frac{x}{2}\right)^{2(k-1)+m+1} \\ &= -J_{m+1}(x),\end{aligned}$$

where in the second sum we may start at $k = 1$ since the $k = 0$ term in the first sum clearly vanishes. Similarly, we see that, for $m > 0$,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k)\left(\frac{x}{2}\right)^{2k+m-1} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!}\left(\frac{x}{2}\right)^{2k+(m-1)} \\ &= J_{m-1}(x),\end{aligned}$$

while if $m = 0$ then we have as before

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k)\left(\frac{x}{2}\right)^{2k+m-1} = -J_1(x).$$

Thus we have, in particular, for $m > 0$,

$$J_m'(x) = \frac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right),$$

while for $m = 0$

$$J_0'(x) = -J_1(x).$$

This proves the first identity. For the second identity, note that, by the foregoing,

$$\begin{aligned} J_{m-1}(x) + J_{m+1}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k-k)\left(\frac{x}{2}\right)^{2k+m-1} \\ &= \frac{2m}{x}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m} = \frac{2m}{x}J_m(x). \end{aligned}$$

The next two follow by adding and subtracting the first two; specifically,

$$J_{m-1}(x) = \frac{1}{2}\left(2J_m'(x) + \frac{2m}{x}J_m(x)\right) = J_m'(x) + \frac{m}{x}J_m(x),$$

while

$$J_{m+1}(x) = \frac{1}{2}\left(\frac{2m}{x}J_m(x) - 2J_m'(x)\right) = -J_m'(x) + \frac{m}{x}J_m(x).$$

Finally, identites 3 and 4 give

$$\frac{d}{dx}\left(x^m J_m(x)\right) = mx^{m-1}J_m(x) + x^m J_m'(x) = x^m\left(\frac{m}{x}J_m(x) + J_m'(x)\right) = x^m J_{m-1}(x),$$

$$\frac{d}{dx}\left(x^{-m} J_m(x)\right) = -mx^{-m-1}J_m(x) + x^{-m} J_m'(x) = x^{-m}\left(-\frac{m}{x}J_m(x) + J_m'(x)\right) = -x^{-m} J_{m+1}(x).$$

This completes the proof. QED.

Identity 5 above gives rise, for example, to the following integral formula:

$$\int x^m J_{m-1}(x)\,dx = x^m J_m(x) + C,$$

which may be used to calculate the coefficients in the expansion of x^m in a Fourier-Bessel series in Bessel functions of order m. This type of expansion is needed on Homework 6.

Finally, we say a few words about the case $\epsilon = -1$, corresponding to oscillatory behaviour in the z direction; explicitly, Z obeys the equation $Z'' = -\lambda^2 Z$, with general solution $Z = a\cos\lambda z + b\sin\lambda z$, while P satisfies the equation

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(-\lambda^2 - \frac{m^2}{\rho^2}\right)P = 0. \tag{6}$$

We recall that with $\epsilon = 1$ the solution to this equation is given by

$$J_m(\lambda\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{\lambda\rho}{2}\right)^{2k+m}.$$

Now it seems reasonable that replacing λ by $i\lambda$ (where $i = \sqrt{-1}$) should give a solution to equation (6); substituting in to the expression above, and dividing by i^m, we obtain the function

$$I_m(\lambda\rho) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}\left(\frac{\lambda\rho}{2}\right)^{2k+m}.$$

This function, known as a *modified Bessel function of order m*, is in fact a solution to equation (6) which is moreover well-behaved (i.e., finite) at $x = 0$. The other linearly independent solution to equation (6) is denoted $K_m(x)$ and will not be discussed here. Assuming that only the $I_m(\lambda\rho)$ factors occur in our separated solutions, a general separated solution to Laplace's equation in this case is of the form

$$I_m(\lambda\rho)\left(a\cos m\phi + b\sin m\phi\right)\left(c\cos\lambda z + d\sin\lambda z\right).$$

Inspecting and comparing the series expansions of $J_m(x)$ and $I_m(x)$, we note the similarity between them and the series for $\sin x$ and $\sinh x$:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \qquad I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m},$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \qquad \sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}.$$

Thus we see that $J_m(x)$ is the parallel for cylindrical coordinates of the oscillatory solution $\sin x$ in rectangular coordinates, while $I_m(x)$ is the parallel for the non-oscillatory (in fact, exponential) solution $\sinh x$. The parallels between these pairs go even deeper, as can be seen from the derivative and recurrence identities satisfied by $I_m(x)$ (see [1], 3.7); but we shall not go any deeper into these here.

The third practice problem for week 6 makes use of the functions $I_m(x)$.

REFERENCES

1. Watson, G. N. A Treatise on the Theory of Bessel Functions, 2nd ed. Cambridge: Cambridge University Press, 1952.